IESC Advanced Summer School on Mathematical Fluid Mechanics,
Cargèse, Corsica 2025/4/11

Regularization of the problem of Vortex reconnection

Yoshi Kimura

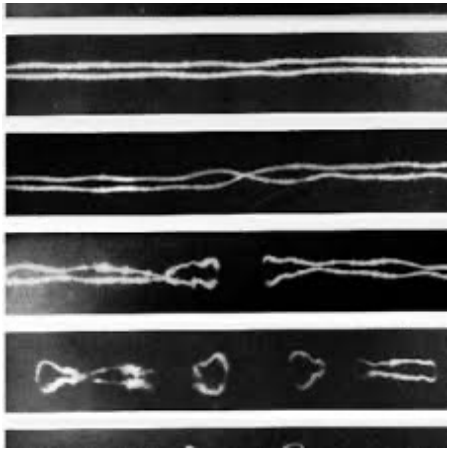
Graduate School of Mathematics, Nagoya University, Japan

Joint work with
H.K. Moffatt (University of Cambridge)
H. Fujiwara (Kyoto University)

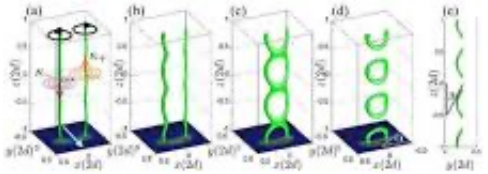
Vortex reconnection

Vortex reconnection is a fundamental process in turbulence which can be easily observed in nature.

This process is always associated with vortex stretching which enhances the magnitude of vorticity sometimes quite significantly.







Many researchers have been fascinated by this problem partially because it may give a clue for solving the problem of a Finite Time Singularity (FTS) of the Navier-Stokes equations:

ex. Pumir & Siggia (1987), Kerr & Hussain (1989), Kida & Takaoka (1994), Hussain & Duraisami (2011) and many others.

Plan of this lecture :

(1) Introduction of Moffatt-Kimura (MK) model for vortex reconnection (15~20 min.)

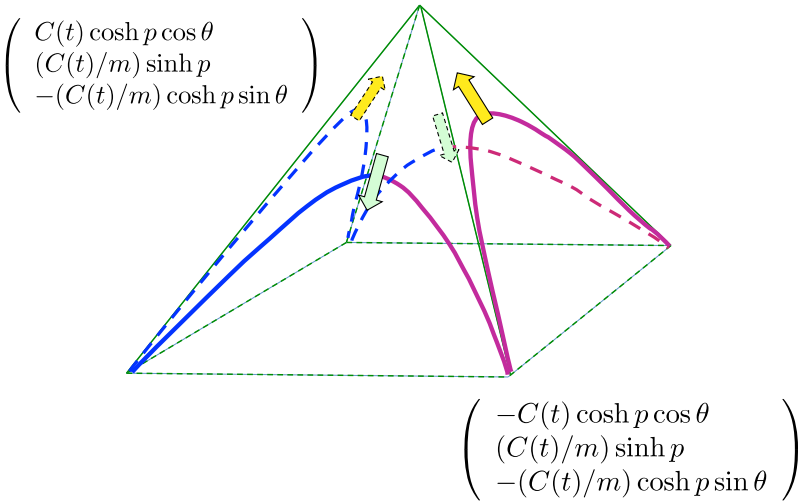
(2) Critical opinions from other researchers (5~10 min.)

(3) Biot-Savart model + regularization by Rosenhead (20 ~ 25 min.)

(4) Questions and comments (? min.)

Tent model of reconnection

Y. Kimura & H.K. Moffatt, A tent model of vortex reconnection under Biot-Savart evolution.
Journal of Fluid Mechanics **834** (2018) R1.



$$\mathbf{u}\left(\mathbf{x}(s)\right) = -\frac{\Gamma}{4\pi}\int \frac{(\mathbf{x}(s)-\mathbf{x}(s'))\times\mathbf{t}(s')}{||\mathbf{x}(s)-\mathbf{x}(s')||^3}ds'$$

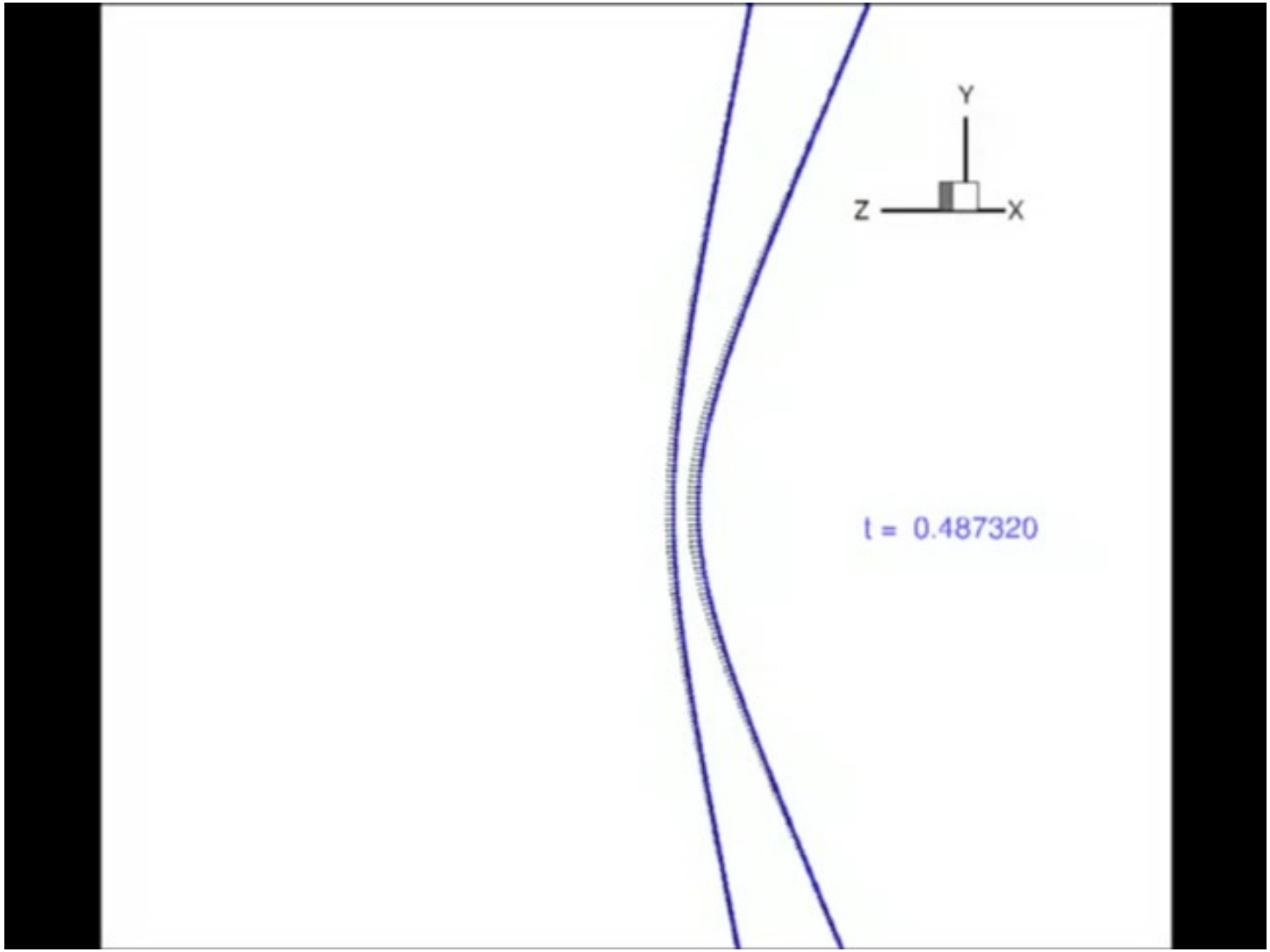
Biot-Savart integral

discretization

$$\mathbf{u}(\mathbf{x}_i) = -\frac{\Gamma}{4\pi}\sum_{\substack{j=0\\ j\neq i}}^{n-1} \frac{(\mathbf{x}_i-\mathbf{x}_j)\times\mathbf{t}_j}{||\mathbf{x}_i-\mathbf{x}_j||^3}\,\Delta s_j \qquad (i=0,1,\cdots,n-1)$$

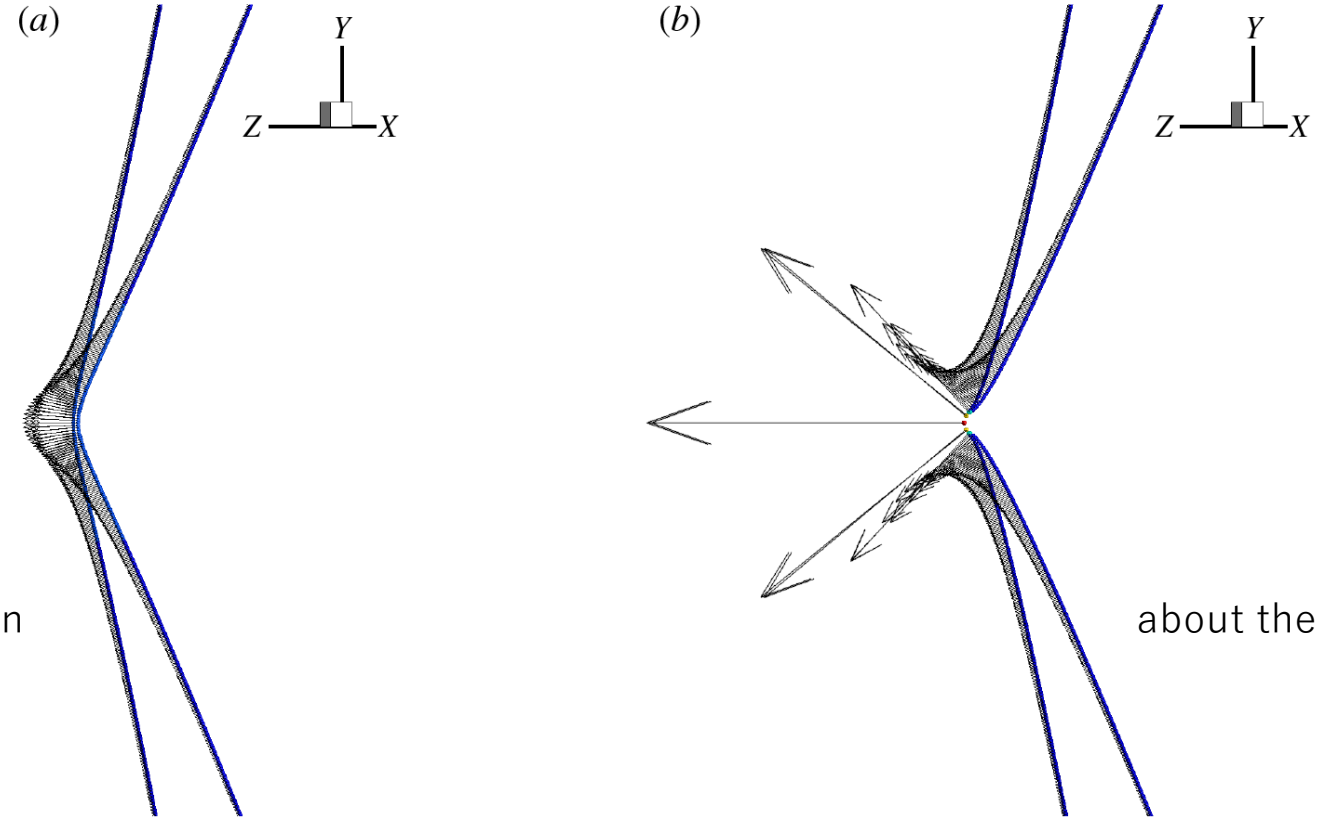
cf.
De Waele, A.T.A.M. & Aarts, R.G.K.M.
Route to vortex reconnection.
Phys. Rev. Lett. **72** (1994) 482-485.

The integral is calculated by a sum, and the singular term is removed (The cut-off method)


Y
Z
X
t = 0.487320

Simulation of tilted vortex hyperbola

(arrows : velocity vectors)



before reconnection

about the time of reconnection

FIGURE 3. Tent-model configuration and velocity vectors at points on the vortices Γ_1 and Γ_2 ($m = 0.35$, $c = 0.1$, $\theta = \pi/4$, $n = 19\,202$); (*a*) $t = 0.449646$; (*b*) $t = 0.449716$.

Scaling properties of the tent model

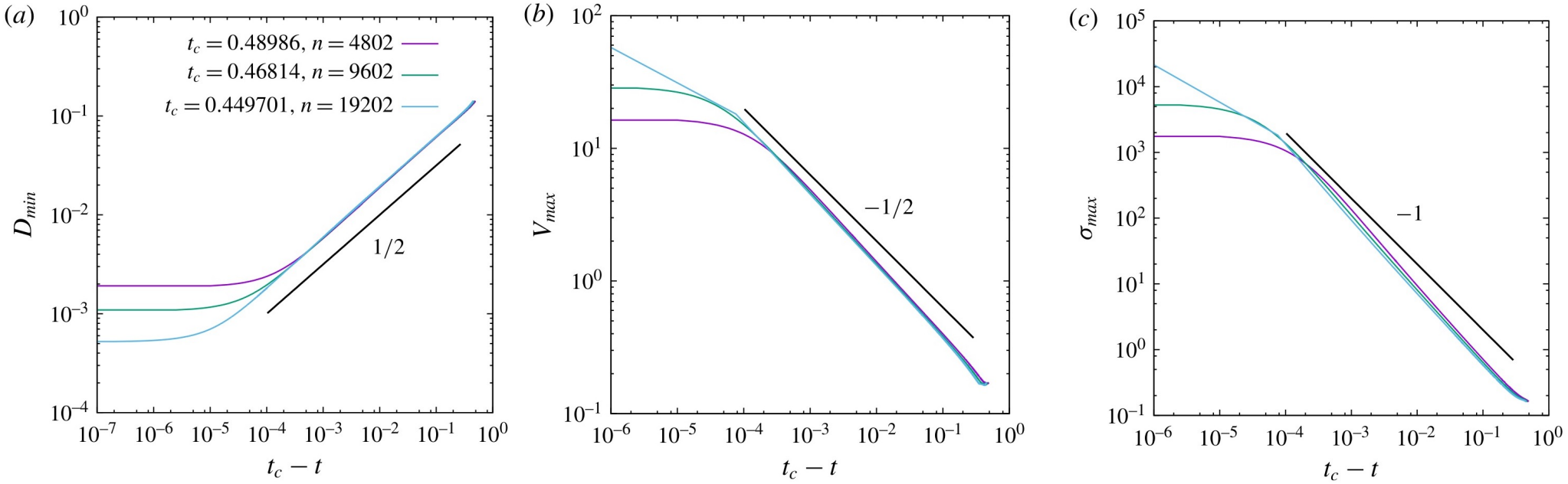


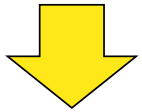
FIGURE 4. Scaling of (*a*) the minimum distance D_{min}, (*b*) the maximum velocity V_{max}, (*c*) the maximum axial strain rate $\sigma_{max} \equiv \boldsymbol{t}(s) \cdot \mathrm{d}\boldsymbol{u}(s)/\mathrm{d}s$ as a function of $t_c - t$ in log–log scale; results of three resolutions, $n = 4802, 9602, 19\,202$ are compared ($m = 0.35$, $c = 0.1$, $\theta = \pi/4$).

Close to the Leray scaling.

Strain around vortices

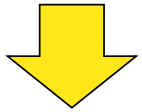
$$\mathbf{u}(q) = -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{(\mathbf{x}(q)-\mathbf{x}(p))\times\mathbf{x}'(p)\;dp}{|\mathbf{x}(q)-\mathbf{x}(p)|^3}$$

Biot-Savart integral



$$\partial_i u_j(\mathbf{x}) = \frac{\Gamma}{4\pi}\int\left[\frac{3\,(x_i - x_i(p))\,[(\mathbf{x}-\mathbf{x}(p))\times d\mathbf{x}(p)]_j}{|\mathbf{x}-\mathbf{x}(p)|^5} + \epsilon_{ijk}\frac{dx_k(p)}{|\mathbf{x}-\mathbf{x}(p)|^3}\right]$$

deformation tensor



$$\mathbf{e} = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^{\mathrm{T}})$$

rate of strain tensor

Three eigenvalues of $\mathbf{e}$: $\lambda_1 \le \lambda_2 \le \lambda_3$

Because of incompressibility : $\lambda_1 + \lambda_2 + \lambda_3 = 0$

$\lambda_1 \le 0, \qquad \lambda_3 \ge 0 \qquad \lambda_2$?

Eigenvalues and eigenvectors of the rate of strain tensor at the tipping point on Γ_1

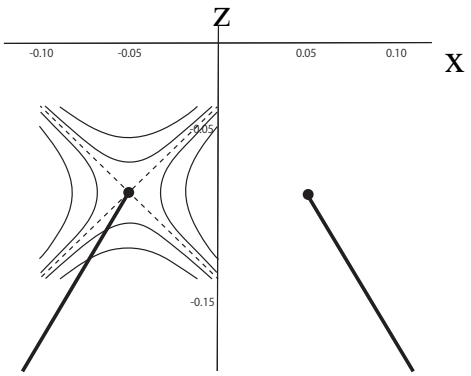
$t = 0.473200$

$\boldsymbol{\Gamma_1 + \Gamma_2}$:			$\boldsymbol{\Gamma_2}$:		
λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
2.4256E+02	2.1026E+01	-2.6359E+02	2.4179E+02	2.0781E+01	-2.6257E+02
$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\begin{pmatrix} -6.8940E-01 \\ 7.4945E-12 \\ 7.2438E-01 \end{pmatrix}$	$\begin{pmatrix} 2.7533E-12 \\ 1.0000E+00 \\ -7.7257E-12 \end{pmatrix}$	$\begin{pmatrix} 7.2438E-01 \\ 3.3316E-12 \\ 6.8940E-01 \end{pmatrix}$	$\begin{pmatrix} -7.0635E-01 \\ -1.7479E-15 \\ 7.0786E-01 \end{pmatrix}$	$\begin{pmatrix} -8.6695E-16 \\ 1.0000E+00 \\ 1.6042E-15 \end{pmatrix}$	$\begin{pmatrix} 7.0786E-01 \\ -5.1942E-16 \\ 7.0635E-01 \end{pmatrix}$

$t = 0.489820$

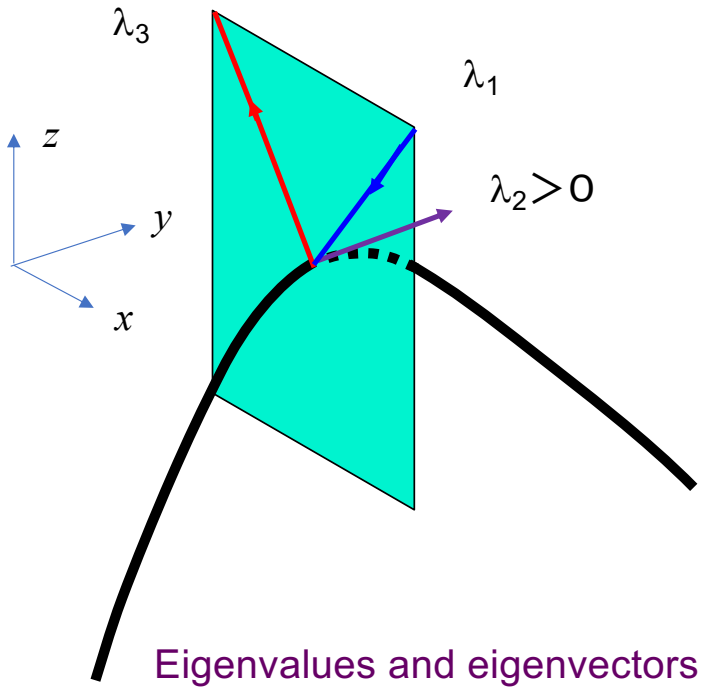
$\boldsymbol{\Gamma_1 + \Gamma_2}$:			$\boldsymbol{\Gamma_2}$:		
λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
3.0615E+04	6.0021E+03	-3.6617E+04	3.1620E+04	6.1659E+03	-3.7786E+04
$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\begin{pmatrix} -6.9195E-01 \\ 4.3931E-14 \\ 7.2194E-01 \end{pmatrix}$	$\begin{pmatrix} 1.4811E-12 \\ 1.0000E+00 \\ 1.3587E-12 \end{pmatrix}$	$\begin{pmatrix} 7.2194E-01 \\ -2.0094E-12 \\ 6.9195E-01 \end{pmatrix}$	$\begin{pmatrix} -7.0653E-01 \\ 7.3238E-13 \\ 7.0768E-01 \end{pmatrix}$	$\begin{pmatrix} 1.0330E-12 \\ 1.0000E+00 \\ -3.5946E-15 \end{pmatrix}$	$\begin{pmatrix} 7.0768E-01 \\ -7.2849E-13 \\ 7.0653E-01 \end{pmatrix}$

TABLE 1. Eigenvalues and eigenvectors of the rate-of-strain tensor at the tipping point on Γ_1 at times $t = 0.47320$ and $t = 0.489820$ corresponding to Figures 7 and 8.

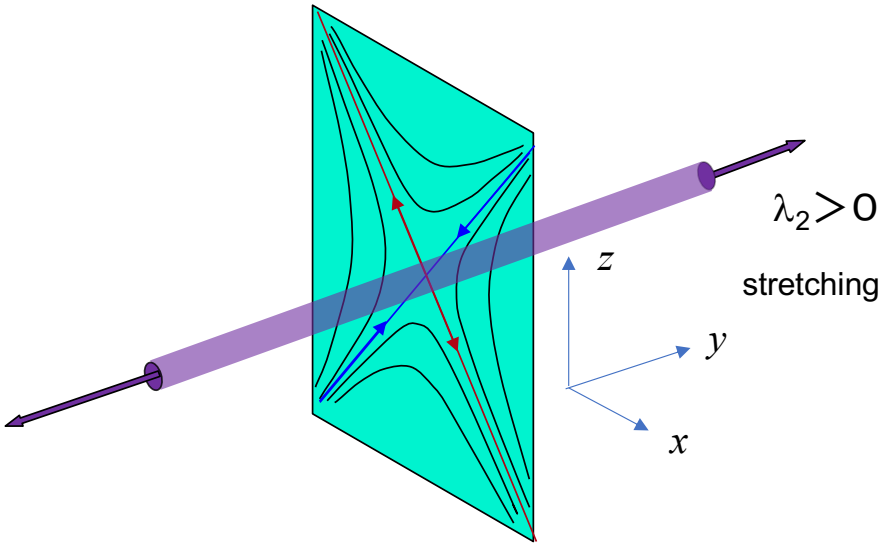


(1) λ_2 is positive and its eigenvector is along the y-axis.
(2) eigenvectors of λ_1 and λ_3 are orthogonal.
(3) at the tip on Γ_1, the rate-of-strain by Γ_2 is dominant in the direction $x = \pm z$

Similarity with non-axisymmetric Burgers vortex



Eigenvalues and eigenvectors of the strain field



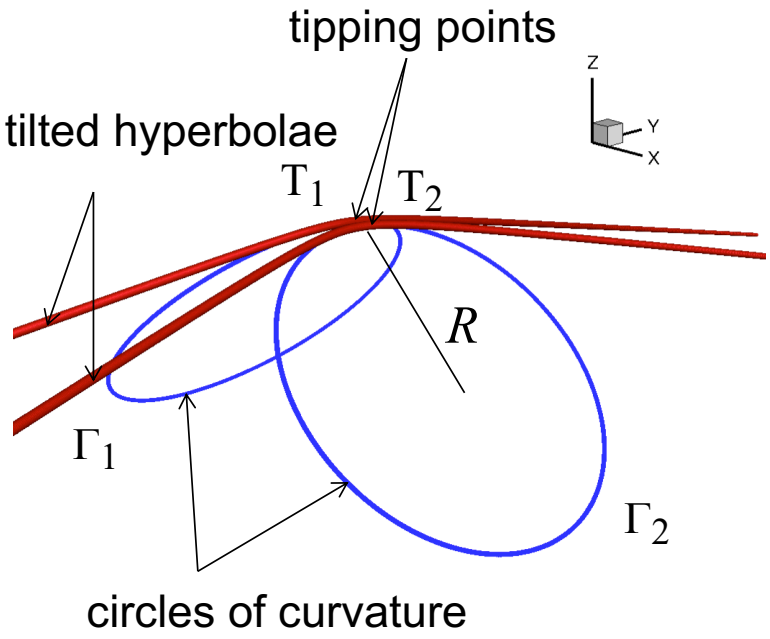
Non-axisymmetric Burgers vortex

$$\mathbf{U} = (\lambda_1 x, \lambda_2 y, \lambda_3 z) \qquad \lambda_2>0$$

$$\lambda_1 \leq \lambda_2 \leq \lambda_3, \quad \lambda_1 + \lambda_2 + \lambda_3 = 0$$

local strain field

Tilted hyperbolae → Tilted vortex rings



$$\mathbf{v}_1 = -\frac{\kappa_0 \Gamma}{4\pi}\,[\ \log(1/\kappa_0\delta)\ +\ \beta\ ]\ \mathbf{b}$$

(self-induced velocity)

$\mathbf{b}$: binormal unit vec.

$\kappa_0 = 1/R$: curvature

$\beta = 0.8283$ for Gaussian core

$\beta = 1.136$ for uniform vorticity core (Saffman 1970)

$$\mathbf{v}_2(\mathbf{x}_{T_1}) = \frac{\Gamma}{4\pi}\oint_{\Gamma_2} \frac{d\mathbf{x}' \wedge (\mathbf{x}_{T_1} - \mathbf{x}')}{|\mathbf{x}_{T_1} - \mathbf{x}'|^3}$$

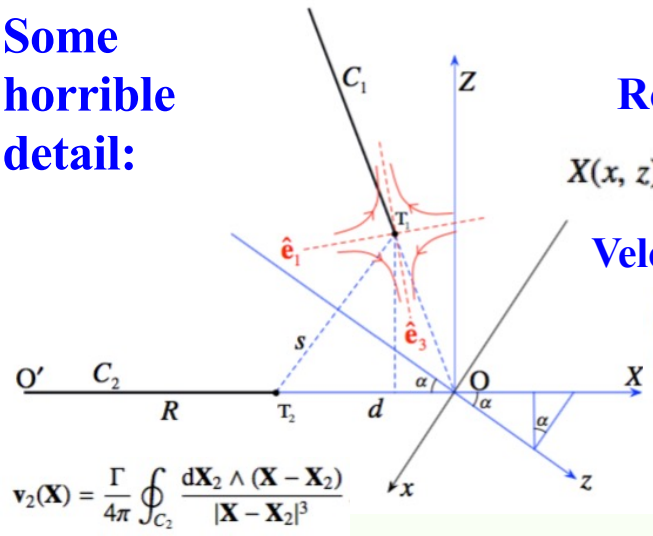
(mutually induced velocity)

can be evaluated in terms of elliptic integrals analytically

Then at the tipping point T_1

$$\mathbf{v} \quad = \quad \mathbf{v}_1 \quad + \quad \mathbf{v}_2$$

Some horrible detail:



$$\mathbf{v}_2(\mathbf{X}) = \frac{\Gamma}{4\pi}\oint_{C_2}\frac{d\mathbf{X}_2\wedge(\mathbf{X}-\mathbf{X}_2)}{|\mathbf{X}-\mathbf{X}_2|^3}$$

Rotate the coordinate system, *x,y,z* —> *X,Y,Z*

$$X(x,z) = R + d - x\sin\alpha + z\cos\alpha, \quad Y = y, \quad Z(x,z) = -x\cos\alpha - z\sin\alpha. \qquad (4.1)$$

Velocity induced by C_2 (now a circle on the plane $Z = 0$)

$$v_{2x}(x,y,z) = -v_{2X}(X(x,z),\, y,\, Z(x,z))\sin\alpha - v_{2Z}(X(x,z),\, y,\, Z(x,z))\cos\alpha,$$
$$v_{2z}(x,y,z) = +v_{2X}(X(x,z),\, y,\, Z(x,z))\cos\alpha - v_{2Z}(X(x,z),\, y,\, Z(x,z))\sin\alpha. \qquad (4.13)$$

$$r^2 = X^2 + Y^2 + Z^2, \quad m = \frac{2R(X^2+Y^2)^{1/2}}{R^2+r^2}, \quad \tan\gamma = Y/X.$$

With $\phi = \theta - \gamma$, the Biot-Savart integral (4.4) can be reduced to the form

$$\mathbf{v}_2(\mathbf{X}) = \frac{\Gamma R}{2\pi(R^2+r^2)^{3/2}}\,[I_1(m)\,Z\cos\gamma,\; I_1(m)\,Z\sin\gamma,\; R\,I_0(m) - I_1(m)\,(X\cos\gamma + Y\sin\gamma)], \quad (4.9)$$

where

$$I_0(m) = \int_0^\pi \frac{d\phi}{(1-m\cos\phi)^{3/2}}, \quad I_1(m) = \int_0^\pi \frac{\cos\phi\, d\phi}{(1-m\cos\phi)^{3/2}}. \qquad (4.10)$$

These integrals may be expressed in terms of complete elliptic integrals $K(k)$, $E(k)$ of the first and second kinds, defined in the notation of Abramowitz & Stegun (1964) by

$$K(k) = \int_0^{\pi/2}(1-k\sin^2\psi)^{-1/2}\,d\psi, \quad E(k) = \int_0^{\pi/2}(1-k\sin^2\psi)^{1/2}\,d\psi. \qquad (4.11)$$

The results are

$$I_0(m) = \frac{-2E(k)}{(m-1)(1+m)^{1/2}}, \quad I_1(m) = \frac{-2[E(k)+(m-1)K(k)]}{m(m-1)(1+m)^{1/2}}, \quad \text{where } k = \frac{2m}{m+1}. \qquad (4.12)$$

The induced velocity field (4.9) is thus expressible in terms of known functions; and it may be verified that this field does indeed satisfy $\nabla\cdot\mathbf{v}_2 = 0$, $\nabla\wedge\mathbf{v}_2 = 0$, identities that must therefore be satisfied at each order under expansion in the neighbourhood of any point.

So we have an explicit formula for the velocity field v_2 induced by the vortex C_2.

We can now extract the asymptotic behaviour of $d\kappa/dt$, s, δ and λ as $s \longrightarrow 0$, near the 'tipping point' T_1

Dynamical system describing the state near the tipping point

The tipping point dynamics is determined completely by

$s(\tau)$: 1/2 of the minimum separation of vortices

$\kappa(\tau)$: curvature at the tipping points

$\delta(\tau)$: core radius

$\gamma(\tau)$: $= \Gamma_s/\Gamma$ = (surviving) circulation/ original circulation

$\tau = (\Gamma/R^2)t$ (dimensionless time)

expression of velocity → $\dfrac{ds}{d\tau} = -\dfrac{\kappa\cos\alpha}{4\pi}\left[\log\left(\dfrac{s}{\delta}\right) + \beta_1\right]$

expression of curvature → $\dfrac{d\kappa}{d\tau} = \dfrac{\kappa\cos\alpha\sin\alpha}{4\pi s^2}$

$\epsilon = 1/R_\Gamma$

$R_\Gamma = \Gamma/\nu$

Vortex Reynolds number

NS eq (Burgers type vortex) + similarity assumption → $\dfrac{d\delta^2}{d\tau} = \epsilon - \dfrac{\kappa\cos\alpha}{4\pi s}\delta^2$

contour integration of velocity → $\dfrac{d\gamma}{d\tau} = -\epsilon\dfrac{s\gamma}{2\sqrt{\pi}\delta^3}\exp[-s^2/4\delta^2]$ when vortices overlap

Derivation of the equation for the separation

$$\frac{ds}{d\tau} = -\frac{\kappa \cos\alpha}{4\pi}\left[\log\left(\frac{s}{\delta}\right) + \beta_1\right]$$

At T_1 : $\mathbf{X} = (R(1 + 2\kappa_0 s \sin\alpha), 0, R\kappa_0 s \cos\alpha)$

$$\begin{cases} v_x = v_{1x} + v_{2x} \sim \dfrac{\kappa_0\Gamma}{4\pi}\left[\log\left(\dfrac{1}{\kappa_0\delta}\right) + \beta - \left(\log\dfrac{4}{\kappa_0 s} - 1\right)\right]\cos\alpha = \dfrac{\kappa_0\Gamma}{4\pi}\left[\log\left(\dfrac{s}{\delta}\right) + \beta_1\right]\cos\alpha \\ v_y = 0 \\ v_z = v_{1z} + v_{2z} \sim \dfrac{\kappa_0\Gamma}{4\pi}\left[\dfrac{1}{\kappa s} - \left\{\log\left(\dfrac{4}{\kappa^2 s\delta}\right) + \beta\right\}\sin\alpha\right] \end{cases}$$

$$\frac{ds}{dt} = -\frac{dx}{dt} = -v_x = -\frac{\kappa\Gamma}{4\pi}\left[\log\left(\frac{s}{\delta}\right) + \beta_1\right]\cos\alpha$$

With dimensionless time $\tau = \Gamma\kappa_0^2 t$, we obtain the equation

Derivation of the equation for curvature

$$\frac{d\kappa}{d\tau} = \frac{\kappa \cos\alpha \sin\alpha}{4\pi s^2}$$

$$\kappa(t) = \frac{|\mathbf{x}'(\theta,t) \times \mathbf{x}''(\theta,t)|}{|\mathbf{x}'(\theta,t)|^3}$$

Taylor expansion around $t = 0$

$$\kappa(t) = \kappa_0 + t\left(\frac{(\mathbf{x}_0' \times \mathbf{x}_0'') \cdot (\mathbf{u}_0' \times \mathbf{x}_0'' + \mathbf{x}_0' \times \mathbf{u}_0'')}{|\mathbf{x}_0' \times \mathbf{x}_0''|^2 |\mathbf{x}_0'|^3} - \frac{3(\mathbf{x}_0' \cdot \mathbf{u}_0')|\mathbf{x}_0' \times \mathbf{x}_0''|}{|\mathbf{x}_0'|^5} \right) + O(t^2)$$

Substituting

$$\mathbf{x}_0' = \frac{1}{\kappa_0}(0,1,0), \qquad \mathbf{x}_0'' = -\frac{1}{\kappa_0}(\sin\alpha, 0, \cos\alpha)$$

$$\mathbf{u}_0 = (u_{2x}, 0, u_{2z}), \quad \mathbf{u}_0' = (0, u_{2y}', 0), \quad \mathbf{u}_0'' = (u_{2x}'', 0, u_{2z}'')$$

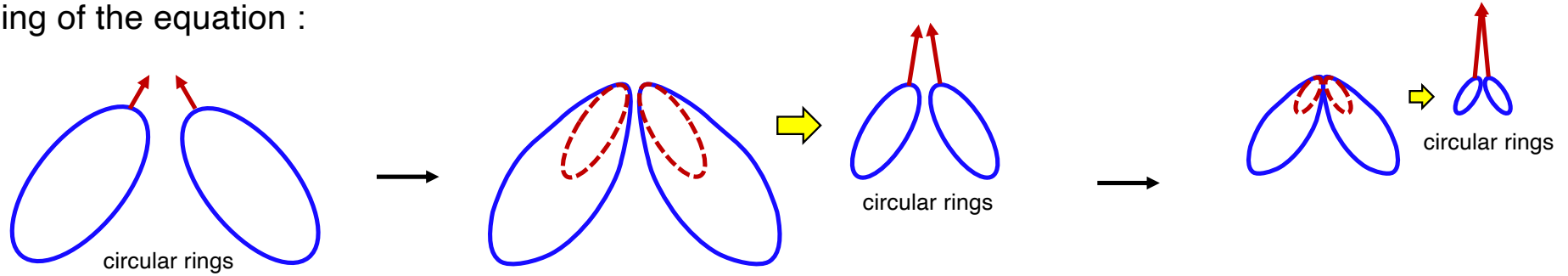
$$(\xi = \kappa s, \quad \tau = \Gamma\kappa^2 t)$$

$$u_{2x} \sim -\frac{\Gamma\kappa_0}{4\pi}\left(\log\frac{4}{\xi} - 1\right)\cos\alpha, \quad u_{2z} \sim \frac{\Gamma\kappa_0}{4\pi\xi}, \quad u_{2y}' \sim \frac{\Gamma\kappa_0}{4\pi\xi}\cos\alpha$$

$$u_{2x}'' \sim \frac{\Gamma\kappa_0 \cos\alpha}{8\pi\xi}, \qquad u_{2z}'' \sim -\frac{\Gamma\kappa_0 \sin\alpha}{4\pi\xi^2} + \frac{\Gamma\kappa_0(1 - 3\cos 2\alpha)}{8\pi\xi}$$

into ——— , you will get to the equation.

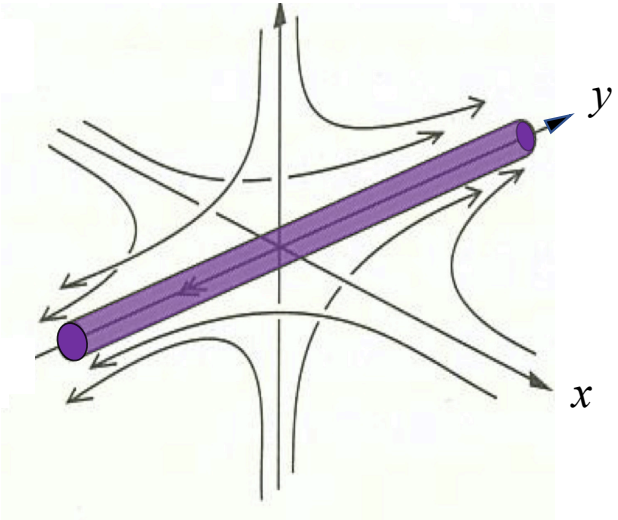
Meaning of the equation :



Derivation of the equation for the core radius δ

$$\frac{d\delta^2}{d\tau} = \epsilon - \frac{\kappa \cos\alpha}{4\pi s}\delta^2$$

Moffatt, H.K, 2000 The interaction of skewed vortex pairs: a model for blow-up of the Navier–Stokes equations *J. Fluid Mech.* **409**, 51- 68.



in cylindrical coordinates: $(r,\ \theta,\ y)$ $\mathbf{U} = \left(-\frac{1}{2}\lambda(t)r,\ 0,\ \lambda(t)y\right)$: straining velocity field

$$\boldsymbol{\omega} = (0, 0, \omega(r,t))$$

$$\frac{\partial\omega}{\partial t} = \frac{\lambda(t)}{2r}\frac{\partial}{\partial r}(r^2\omega) + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\omega}{\partial r}\right) \quad (*)$$

$$\omega(r,t) = \frac{\Gamma(\mu-1)}{4\pi\nu(t_c-t)}\exp\left(\frac{-(\mu-1)r^2}{4\nu(t_c-t)}\right)$$

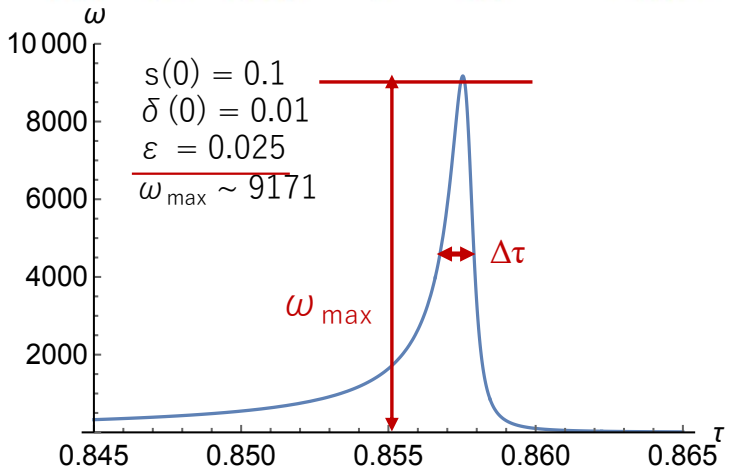
$$\lambda(t) = \frac{\mu}{t_c - t}, \qquad 0 < t < t_c$$

substituting $\omega(r,t) = \frac{1}{4\pi\delta^2(t)}\exp\left(-\frac{r^2}{4\delta^2(t)}\right)$ into $(*)$ → $\frac{d}{dt}\delta^2(t) = \nu - \lambda(t)\delta^2(t)$

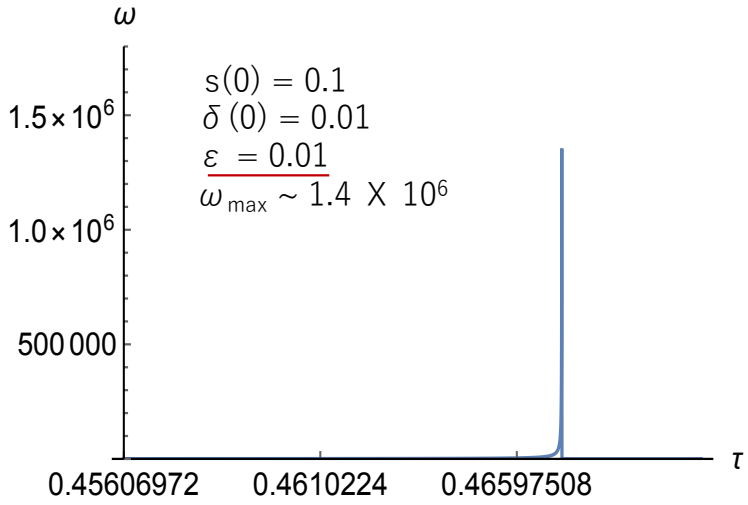
Expression of velocity → The rate of strain tensor → Eigenvalues → λ(t)

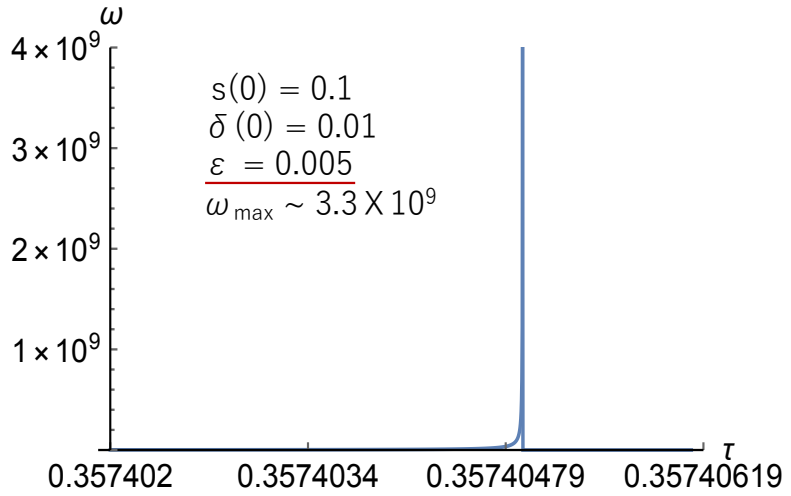
Our dynamical system :

$$\frac{ds}{d\tau} = -\frac{\gamma\kappa\cos\alpha}{4\pi}\left[\log\left(\frac{s}{\delta}\right)+\beta_1\right], \quad \frac{d\kappa}{d\tau} = \frac{\gamma\kappa\cos\alpha\sin\alpha}{4\pi s^2}, \quad \frac{d\delta^2}{d\tau} = \epsilon - \frac{\gamma\kappa\cos\alpha}{4\pi s}\delta^2, \quad \frac{d\gamma}{d\tau} = -\epsilon\frac{s\gamma}{2\sqrt{\pi}\,\delta^3}\exp[-s^2/4\delta^2]$$



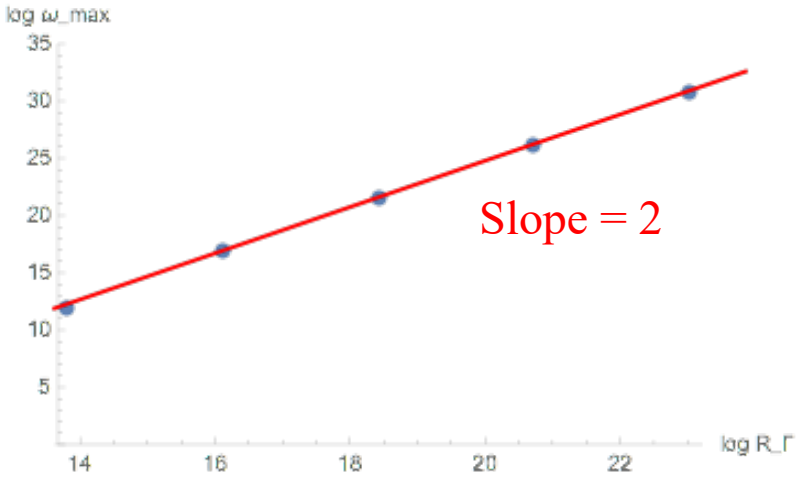
As ε becomes smaller (Re becomes larger)
$\Delta\tau$ becomes smaller and ω_{max} becomes larger.

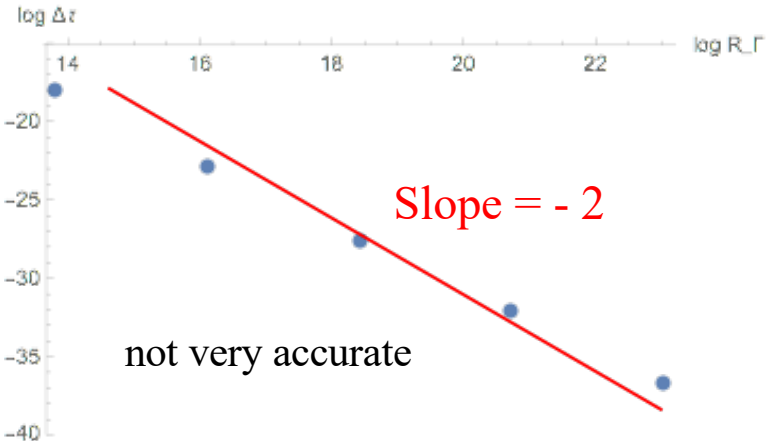




ω_{max}, δ_{min} and $\Delta\tau$ as functions of R_Γ $(\Gamma \sim \omega_{max}\,\delta_{min}^2)$

from numerical integration of dynamical system with s(0)=0.1

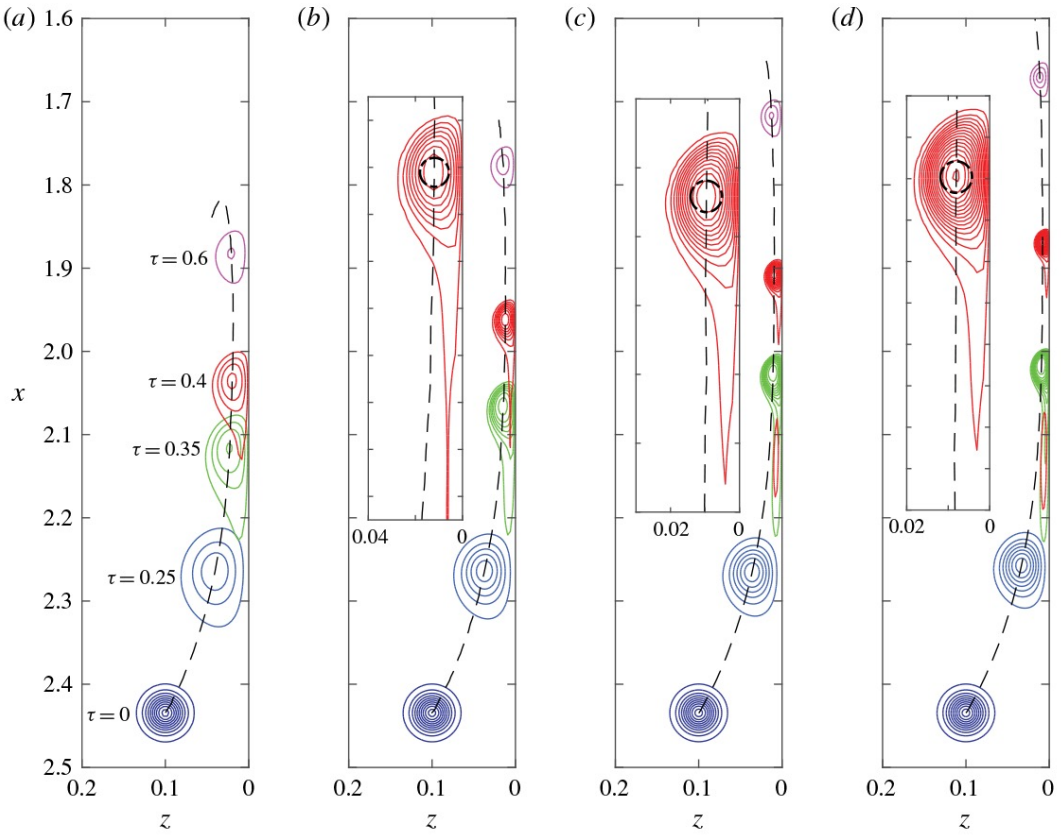




$\omega_{max} \sim R_\Gamma^{\,2}$ so $\delta_{min} \sim R_\Gamma^{\,-1}$ $\Delta\tau \sim R_\Gamma^{\,-2}$ $(R_\Gamma = 1/\varepsilon)$

- Reconnection becomes more and more intermittent with the Reynolds number.

Critical opinions from other researchers



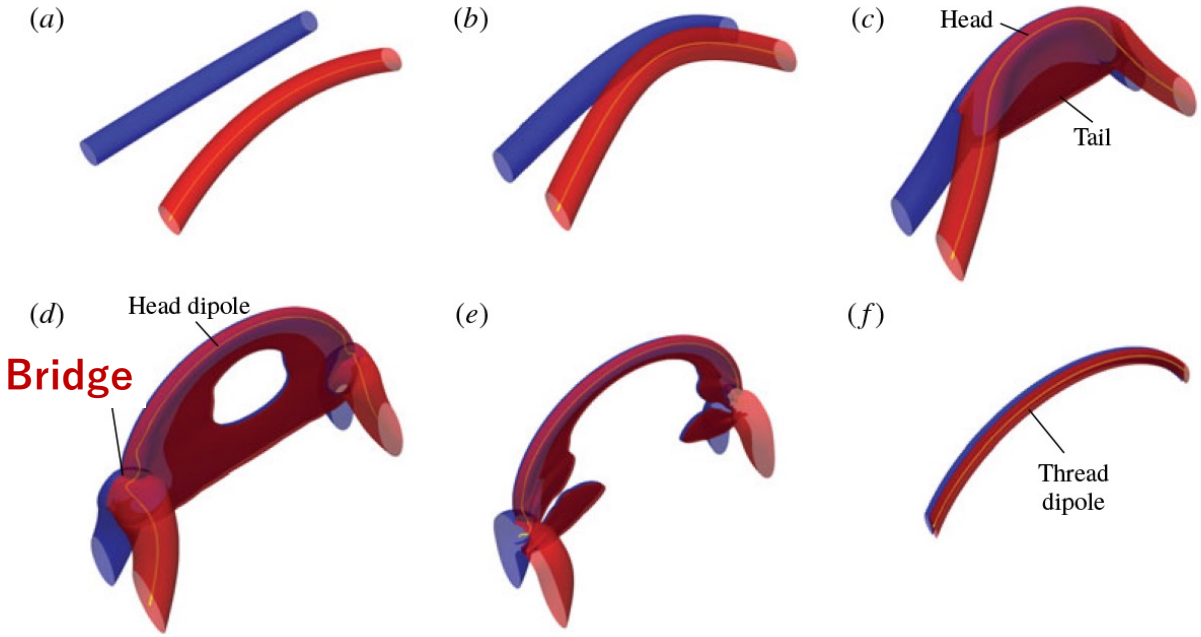
On singularity formation via viscous vortex reconnection

J. Yao & F. Hussain

J. Fluid Mech. (2020), vol. **888**, R2, doi:10.1017/jfm.2020.58

Significant flattening of the core should suppress the vortex stretching.

Other researchers who have a similar criticism:
M. Brenner, A. Pumir, R. Kerr etc.



On singularity formation via viscous vortex reconnection

J. Yao & F. Hussain

J. Fluid Mech. (2020), vol. **888**, R2, doi:10.1017/jfm.2020.58

Their claim :

(1) Core deformation by overlapping of vortices
(2) Bridging of vortices

} two missing physical mechanisms in the MK model

Our refute :

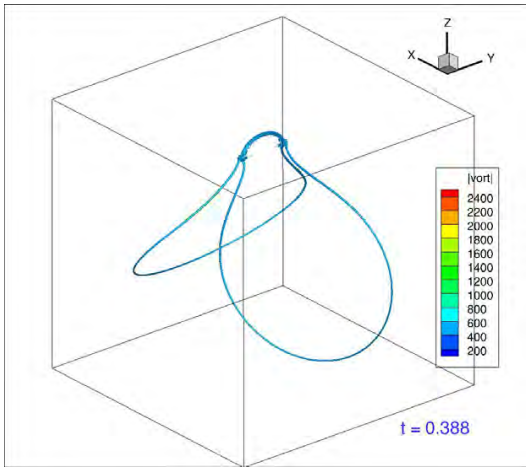
By choosing initial separation, core radius and Reynolds number, we can keep the core shape close to circular. → Moffatt and Kimura, *JFM* (2023) **967**, R1.

Biot-Savart model + regularization by Rosenhead

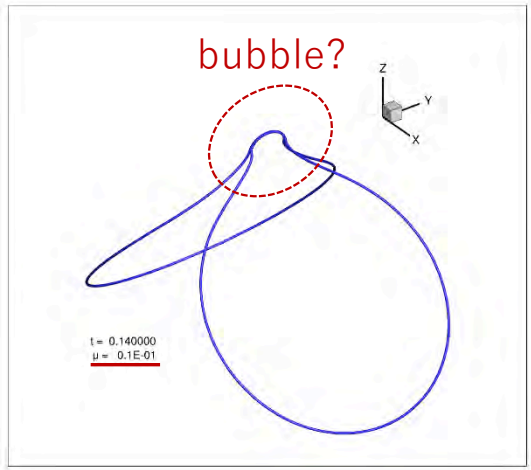
$$\mathbf{u}(\mathbf{X}(s)) = -\frac{\Gamma}{4\pi}\int \frac{\mathbf{t}(s')\times(\mathbf{X}(s)-\mathbf{X}(s')}{(|\mathbf{X}(s)-\mathbf{X}(s')|^2+\mu^2)^{3/2}}\,ds'$$

Regularization parameter

Motivation of this study :



DNS



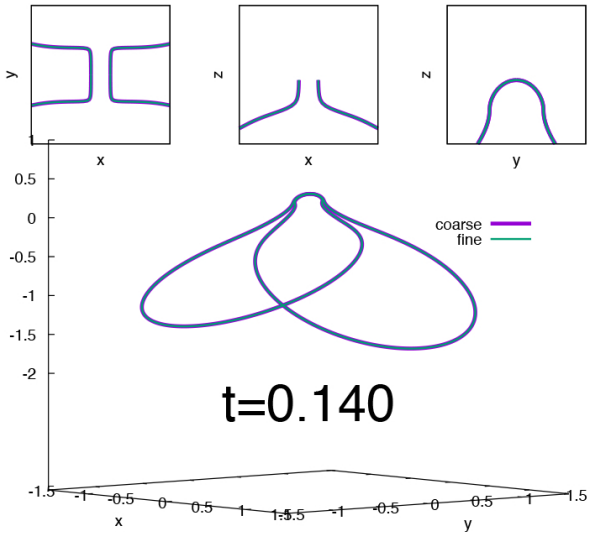
Biot-Savart + Rosenhead regularization

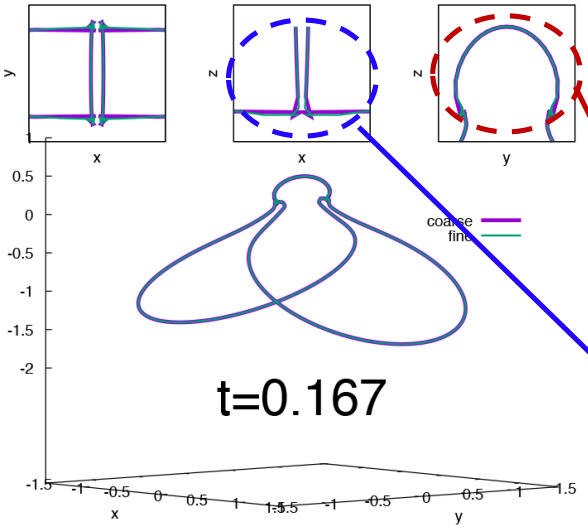
Rosenhead regularization with relatively large parameter value would produce some artifact (bubble) similar with the DNS calculation of the NS.

Numerical investigation of the 3D regularized Biot-Savart model towards vortex reconnection

YK & H. Fujiwara, in preparation

$\mu = 0.01$





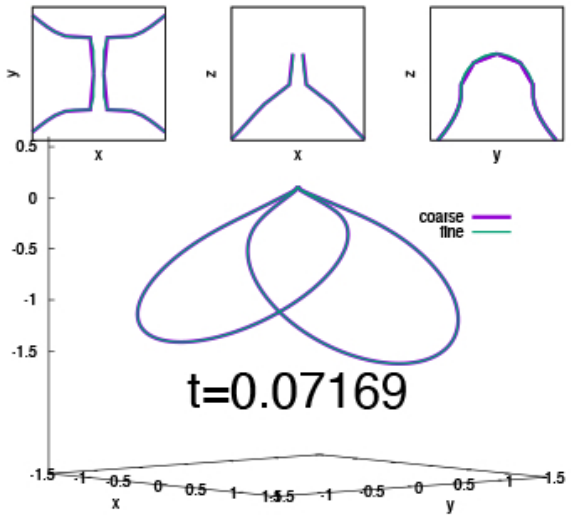
bubble

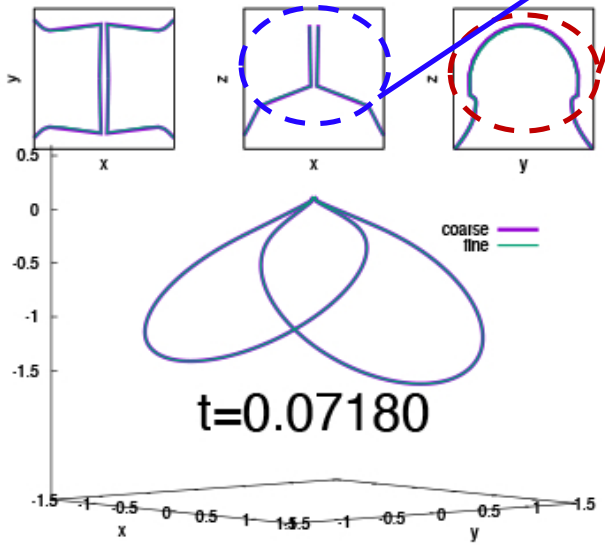
chimney

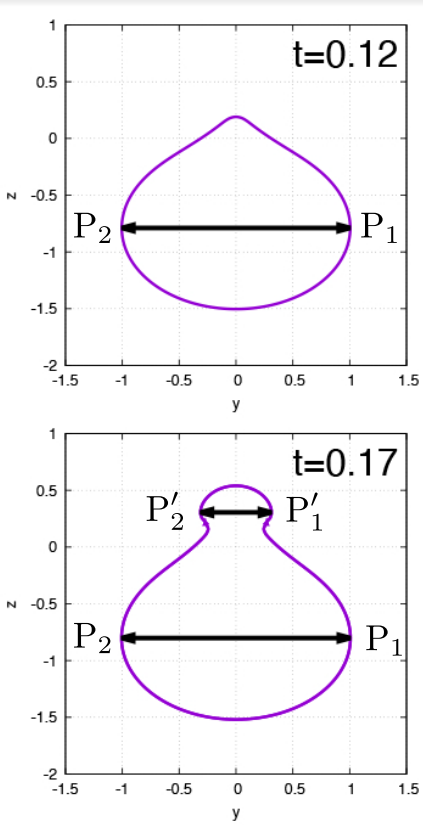
grow in time

How and Why ?

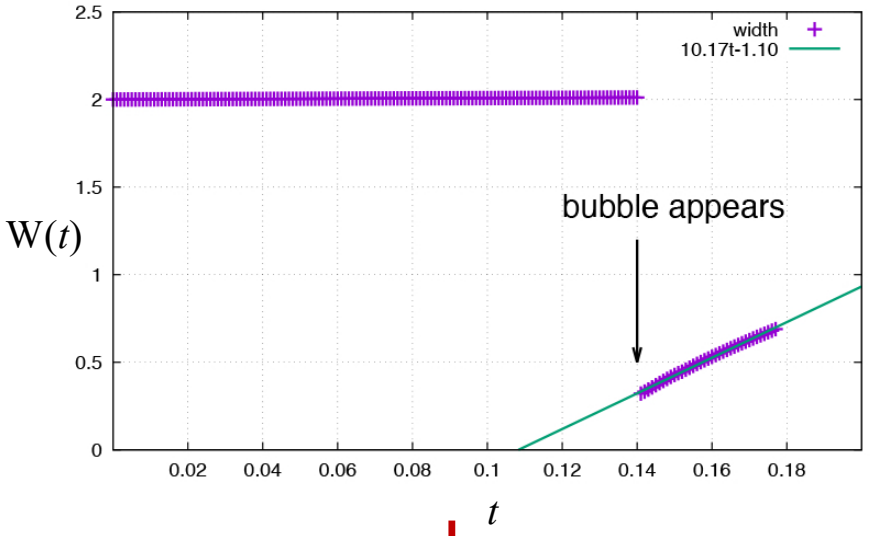
$\mu = 0.001$



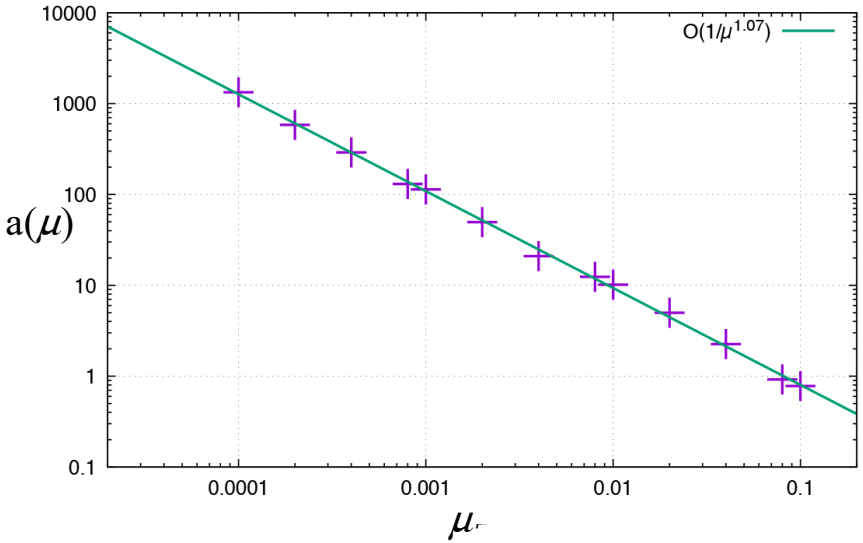




Checking the local maximum, and taking the interval at the largest in z. → $W(t)$

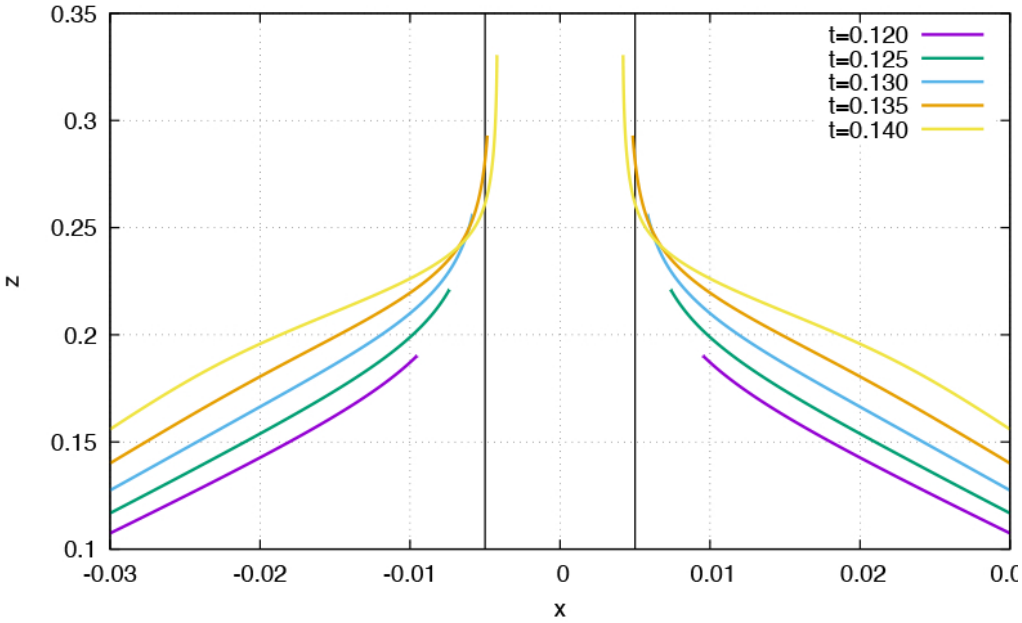


The growth of a bubble Is always linear in time, $W(t) = a(\mu)t + b$, but the growth rate differs.

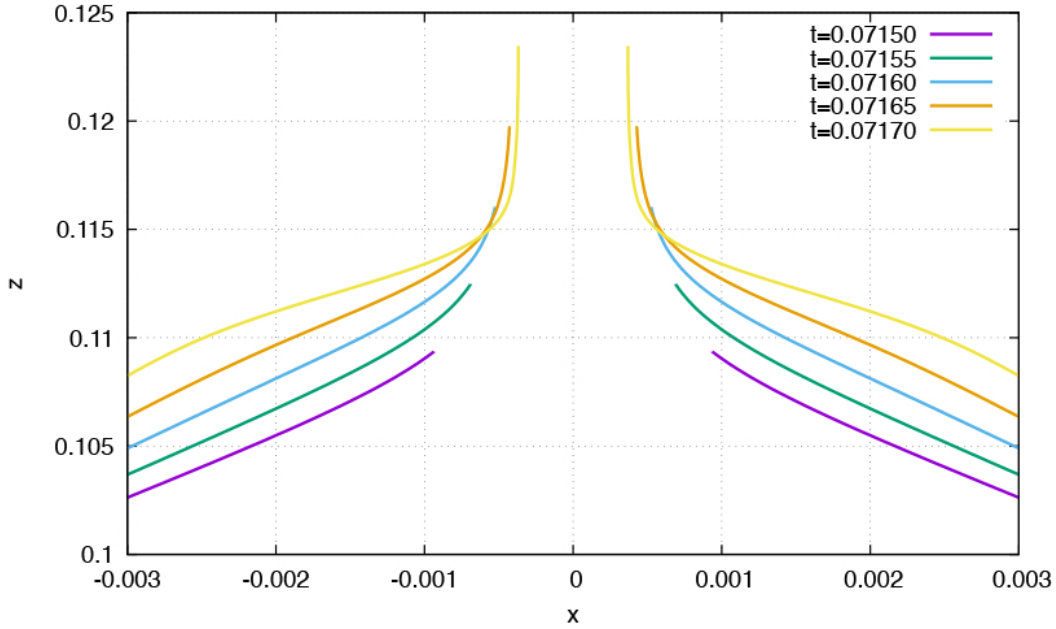


The growth rate, $a(\mu)$ is inversely proportional to μ.

Growth of the chimney



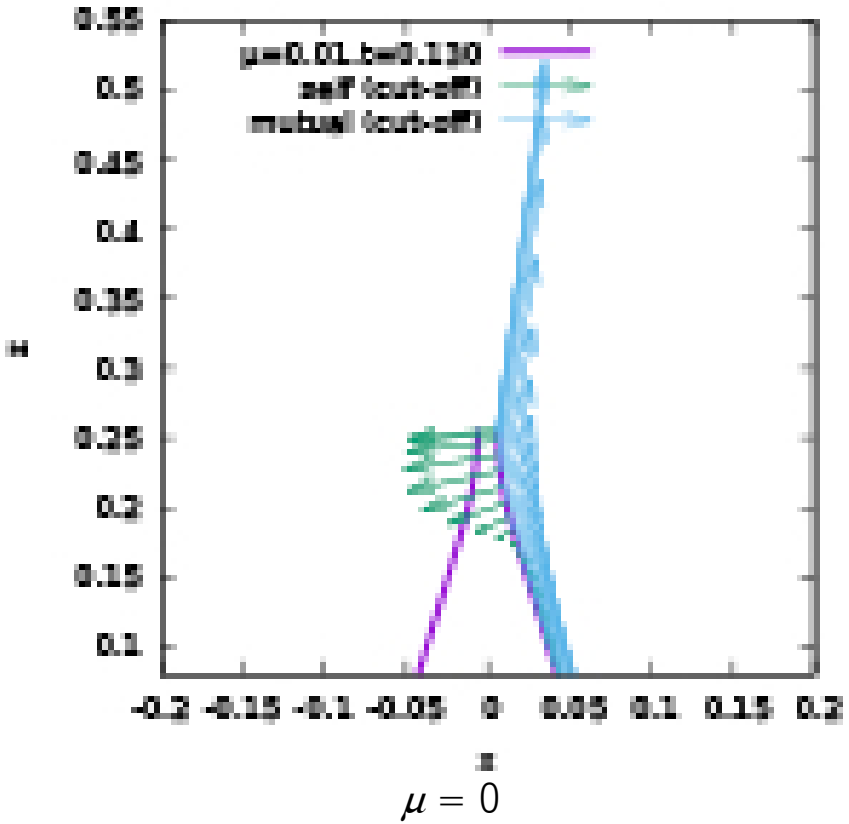
$\mu = 0.01$



$\mu = 0.001$

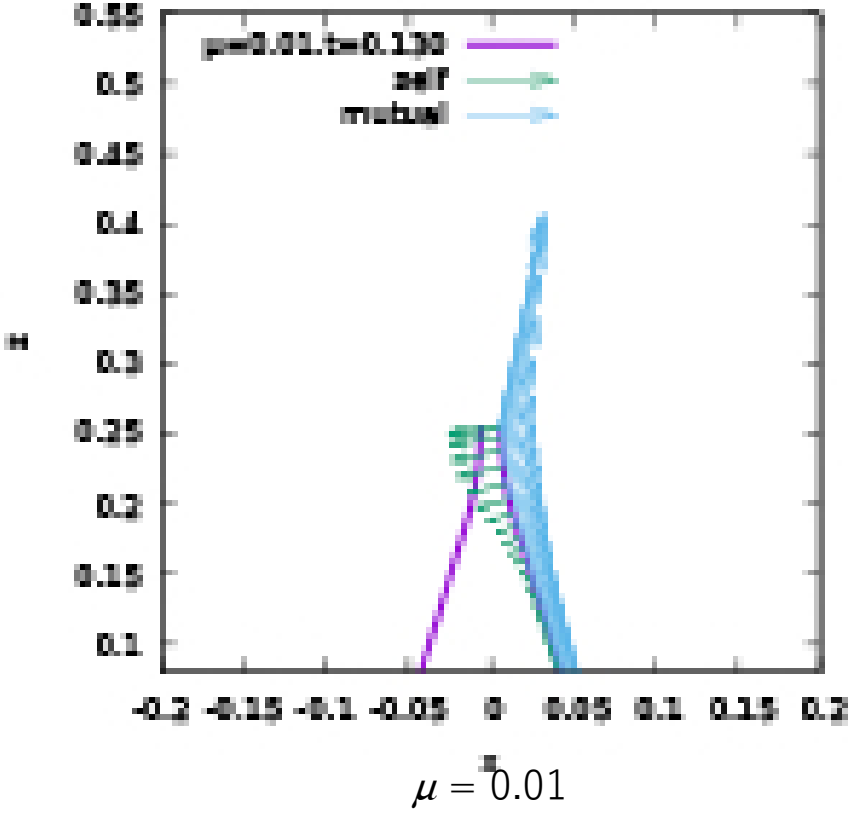
Around the time when the spacing of the tipping points ~ μ , vortices stop approaching and tend to move only vertically.

Green : projection of the self velocity, Blue : projection of the mutual velocity



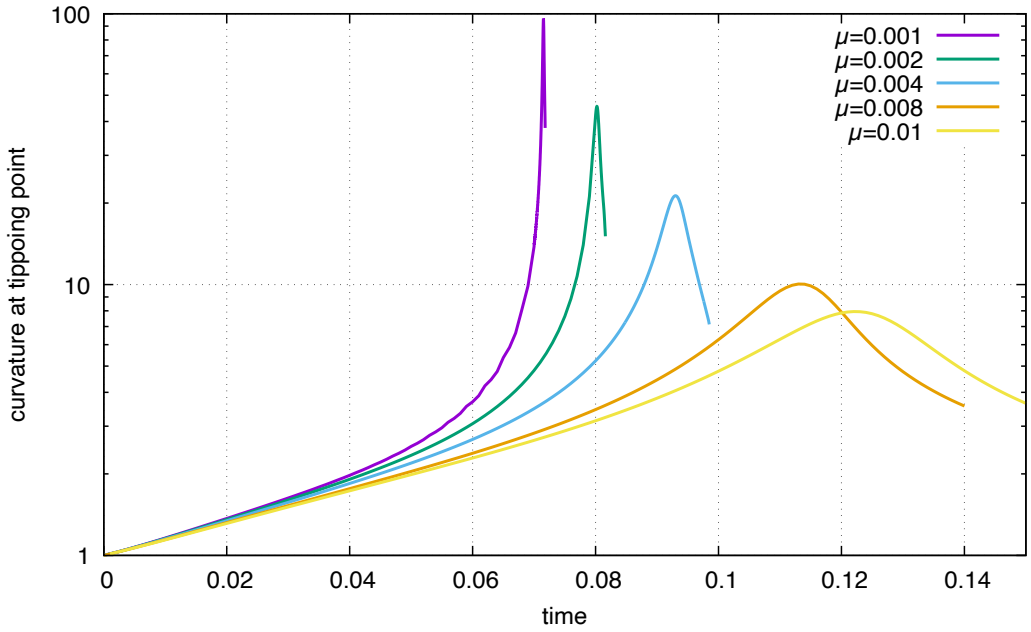
$\mu = 0$

The self velocity has larger x component than the mutual one. → vortices keep approaching.

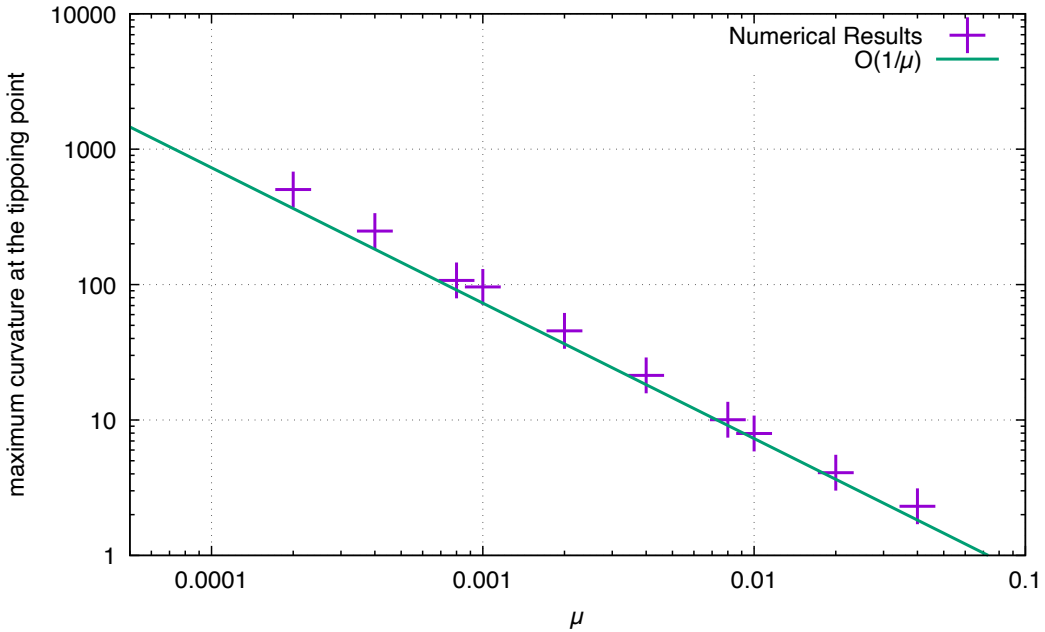


$\mu = 0.01$

Both the self and the mutual velocity decrease, but the self velocity has larger degree of decrease, and the x components almost cancel. → vortices move vertically.

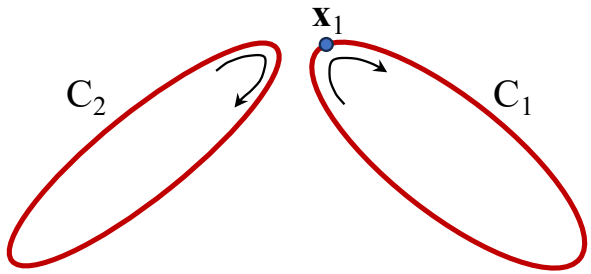


Curvature at the tipping point becomes very sharp with μ



Peak curvature is inversely proportional with μ

Decomposition of the velocity



$$\mathbf{u}(\mathbf{x}_1) = \frac{\Gamma}{4\pi}\oint_{C1+C2} \frac{d\mathbf{x}\wedge(\mathbf{x}_1-\mathbf{x})}{\{|\mathbf{x}-\mathbf{x}_1|^2+\mu^2\}^{3/2}}$$

$$= \frac{\Gamma}{4\pi}\oint_{C1} \frac{d\mathbf{x}\wedge(\mathbf{x}_1-\mathbf{x})}{\{|\mathbf{x}-\mathbf{x}_1|^2+\mu^2\}^{3/2}} + \frac{\Gamma}{4\pi}\oint_{C2} \frac{d\mathbf{x}\wedge(\mathbf{x}_1-\mathbf{x})}{\{|\mathbf{x}-\mathbf{x}_1|^2+\mu^2\}^{3/2}}$$

$\mathbf{u}_{C_1}(\mathbf{x}_1)$: self induced velocity

$\mathbf{u}_{C_2}(\mathbf{x}_1)$: mutually induced velocity

Localized Induction Approximation

Da Rios (1906), Arms & Hama (1965)

Analytic solution with the complete elliptic integrals.

$\mathbf{u}_{C_1}(\mathbf{x}_1)$: self induced velocity

$$\mathbf{u}_{C1}(\mathbf{x}_1(q)) = \frac{\Gamma}{4\pi}\int \frac{(\partial\mathbf{x}/\partial s)(p)\times(\mathbf{x}_1(q)-\mathbf{x}(p))}{\{|\mathbf{x}_1(q)-\mathbf{x}(p)|^2+\mu^2\}^{3/2}}\,dp.$$

Setting $p = q+\xi$ where $|\xi| \ll 1$, we have

Substituting

$$\begin{cases} \mathbf{x}(p)-\mathbf{x}_1(q) = \mathbf{x}(q+\xi)-\mathbf{x}_1(q) = \left(\mathbf{x}_1(q)+\dfrac{d\mathbf{x}_1}{ds}(q)\xi+\dfrac{1}{2}\dfrac{d^2\mathbf{x}_1}{ds^2}(q)\xi^2+\cdots\right)-\mathbf{x}_1(q) \\ \qquad\qquad = \dfrac{d\mathbf{x}_1}{ds}(q)\xi+\dfrac{1}{2}\dfrac{d^2\mathbf{x}_1}{ds^2}(q)\xi^2+\cdots \\ \dfrac{d\mathbf{x}}{ds}(p) = \dfrac{d\mathbf{x}_1}{ds}(q)+\dfrac{d^2\mathbf{x}_1}{ds^2}(q)\xi+\cdots \end{cases}$$

Upper limit ~ Large scale

$$\mathbf{u}_{C1}(\mathbf{x}_1(q)) = \frac{\Gamma}{4\pi}\kappa(q)\mathbf{b}(q)\int_0^{L}\frac{\left|\frac{d\mathbf{x}}{ds}(q)\right|^3|\xi|^2}{\left(\left|\frac{d\mathbf{x}}{ds}(q)\right|^2|\xi|^2+\mu^2\right)^{3/2}}d\xi$$

where $\left(\dfrac{d\mathbf{x}_1}{ds}(q)\times\dfrac{d^2\mathbf{x}_1}{ds^2}(q)\right)\Big/\left|\dfrac{d\mathbf{x}_1}{ds}(q)\right|^3 = \kappa(q)$ (curvature) $\mathbf{b}(q)$ (binormal unit vector)

$$\mathbf{u}_{C1}(\mathbf{x}_1(q)) = \frac{\Gamma}{4\pi}\kappa(q)\mathbf{b}(q)\int_0^L \frac{\left|\frac{d\mathbf{x}_1}{ds}(q)\right|^3 |\xi|^2}{\left(\left|\frac{d\mathbf{x}_1}{ds}(q)\right|^2 |\xi|^2 + \mu^2\right)^{3/2}} d\xi$$

$$= \frac{\Gamma}{4\pi}\kappa(q)\mathbf{b}(q)\left[\sinh^{-1}\left(\left|\frac{d\mathbf{x}_1}{ds}(q)\right|\left(\frac{L}{\mu}\right)\right) - \frac{\left|\frac{d\mathbf{x}_1}{ds}(q)\right|\left(\frac{L}{\mu}\right)}{\sqrt{\left|\frac{d\mathbf{x}_1}{ds}(q)\right|^2\left(\frac{L}{\mu}\right)^2 + 1}}\right]$$

$$= \frac{\Gamma}{4\pi}\kappa(q)\mathbf{b}(q)\left[\sinh^{-1}\left(\frac{L}{\mu}\right) - \frac{1}{\sqrt{1+(\mu/L)^2}}\right]$$

(we suppose s is arclength → $\left|\frac{d\mathbf{x}_1}{ds}(q)\right| = 1$)

On the other hand, for the Biot-Savart case (without regularization) we have

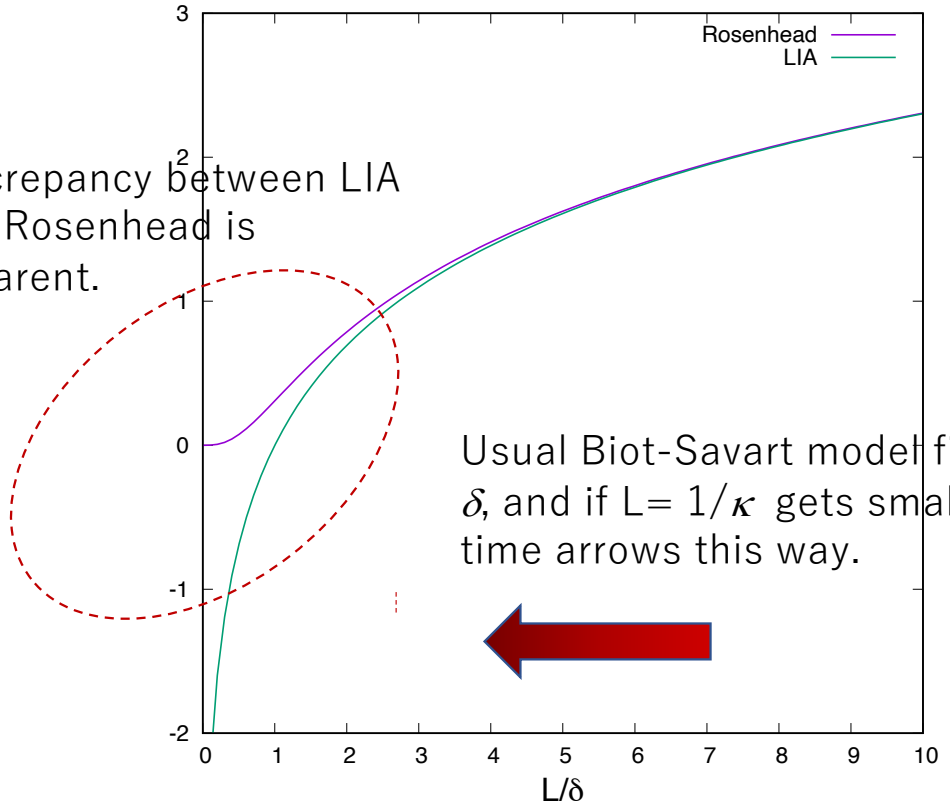
$$\mathbf{u}(\mathbf{x}(q)) = \frac{\Gamma}{4\pi}\int \frac{(\partial\mathbf{x}/\partial s)(p)\times(\mathbf{x}(q)-\mathbf{x}(p))}{|\mathbf{x}(q)-\mathbf{x}(p)|^3}\, dp. \longrightarrow \frac{\Gamma}{4\pi}\kappa(q)\mathbf{b}(q)\log\left(\frac{L}{\delta}\right) \quad (\delta > 0)$$

Small scale ~ core radius

For comparison, we require these two formulae match in the region $L/\mu \to \infty$

If the following asymptotic relation is used

$$\lim_{L/\mu\to\infty}\left[\sinh^{-1}\left(\frac{L}{\mu}\right)-\frac{L}{\sqrt{L^2+\mu^2}}\right]=\log\left(\frac{2L}{\mu}\right)-1=\log\left(\frac{2L}{\mu e}\right)\sim\ \log\left(\frac{L}{\delta}\right)$$



Discrepancy between LIA and Rosenhead is apparent.

Usual Biot-Savart model fixes δ, and if L= $1/\kappa$ gets smaller, time arrows this way.

$$2L/(\mu e)\sim L/\delta\ \rightarrow\ \underline{\mu\sim(2/e)\delta}$$

optimized value of μ (based on δ for asymptotic equivalence).

$\mathbf{u}_{C_2}(\mathbf{x}_1) = (u_{C_2}(\mathbf{x}_1), v_{C_2}(\mathbf{x}_1), w_{C_2}(\mathbf{x}_1))$: mutually induced velocity

For $\mathbf{x}_1 = (x_1, y_1, z_1)$, we define the following coordinate transformation

$$\mathbf{X} = (X(x_1, z_1), Y, Z(x_1, z_1)) = (R + d - x_1 \sin\alpha + z_1 \cos\alpha, y_1, -x_1 \cos\alpha - z_1 \sin\alpha)$$

$$\begin{cases} u_{C_2}(x_1, y_1, z_1) = -U(X(x_1, z_1), Y, Z(x_1, z_1)) \sin\alpha - W(X(x_1, z_1), Y, Z(x_1, z_1)) \cos\alpha \\ v_{C_2}(x_1, y_1, z_1) = V(X(x_1, z_1), Y, Z(x_1, z_1)) \\ w_{C_2}(x_1, y_1, z_1) = U(X(x_1, z_1), Y, Z(x_1, z_1)) \cos\alpha - W(X(x_1, z_1), Y, Z(x_1, z_1)) \sin\alpha \end{cases}$$

$$\begin{cases} U(X,Y,Z) = \dfrac{\Gamma R}{2\pi(R^2 + r^2)^{3/2}} Z \cos\gamma I_1(m,\xi) \\ V(X,Y,Z) = \dfrac{\Gamma R}{2\pi(R^2 + r^2)^{3/2}} Z \sin\gamma I_1(m,\xi) \\ W(X,Y,Z) = \dfrac{\Gamma R}{2\pi(R^2 + r^2)^{3/2}} [R I_0(m,\xi) - (X\cos\gamma + Y\sin\gamma) I_1(m,\xi).] \end{cases}$$

$$r^2 = X^2 + Y^2 + Z^2$$

$$m = \frac{2R(X^2 + y^2)^{1/2}}{R^2 + r^2}$$

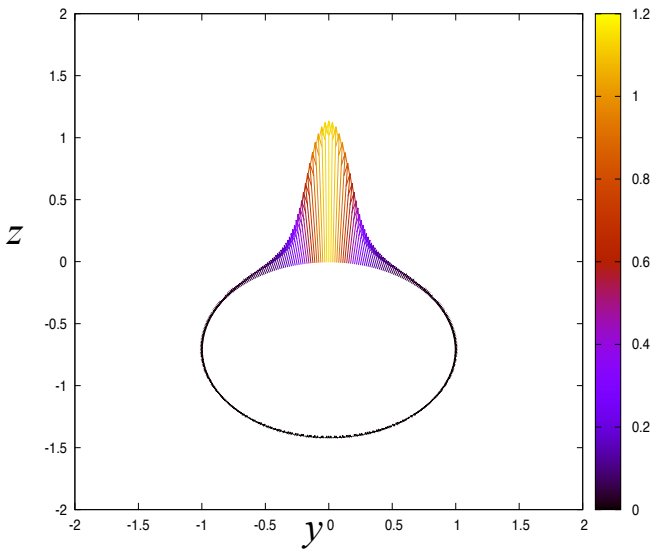
$$\tan\gamma = Y/X$$

$$\xi^2 = \mu^2/(R^2 + r^2)$$

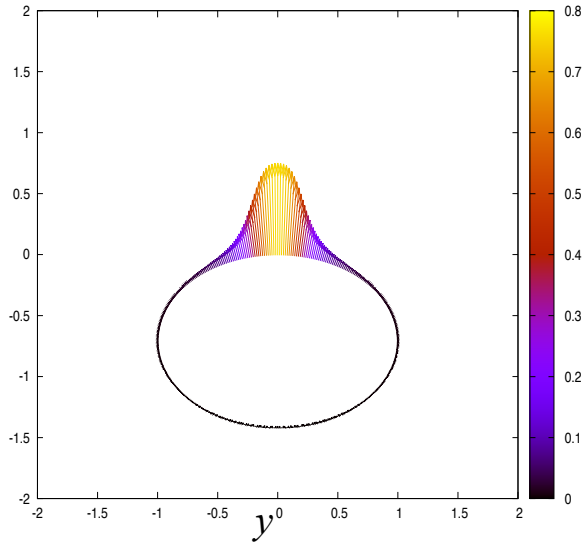
$$I_0(m,\xi) = \int_0^\pi \frac{d\phi}{(1 + \xi^2 - m\cos\phi)^{3/2}} = \frac{2E\left(\frac{2m}{1+\xi^2+m}\right)}{(1 + \xi^2 - m)\sqrt{1 + \xi^2 + m}}$$

$$I_1(m,\xi) = \int_0^\pi \frac{\cos\phi d\phi}{(1 + \xi^2 - m\cos\phi)^{3/2}} = \frac{2\left[(1 + \xi^2)E\left(\frac{2m}{1+\xi^2+m}\right) - (1 + \xi^2 - m)K\left(\frac{2m}{1+\xi^2+m}\right)\right]}{(1 + \xi^2 - m)m\sqrt{1 + \xi^2 + m}}$$

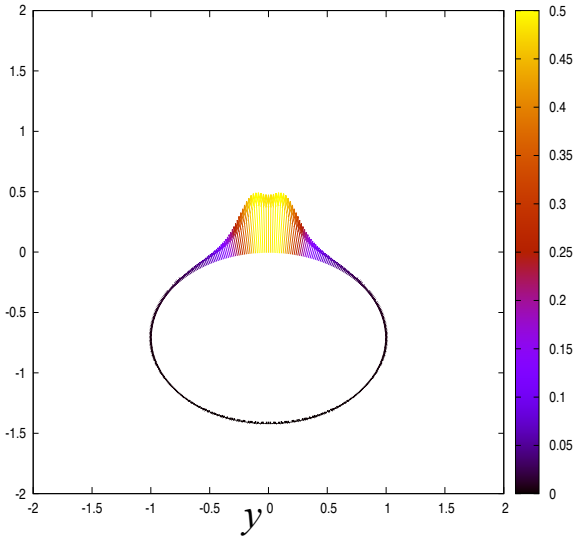
Projection of initial velocity vectors on the yz plane



$\mu = 0.005$



$\mu = 0.01$



$\mu = 0.015$

As μ gets larger, the tips of the velocity vectors are trimmed, and they become more and more uniform.


IUTAM SYMPOSIUM
TOPOLOGICAL FLUID MECHANICS



Keith,

Please stay young, healthy and active !

Thank you !