Rotating Fluids: 5. Heat and Momentum Transport in Rotating Turbulent Convection

Susanne Horn

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5. Momentum and Heat Transport in Rotating Turbulent Convection

- heat and momentum transport
- connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings

Momentum and Heat Transport in Rotating Turbulent Convection

Nusselt and Reynolds Numbers

$$\mathbf{Re}(\mathbf{Ra}, \mathbf{Pr}, \mathbf{Ek}) = ?$$

$$\mathbf{Nu}(\mathbf{Ra}, \mathbf{Pr}, \mathbf{Ek}) = ?$$

Rotating Rayleigh–Bénard convection: Governing equations

- incompressible continuity equation:

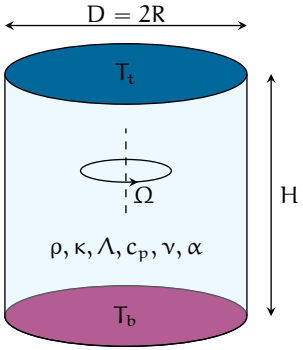
$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

- Navier–Stokes equations:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + 2\Omega \mathbf{u} \times \hat{\mathbf{e}}_z + \alpha g (T - T_m)\hat{\mathbf{e}}_z$$

- energy equation:

$$\partial_t T + \mathbf{u} \cdot \boldsymbol{\nabla} T = \kappa \boldsymbol{\nabla}^2 T$$



- isothermal top and bottom boundary conditions, adiabatic sidewall

$$T\big|_{z=H} = T_t, \qquad T\big|_{z=0} = T_b, \qquad \partial_r T\big|_{r=R} = 0$$

- impenetrable and no-slip velocity boundary conditions on all walls

$$\mathbf{u}\big|_{z=H} = \mathbf{u}\big|_{z=0} = \mathbf{u}\big|_{r=R} = 0$$

Momentum Transport - the Reynolds Number

- Reynolds number is ratio of inertial forces to viscous forces:

$$Re = \frac{UH}{\nu}$$

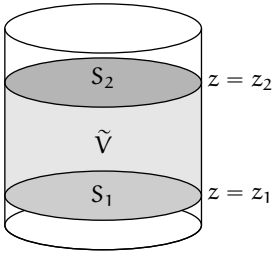
- velocity and length scale depend on set-up $\rightsquigarrow$ rapidly or slowly rotating
- related to turbulence, but no theorems to prove it

Heat Transport - Analytical Relations for the Nusselt Number

- energy equation $\partial_t T + \mathbf{u}\cdot\boldsymbol{\nabla} T = \kappa \boldsymbol{\nabla}^2 T$

$\rightsquigarrow$ average in time and over a disk with thickness $(z_2 - z_1)$, i.e., a volume $\widetilde{V}$

$$
\begin{aligned}
\langle \partial_t T\rangle_{t,\widetilde{V}} &= \partial_t \langle T\rangle_{t,\widetilde{V}} = 0 \\
\langle \mathbf{u}\cdot\boldsymbol{\nabla} T\rangle_{t,\widetilde{V}} &= \langle \boldsymbol{\nabla}\cdot(\mathbf{u}T)\rangle_{t,\widetilde{V}} - \langle T \underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}}_{=0}\rangle_{t,\widetilde{V}} \\
&= \frac{1}{\widetilde{V}}\left\langle \int_V \mathrm{d}V\, \boldsymbol{\nabla}\cdot(\mathbf{u}T)\right\rangle_t \qquad \text{(Gauss's theorem)} \\
&= \frac{1}{\widetilde{V}}\left\langle \oint_S \mathrm{d}\mathbf{S}\,(\mathbf{u}T)\right\rangle_t \qquad (\mathbf{u}|_{r=R} = 0) \\
&= \frac{1}{z_2 - z_1}\left(\langle u_z T\rangle_{t,S_2} - \langle u_z T\rangle_{t,S_1}\right) \\
\kappa\langle \boldsymbol{\nabla}^2 T\rangle_{t,\widetilde{V}} &= \kappa\langle \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} T\rangle_{t,\widetilde{V}} \\
&= \kappa\frac{1}{\widetilde{V}}\left\langle \oint_S \mathrm{d}\mathbf{S}\,(\boldsymbol{\nabla} T)\right\rangle_t \qquad (\partial_r T|_{r=R} = 0) \\
&= \kappa\frac{1}{z_2 - z_1}\left(\langle \partial_z T\rangle_{t,S_2} - \langle \partial_z T\rangle_{t,S_1}\right)
\end{aligned}
$$



Heat Transport - Analytical Relations for the Nusselt Number

- for arbitrary S_1, S_2 holds

$$\langle u_z T\rangle_{t,S_2} - \kappa\langle\partial_z T\rangle_{t,S_2} = \langle u_z T\rangle_{t,S_1} - \kappa\langle\partial_z T\rangle_{t,S_1}$$

$$\Rightarrow \qquad \langle u_z T\rangle_{t,S} - \kappa\langle\partial_z T\rangle_{t,S} = \text{const.}$$

$\rightsquigarrow$ heat flux is the same in any horizontal plane S

- normalisation with total conductive heat flux $\kappa\Delta/H$:

$$Nu \equiv \frac{\langle u_z T\rangle_{t,S} - \kappa\langle\partial_z T\rangle_{t,S}}{\kappa\Delta/H}$$

- at the top and bottom boundaries $u_z = 0$ $\Rightarrow$

$$Nu = -\frac{H}{\Delta}\langle\partial_z T\rangle_{t,S_{z=0}} = -\frac{H}{\Delta}\langle\partial_z T\rangle_{t,S_{z=H}}$$

- averaging over the entire volume V, i.e., additionally integrating in z-direction:

$$\int_0^H dz\, Nu = \int_0^H dz \frac{\langle u_z T\rangle_{t,S} - \kappa\langle\partial_z T\rangle_{t,S}}{\kappa\Delta/H} \qquad \Rightarrow \qquad Nu = 1 + \frac{H}{\kappa\Delta}\langle u_z T\rangle_{t,V}$$

Heat Transport - Analytical Relations for the Nusselt Number

- energy equation $\partial_t T + \mathbf{u}\cdot\boldsymbol{\nabla} T = \kappa\boldsymbol{\nabla}^2 T$

$\rightsquigarrow$ multiply by T and average in time and over entire volume V

$$\langle T\partial_t T\rangle_{t,V} = \langle \tfrac{1}{2}\partial_t T^2\rangle_{t,V} = 0$$

$$\langle T\mathbf{u}\cdot\boldsymbol{\nabla} T\rangle_{t,V} = \frac{1}{2}\big(\langle\boldsymbol{\nabla}\cdot(\mathbf{u}T^2)\rangle_{t,V} - \langle T^2\underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}}_{=0}\rangle_{t,V}\big) = \frac{1}{V}\left\langle\oint_S \mathrm{d}\mathbf{S}\,(\mathbf{u}T^2)\right\rangle_t = 0$$

$$\begin{aligned}
\kappa\langle T\boldsymbol{\nabla}^2 T\rangle_{t,V} &= \kappa\langle\boldsymbol{\nabla}\cdot(T\boldsymbol{\nabla} T)\rangle_{t,V} - \kappa\langle\boldsymbol{\nabla} T\cdot\boldsymbol{\nabla} T\rangle_{t,V}\\
&= \kappa\frac{1}{V}\left\langle\oint_S \mathrm{d}\mathbf{S}\,(T\boldsymbol{\nabla} T)\right\rangle_t - \underbrace{\kappa\langle(\boldsymbol{\nabla} T)^2\rangle_{t,V}}_{\text{thermal dissipation rate } \epsilon_\theta}\\
&= \kappa\frac{\pi R^2}{V}\left(\langle T_t\partial_z T\rangle_{t,S_{z=H}} - \langle T_b\partial_z T\rangle_{t,S_{z=0}}\right) - \epsilon_\theta \qquad (Nu = -\tfrac{H}{\Delta}\langle\partial_z T\rangle_{t,S_{z=0,H}})\\
&= \kappa\frac{\Delta^2}{H^2}Nu - \epsilon_\theta
\end{aligned}$$

$\rightsquigarrow$ relationship between thermal dissipation and Nusselt number:

$$\epsilon_\theta = \kappa\frac{\Delta^2}{H^2}Nu$$

Heat Transport - Analytical Relations for the Nusselt Number*

- Navier–Stokes equations $\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{u} + 2\Omega\mathbf{u}\times\hat{\mathbf{e}}_z + \alpha g(T - T_m)\hat{\mathbf{e}}_z$

$\rightsquigarrow$ multiply by $\mathbf{u}$ and average in time and over entire volume V

$$
\begin{aligned}
\langle \mathbf{u}\cdot\partial_t\mathbf{u}\rangle_{t,V} &= \partial_t\langle \tfrac{1}{2}\mathbf{u}^2\rangle_{t,V} = 0 \\
\langle \mathbf{u}\cdot(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\rangle_{t,V} &= \langle \mathbf{u}\cdot\tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u})\rangle_{t,V} - \langle \mathbf{u}\cdot\big(\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})\big)\rangle_{t,V} \\
&= \tfrac{1}{2}\langle \mathbf{u}\cdot\boldsymbol{\nabla}\|\mathbf{u}\|^2\rangle_{t,V} - \langle(\boldsymbol{\nabla}\times\mathbf{u})\cdot\underbrace{(\mathbf{u}\times\mathbf{u})}_{=0}\rangle_{t,V} \\
&= \tfrac{1}{2}\langle \boldsymbol{\nabla}\cdot(\mathbf{u}\|\mathbf{u}\|^2)\rangle_{t,V} - \tfrac{1}{2}\langle\|\mathbf{u}\|^2\underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}}_{=0}\rangle_{t,V} \\
&= \frac{1}{V}\left\langle \oint_S \mathrm{d}\mathbf{S}\,\mathbf{u}\big(\tfrac{1}{2}\|\mathbf{u}\|^2\big)\right\rangle_t = 0 \\
-\langle \mathbf{u}\boldsymbol{\nabla} p\rangle_{t,V} &= -\langle\boldsymbol{\nabla}\cdot(\mathbf{u}p)\rangle_{t,V} - \langle p\underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}}_{=0}\rangle_{t,V} = -\frac{1}{V}\left\langle\oint_S \mathrm{d}\mathbf{S}\,\mathbf{u}p\right\rangle_t = 0
\end{aligned}
$$

*$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A})$
$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$

Heat Transport - Analytical Relations for the Nusselt Number*

- Navier–Stokes equations $\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{u} + 2\Omega\mathbf{u}\times\hat{\mathbf{e}}_z + \alpha g(T - T_m)\hat{\mathbf{e}}_z$

$\rightsquigarrow$ multiply by $\mathbf{u}$ and average in time and over entire volume V

$$
\begin{aligned}
\nu\langle \mathbf{u}\cdot\boldsymbol{\nabla}^2\mathbf{u}\rangle_{t,V} &= \nu\langle \mathbf{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{u}\rangle_{t,V} \\
&= \nu\Big\langle \boldsymbol{\nabla}\cdot\big(\boldsymbol{\nabla}\mathbf{u}\cdot\mathbf{u}\big)\Big\rangle_{t,V} - \nu\big\langle \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}\big\rangle_{t,V} \\
&= \nu\frac{1}{V}\underbrace{\left\langle \oint_S \mathrm{d}\mathbf{S}\,\big(\boldsymbol{\nabla}\mathbf{u}\cdot\mathbf{u}\big)\right\rangle_t}_{=0} \overset{\text{viscous dissipation rate } \epsilon_u}{\overset{\downarrow}{-\nu}}\big\langle \|\boldsymbol{\nabla}\mathbf{u}\|_F^2\big\rangle_{t,V} \\
&= -\epsilon_u \\
\langle \mathbf{u}\cdot(2\Omega\mathbf{u}\times\hat{\mathbf{e}}_z)\rangle_{t,V} &= \langle 2\Omega\hat{\mathbf{e}}_z\cdot(\mathbf{u}\times\mathbf{u})\rangle_{t,V} = 0
\end{aligned}
$$

*Frobenius inner product $\mathbf{A}:\mathbf{B} = \sum_{ij} A_{ij}B_{ij} = \mathbf{tr}(\mathbf{A}^T\mathbf{B}) = \mathbf{tr}(\mathbf{A}\mathbf{B}^T)$; Frobenius norm $\|\mathbf{A}\|_F = \sqrt{\mathbf{A}:\mathbf{A}} = \sqrt{\mathbf{tr}(\mathbf{A}\mathbf{A}^T)}$

Heat Transport - Analytical Relations for the Nusselt Number

- Navier–Stokes equations $\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu\boldsymbol{\nabla}^2\mathbf{u} + 2\Omega\mathbf{u}\times\hat{\mathbf{e}}_z + \alpha g(T - T_m)\hat{\mathbf{e}}_z$

$\rightsquigarrow$ multiply by $\mathbf{u}$ and average in time and over entire volume V

$$\begin{aligned}
\langle \mathbf{u}\cdot\alpha g(T-T_m)\hat{\mathbf{e}}_z\rangle_{t,V} &= \alpha g\langle u_z T\rangle_{t,V} - \alpha g T_m \underbrace{\langle u_z\rangle_{t,V}}_{=0} \qquad (Nu = 1 + \tfrac{H}{\kappa\Delta}\langle u_z T\rangle_{t,V}) \\
&= \alpha g(Nu-1)\frac{\kappa\Delta}{H} \\
&= \frac{\nu^3}{H^4}(Nu-1)RaPr^{-2}
\end{aligned}$$

$\rightsquigarrow$ relationship between viscous dissipation and Nusselt number: $\epsilon_u = \frac{\nu^3}{H^4}(Nu-1)RaPr^{-2}$

Heat Transport - Analytical Relations for the Nusselt Number

- rearrange equation for ϵ_u

$$Nu = 1 + \underbrace{\frac{H^4}{\nu^3}\frac{Pr^2}{Ra}\epsilon_u}_{\geqslant 0}$$

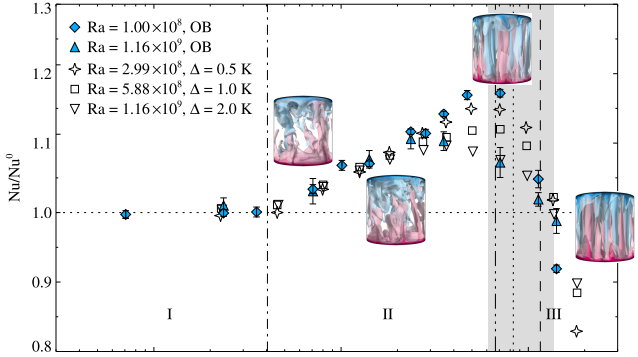
- compare to

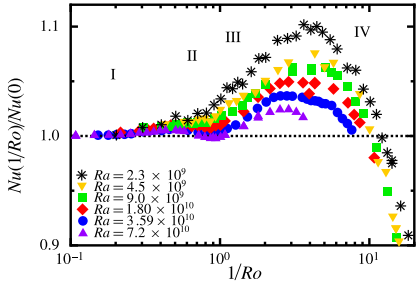
$$Nu = 1 + \frac{H}{\kappa\Delta}\langle u_z T\rangle_{t,V}$$

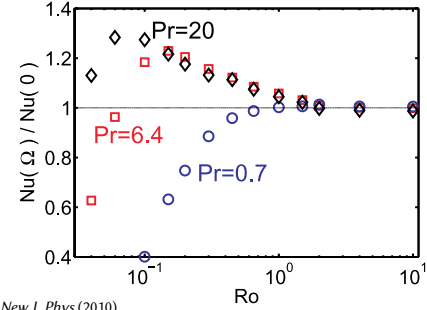
- pure conduction state means $Nu = 1$, as soon as convection sets in $Nu > 1$

$\rightsquigarrow$ rotation appears to not affect the heat transport

Numerical Simulations and Experiments in Water*



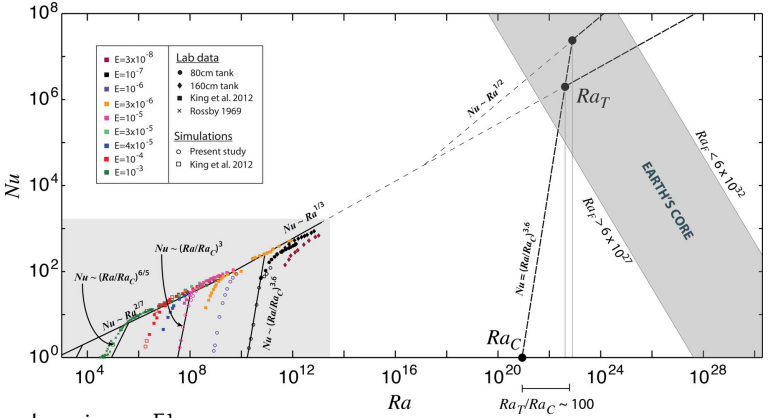




⇝ Ekman pumping leads to heat transport enhancement

*Horn, S., Shishkina, O., *Phys. Fluids* (2014). Weiss, S., Ahlers, G., *J. Fluid Mech.* (2011); Stevens, R.J.A.M. et al.,*New J. Phys.*(2010)

Comparison between laboratory-numerical heat transfer data and estimated ranges of heat transfer parameters in Earth's core†

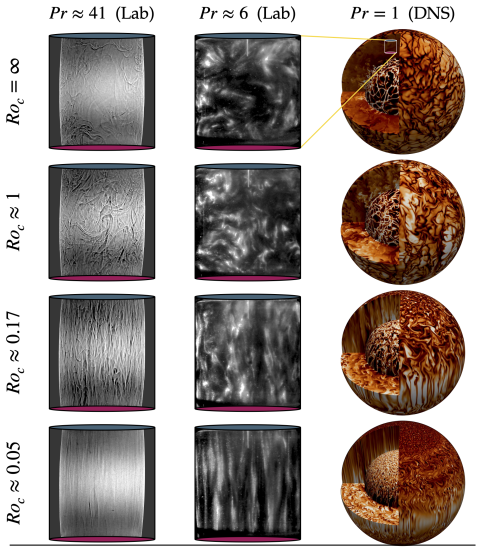


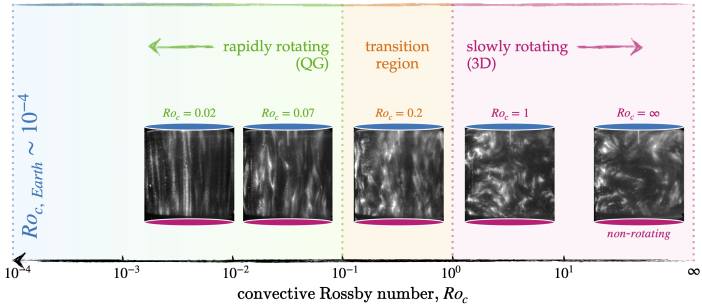
- different scalings depening on Ek
- laboratory experiments and direct numerical simulations are still far from realistic values

⇝ we do not just need rotation but also magnetic fields!

†Cheng et al., *Geophys. J. Int.*(2015)

“Ultimate Turbulence” = Diffusivity-Free Scalings of Heat/Momentum Transport‡





- if flow is turbulent enough, diffusive terms will not matter anymore
- ⇝ scalings of Nu and Re become independent of κ and ν
- ⇝ different for rapidly and slowly rotating systems

‡Abbate, J.A., PhD thesis (2024)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings[†]

- dimensional governing equations:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{u} &= 0\\
\partial_t\mathbf{u}+\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} &= -\boldsymbol{\nabla}p+\nu\boldsymbol{\nabla}^2\mathbf{u}+2\mathbf{u}\times\boldsymbol{\Omega}+\alpha\theta g\hat{\mathbf{e}}_z\\
\partial_t\theta+(\mathbf{u}\cdot\boldsymbol{\nabla}\theta)' &= \kappa\boldsymbol{\nabla}^2\theta+(\hat{\mathbf{e}}_z\cdot\mathbf{u})\partial_z\bar{T}\\
\partial_t\bar{T}+\boldsymbol{\nabla}\cdot\overline{(\mathbf{u}\theta)} &= \kappa\boldsymbol{\nabla}^2\bar{T}
\end{aligned}$$

- splitting temperature into mean and fluctuating part $T=\bar{T}+\theta$, where $\bar{T}=\langle T\rangle_h$
- $(\mathbf{u}\cdot\boldsymbol{\nabla}\theta)=\mathbf{u}\cdot\boldsymbol{\nabla}\theta-\boldsymbol{\nabla}\cdot\overline{(\mathbf{u}\theta)}$
- characteristic scales: length ℓ, velocity U, temperature anomaly ϑ

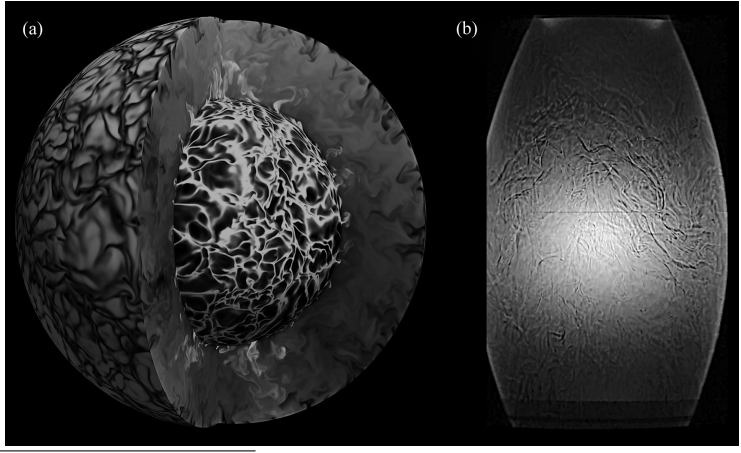
$\rightsquigarrow$ Nusselt number ($Nu\gg1$): $Nu\sim\dfrac{U\vartheta H}{\kappa\Delta}$

$\rightsquigarrow$ Reynolds number: $Re_\ell=\dfrac{U\ell}{\nu}$

[†] Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings†

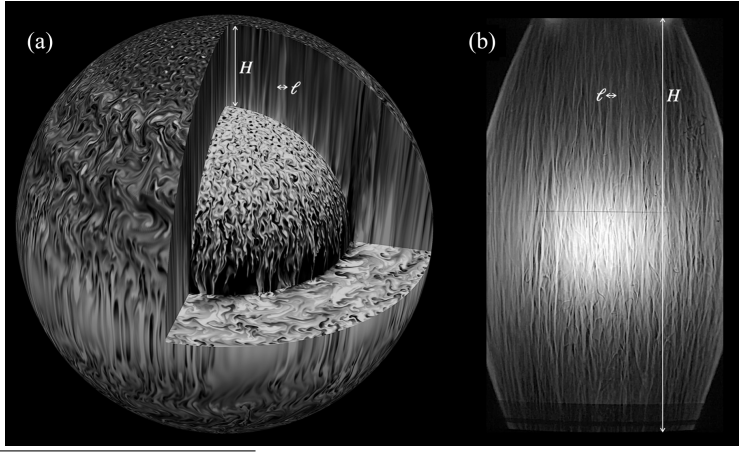
Slowly rotating limit (SRL)



†Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings‡

Rapidly rotating limit (RRL)



‡Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings[§]

Slowly rotating limit (SRL)

- inertial forces greatly exceed the Coriolis force

$$\begin{aligned} \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} &\gg 2\boldsymbol{\Omega}\times\mathbf{u} \\ \frac{U^2}{\ell} &\gg 2\Omega U \end{aligned}$$

$\rightsquigarrow$ local Rossby number:

$$\mathrm{Ro}_\ell = \frac{U}{2\Omega\ell} \gg 1$$

vs.

Rapidly rotating limit (RRL)

- Coriolis force dominates over inertial forces

$$\begin{aligned} \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} &\ll 2\boldsymbol{\Omega}\times\mathbf{u} \\ \frac{U^2}{\ell} &\ll 2\Omega U \end{aligned}$$

$\rightsquigarrow$ local Rossby number:

$$\mathrm{Ro}_\ell = \frac{U}{2\Omega\ell} \ll 1$$

[§] Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings†

Slowly rotating convection $\mathrm{Ro}_\ell \gg 1$

- free-fall inertial balance:

$$\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u} \sim \alpha\theta g\hat{\mathbf{e}}_z \qquad \Leftarrow \qquad \frac{U^2}{\ell} \sim g\alpha\vartheta$$

- take $\vartheta \sim \Delta$ and $\ell \sim H$, then

$$U = \sqrt{\alpha g \Delta H} \equiv U_{ff} \quad \text{free-fall velocity}$$

$\rightsquigarrow$ local Rossby number:

$$\begin{aligned} \mathrm{Ro}_\ell &= \frac{U}{2\Omega\ell} \sim \frac{U_{ff}}{2\Omega H} \\ &= \frac{\sqrt{\alpha g \Delta H}}{2\Omega H} \\ &\equiv \mathrm{Ro}_c \quad \text{convective Rossby number} \end{aligned}$$

† Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings†

Rapidly rotating convection $\mathrm{Ro}_\ell \ll 1$

- start from vorticity equation: turbulent CIA (Coriolis-Inertia-Archimedean) vorticity balance
  - Coriolis: only stretching of the background vorticity, $2\Omega\partial_z\mathbf{u}$, can occur on the system scale:

$$2\Omega\partial_z\mathbf{u} \sim \frac{2\Omega U}{H}$$

  - Inertial: inertial advection acts on a local scale:

$$\mathbf{u}\cdot\boldsymbol{\nabla}\boldsymbol{\omega} \sim \frac{U^2}{\ell^2}$$

  - Archimedean: buoyancy acts on local scale:

$$\boldsymbol{\nabla}\times(\mathbf{g}\alpha\theta) \sim \frac{g\alpha\vartheta}{\ell}$$

† Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings†

Rapidly rotating convection $\mathrm{Ro}_\ell \ll 1$

- balance Coriolis and Inertia $\rightsquigarrow$ reflects highly anisotropic length scales

$$\frac{\ell}{H} \sim \frac{U}{2\Omega\ell} \equiv \mathrm{Ro}_\ell$$

- unstable mean temperature gradient Δ/H is sustained in bulk, fluctuating thermal energy:

$$(\mathbf{u}\cdot\boldsymbol{\nabla}\theta)' \sim (\hat{\mathbf{e}}_z\cdot\mathbf{u})\partial_z\bar{T} \quad \Rightarrow \quad \frac{U\vartheta}{\ell} \sim \frac{U\Delta}{H}$$

- thus, thermal anomaly scales as

$$\frac{\vartheta}{\Delta} \sim \frac{\ell}{H} \sim \mathrm{Ro}_\ell$$

- balance Coriolis and Archimedean:

$$\frac{2\Omega U}{H} \sim \frac{g\alpha\vartheta}{\ell} \quad \Rightarrow \quad U \sim \frac{g\alpha\vartheta H}{2\Omega\ell} = \frac{g\alpha\Delta}{2\Omega}\frac{H}{\ell}\frac{\vartheta}{\Delta} = \frac{g\alpha\Delta}{2\Omega} \equiv U_{TW} \quad \text{thermal wind velocity}$$

- local Rossby number is

$$\mathrm{Ro}_\ell \sim \frac{U_{TW}}{2\Omega\ell} = \frac{g\alpha\Delta}{4\Omega^2\ell} = \frac{g\alpha\Delta}{4\Omega^2 H}\frac{1}{\mathrm{Ro}_\ell} = \frac{\sqrt{\alpha g\Delta H}}{2\Omega H} = \mathrm{Ro}_c = \sqrt{\frac{\mathrm{Ek}^2\mathrm{Ra}}{\mathrm{Pr}}}$$

† Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Connections between nonrotating, slowly rotating, and rapidly rotating turbulent convection transport scalings†

Heat and Momentum Transport

Slowly rotating limit (SRL) **vs.** **Rapidly rotating limit (RRL)**

$$\begin{aligned} \mathrm{Re}_\ell &= \frac{U_{ff}\ell}{\nu} \sim \frac{g\alpha\Delta H^3}{\nu} \\ &= \left(\frac{Ra}{Pr}\right)^{1/2} = \mathrm{Re}_{ff} \\ Nu &= \frac{U\vartheta H}{\kappa\Delta} = \frac{U_{ff}H}{\kappa} \\ &= (Ra\,Pr)^{1/2} \end{aligned}$$

$$\begin{aligned} \mathrm{Re}_\ell &= \frac{U_{TW}\ell}{\nu} \sim \frac{g\alpha\Delta}{2\Omega}\frac{\ell}{\nu} \\ &= Ro_c^2 \mathrm{Re}_{ff} = Ro_c^2\left(\frac{Ra}{Pr}\right)^{1/2} \\ &= \left(\frac{Ra}{Pr}\right)^{3/2} Ek^2 \\ Nu &= \frac{U_{TW}}{\kappa}\frac{\vartheta H}{\Delta} = \frac{U_{TW}\ell}{\kappa} \\ &= Ro_c^2 (Ra\,Pr)^{1/2} \\ &= \left(\frac{Ra^3}{Pr}\right)^{1/2} Ek^2 \end{aligned}$$

† Aurnou, J.M., Horn, S., Julien K., *Phys. Rev. Res.* (2020)

Take Away Messages

- Think before you compute.
- Keep in mind that all the equations have a fluid dynamical/physical meaning.
- Make sure you know what your approximations entail.
- Know your scales (length, time, velocity, …).