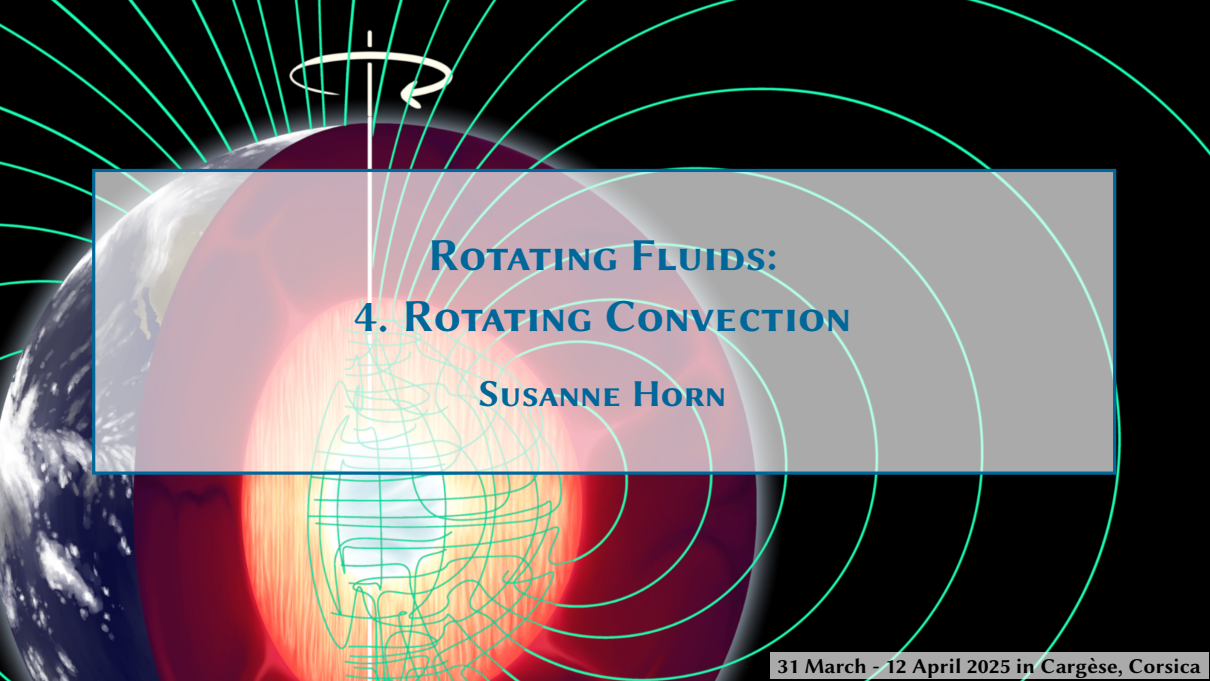


Rotating Fluids:
4. Rotating Convection

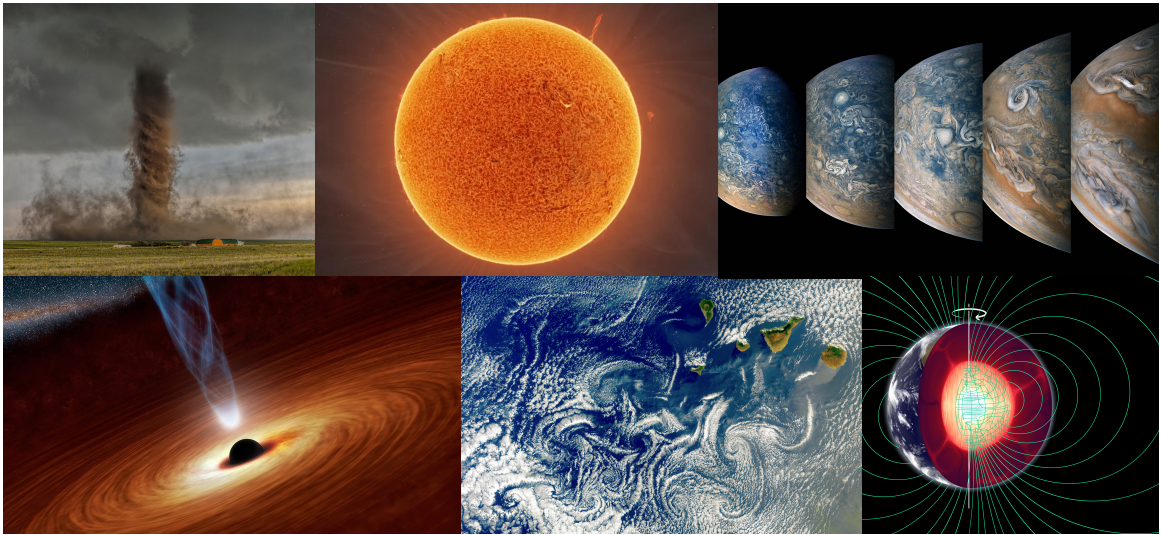
Susanne Horn

31 March - 12 April 2025 in Cargèse, Corsica

4. Rotating Convection

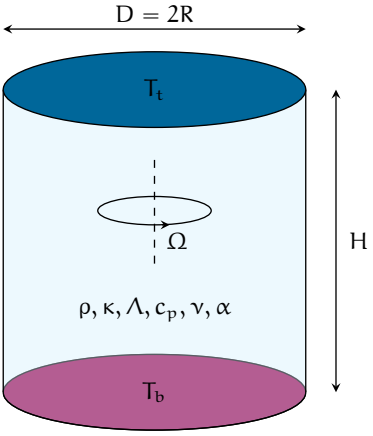
- Rotating Rayleigh-Bénard Convection
- Linear Stability Analysis
- Relevance for Nonlinear Problems
- Centrifugal Buoyancy

Rotating Flows in Nature



All these flows are not just controlled by rotation, but also convection $\rightsquigarrow$ **rotating convection**

Rotating Rayleigh–Bénard convection





Lord Rayleigh
John William Strutt
(1842–1919)



Henri Bénard
(1874–1939)

- layer heated from below, cooled from above
- incompressible fluid
- default experimental/numerical BCs: isothermal, impermeable, no-slip top and bottom; insulating, impermeable, no-slip sidewall

Governing Equations - Oberbeck-Boussinesq Approximation

- incompressible (solenoidal) continuity equation:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

- Navier–Stokes equations:

$$\mathrm{D_t}\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + 2\Omega \mathbf{u} \times \hat{\mathbf{e}}_z + g\alpha(T - T_m)\hat{\mathbf{e}}_z - \Omega^2 r \alpha (T - T_m)\hat{\mathbf{e}}_r$$

viscous; gravitational buoyancy; pressure gradient; Coriolis; centrifugal buoyancy

- temperature equation:

$$\mathrm{D_t} T = \kappa \boldsymbol{\nabla}^2 T$$

Non-dimensionalisation

- non-dimensionalisation: replace every variable by scaled counterpart $\tilde{\chi} = \dfrac{\chi}{\chi_{ref}}$:

$$x_{ref} = H, \quad t_{ref} = \frac{H^2}{\kappa} = \frac{x_{ref}}{u_{ref}}, \quad u_{ref} = \frac{\kappa}{H}, \quad p_{ref} = \rho u_{ref}^2 = \rho \frac{\kappa^2}{H^2}, T_{ref} = \Delta$$

- dimensionless form:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ D_t \mathbf{u} &= -\boldsymbol{\nabla} p + Pr \boldsymbol{\nabla}^2 \mathbf{u} + \frac{Pr}{Ek} \mathbf{u} \times \hat{\mathbf{e}}_z + Ra\, Pr\, T \hat{\mathbf{e}}_z \\ D_t T &= \boldsymbol{\nabla}^2 T \end{aligned}$$

- control parameters:

$$\text{Rayleigh number: } Ra = \frac{\alpha g \Delta H^3}{\kappa \nu}$$

$$\text{Prandtl number: } Pr = \frac{\nu}{\kappa}$$

$$\text{Ekman number: } Ek = \frac{\nu}{2\Omega H^2}$$

Detour: Velocity Potentials - Toroidal and Poloidal Field*

- velocity field $\mathbf{u}$ is solenoidal

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

- decomposition into poloidal field $\boldsymbol{\pi}$ and toroidal field $\boldsymbol{\tau}$ (aka Mie decomposition) with defining scalars ξ and ψ, respectively

$$\mathbf{u} = \boldsymbol{\pi} + \boldsymbol{\tau} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\xi \boldsymbol{e}_z) + \boldsymbol{\nabla} \times (\psi \boldsymbol{e}_z)$$

- in Cartesian component notation

$$u_x = \pi_x + \tau_x = \partial_x \partial_z \xi + \partial_y \psi \quad (1)$$
$$u_y = \pi_y + \tau_y = \partial_y \partial_z \xi - \partial_x \psi \quad (2)$$
$$u_z = \pi_z + \tau_z = -\partial_{xx}^2 \xi - \partial_{yy}^2 \xi = -\Delta_{xy} \xi \quad (3)$$

- combine ∂_y eq. (1) and ∂_z eq. (2)

$$\omega_z = \partial_x u_y - \partial_y u_x = -\partial_{xx}^2 \psi - \partial_{yy}^2 \psi = -\Delta_{r\phi} \psi$$

*Backus, G., Poloidal and toroidal fields in geomagnetic field modeling, *Rev. Geophys.* (1986); Boronski, P., Tuckerman, L., Poloidal-toroidal decomposition in a finite cylinder. I: influence matrices for the magnetohydrodynamic equations. *J. Comput. Phys.* (2007); Horn, S., Shishkina, O., Toroidal and poloidal energy in rotating Rayleigh-Bénard convection, *J. Fluid Mech.* (2015)

Detour: Velocity Potentials - Toroidal and Poloidal Field ctd.

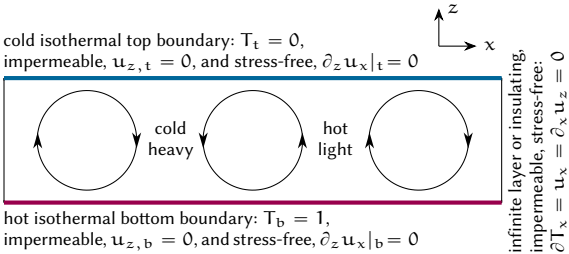
- two 2D Poisson equations

$$\begin{aligned} \Delta_{r\phi}\xi + u_z &= 0 \\ \Delta_{r\phi}\psi + \omega_z &= 0 \end{aligned}$$

- toroidal potential ψ only determined up to an arbitrary horizontal harmonic function
- poloidal potential ξ only determined up to an arbitrary function of z
- gauge freedom for boundary conditions
- we can express flow in variables of u_z and ω_z (or ξ and ψ)

Linear Stability Analysis

Nonrotating Rayleigh-Bénard Convection



- onset is steady/stationary: instability sets in as exponential growth without oscillations
- onset in form of rolls at critical Rayleigh number and wavenumber

$$Ra_c = \frac{\alpha g \Delta H^3}{\kappa \nu} = \frac{27\pi^4}{4} = 657.5, \qquad k_c = \frac{\pi}{\sqrt{2}}$$

- independent of Prandtl number $Pr = \frac{\nu}{\kappa}$
- Ra_c is higher for other BCs (free-slip, no-slip: $Ra_c = 1101$, no-slip, no-slip: $Ra_c = 1708$)

Linear Stability Analysis
Rotating Rayleigh-Bénard Convection

- Taylor-Proudman: in an inviscid fluid: no motion, no-matter how high Δ

$\rightsquigarrow$ 2 options to "break" the TP constraint

1. viscosity
2. thermal-inertial oscillations[†]

[†] Zhang, K., Roberts, P.H. Thermal inertial waves in a rotating fluid layer: exact and asymptotic solutions. *Phys. Fluids* (1997)

Linear Stability Analysis

Perturbation Equations

- consider equilibrium solution (basic state) where
  - fluid is at rest $\mathbf{u} = 0$
  - temperature is linearly stratified (conductive state):

$$T_c = 1 - z$$

- consider an infinitesimal perturbation (denoted by primes):

$$\begin{aligned}
\text{velocity:} \quad u_x &= 0 + u_x' \\
u_y &= 0 + u_y' \\
u_z &= 0 + u_z' \\
\text{pressure:} \quad p &= \tilde{P}(z) + \tilde{p}'(x, y, z, t) \\
\text{temperature:} \quad T &= T_c(z) + \theta(x, y, z, t)
\end{aligned}$$

$\rightsquigarrow$ only need to "worry" about the advective term in temperature equation $D_t T = \partial_t T + u \cdot \boldsymbol{\nabla} T = \boldsymbol{\nabla}^2 T$

$\rightsquigarrow$ $\mathbf{u} \cdot \boldsymbol{\nabla}(T_c + \theta) = -u_z + \mathbf{u} \cdot \boldsymbol{\nabla}\theta$

Linear Stability Analysis

Perturbation Equations

- drop advective ("non-linear") terms (and the primes)
- perturbation equations are

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} &= -\boldsymbol{\nabla} p + \mathrm{Pr} \boldsymbol{\nabla}^2 \mathbf{u} + \frac{\mathrm{Pr}}{\mathrm{Ek}} \mathbf{u} \times \hat{\mathbf{e}}_z + \mathrm{Ra}\,\mathrm{Pr}\,\theta \hat{\mathbf{e}}_z \\ \partial_t \theta &= \boldsymbol{\nabla}^2 \theta + u_z \end{aligned}$$

- take curl of Navier-Stokes perturbation equations $\rightsquigarrow$ eliminate pressure

$$\frac{1}{\mathrm{Pr}} \partial_t \omega = \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \frac{1}{\mathrm{Ek}} \underbrace{(\hat{\mathbf{e}}_z \cdot \boldsymbol{\nabla})}_{\partial_z} \mathbf{u} - \mathrm{Ra}(\hat{\mathbf{e}}_z \times \boldsymbol{\nabla} \theta)$$

Linear Stability Analysis

Perturbation Equations

- take curl again:

$$\boldsymbol{\nabla} \times \left[\frac{1}{\mathrm{Pr}} \partial_t \boldsymbol{\omega} = \boldsymbol{\nabla}^2 \boldsymbol{\omega} + \frac{1}{\mathrm{Ek}} \underbrace{(\hat{\mathbf{e}}_z \cdot \boldsymbol{\nabla})}_{\partial_z} \mathbf{u} - \mathrm{Ra} (\hat{\mathbf{e}}_z \times \boldsymbol{\nabla} \theta) \right]$$

- term by term:

$$\begin{aligned}
\boldsymbol{\nabla} \times \boldsymbol{\omega} &= \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{u}) \\
&= \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \mathbf{u}) - \boldsymbol{\nabla}^2 \mathbf{u} \\
&= -\boldsymbol{\nabla}^2 \mathbf{u} \\
\boldsymbol{\nabla} \times (\boldsymbol{\nabla}^2 \boldsymbol{\omega}) &= \boldsymbol{\nabla}^2 (\boldsymbol{\nabla} \times \boldsymbol{\omega}) \\
&= -\boldsymbol{\nabla}^4 \mathbf{u} \\
\boldsymbol{\nabla} \times (\hat{\mathbf{e}}_z \cdot \boldsymbol{\nabla}) \mathbf{u} &= \partial_z \boldsymbol{\omega} \\
\boldsymbol{\nabla} \times (\hat{\mathbf{e}}_z \times \boldsymbol{\nabla} \theta) &= \partial_z \boldsymbol{\nabla} \theta - \hat{\mathbf{e}}_z \cdot \boldsymbol{\nabla} \theta
\end{aligned}$$

$\rightsquigarrow$ full equation becomes:

$$-\frac{1}{\mathrm{Pr}} \partial_t \boldsymbol{\nabla}^2 \mathbf{u} = -\boldsymbol{\nabla}^4 \mathbf{u} + \frac{1}{\mathrm{Ek}} \partial_z \boldsymbol{\omega} - \mathrm{Ra} \left(\partial_z \boldsymbol{\nabla} \theta - \hat{\mathbf{e}}_z \cdot \boldsymbol{\nabla} \theta \right)$$

Linear Stability Analysis

Perturbation Equations

- now multiply both equations by $\hat{\mathbf{e}}_z$ (or simply take the z-component)

$$\begin{aligned} \frac{1}{\mathrm{Pr}}\partial_t \omega_z &= \boldsymbol{\nabla}^2 \omega_z + \frac{1}{\mathrm{Ek}}\partial_z u_z \\ \frac{1}{\mathrm{Pr}}\partial_t \boldsymbol{\nabla}^2 u_z &= \boldsymbol{\nabla}^4 u_z - \frac{1}{\mathrm{Ek}}\partial_z \omega_z + \mathrm{Ra}\boldsymbol{\nabla}_\perp^2 \theta \\ \partial_t \theta &= \boldsymbol{\nabla}^2 \theta + u_z \end{aligned}$$

- solution is only required for θ, u_z and ω_z

Linear Stability Analysis

Normal mode ansatz

- analyse perturbation into 2D periodic waves:

$$\begin{aligned} u_z &= W(z)\exp(i(k_x x + k_y y) + \sigma t) \\ \theta &= \Theta(z)\exp(i(k_x x + k_y y) + \sigma t) \\ \omega_z &= Z(z)\exp(i(k_x x + k_y y) + \sigma t) \end{aligned}$$

$W(z), \Theta(z), Z(z) \ldots$ mode shapes

$k = \sqrt{k_x^2 + k_y^2} \ldots$ horizontal wavenumber

$\sigma = \sigma_r + i\sigma_i \ldots$ complex growth rate, σ_r is the temporal growth rate, σ_i is frequency

$\rightsquigarrow$ assume an wavelike solution and seek the solution that grows in time

- $\sigma_r > 0$: unstable (exponentially increasing in time)
- $\sigma_r < 0$: stable (exponentially decreasing in time)
- $\sigma_r = 0$: neutral

Linear Stability Analysis

Normal mode ansatz

- advantage: functions make differentiations easy

$$\partial_t \to \sigma, \quad \partial_{xx}^2 + \partial_{yy}^2 \to -k^2, \quad \boldsymbol{\nabla}^2 \to d_{zz}^2 - k^2$$

$\rightsquigarrow$ hence

$$\begin{aligned} \left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right) Z &= \frac{1}{Ek} d_z W \\ \left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right)(d_{zz}^2 - k^2) W &= -\frac{1}{Ek} d_z Z - Ra k^2 \Theta \\ \left(\sigma - d_{zz}^2 + k^2\right) \Theta &= W \end{aligned}$$

- with free-slip and isothermal boundary conditions:

$$\Theta = W = d_z Z = d_{zz}^2 Z = 0 \text{ at } z = 0 \text{ and } z = 1$$

Linear Stability Analysis

Stationary Onset (Viscous Mode)

- assume onset is steady/stationary: instability sets in as exponential growth without oscillations $\sigma_i = 0$
- look for marginal stable state $\sigma_r = 0 \rightsquigarrow \sigma = 0$
- equations simplify significantly:

$$\left(-d_{zz}^2 + k^2\right) Z = \frac{1}{Ek} d_z W \tag{4}$$

$$\left(-d_{zz}^2 + k^2\right)^2 W = \frac{1}{Ek} d_z Z + Ra k^2 \Theta \tag{5}$$

$$\left(-d_{zz}^2 + k^2\right) \Theta = W \tag{6}$$

- Pr dependence drops out

$\rightsquigarrow$ eliminate Z and Θ from eq. (5) by operating $\left(-d_{zz}^2 + k^2\right)$ on it and using eq. (4) and (6)

- we obtain a simple ODE:

$$\left(-d_{zz}^2 + k^2\right)^3 W = \frac{1}{Ek^2} \partial_{zz}^2 W + Ra k^2 W$$

Linear Stability Analysis

Stationary Onset (Viscous Mode)

- ODE for W:

$$\left(-d_{zz}^2 + k^2\right)^3 W = \frac{1}{Ek^2} d_{zz}^2 W + Ra k^2 W$$

- BCs, $W = d_z Z = d_{zz}^2 W = d_{zzzz}^4 W = 0$ at $z = 0$ and $z = 1$

$\rightsquigarrow$ all even derivatives $d_z^n W$ vanish for $z = 0$ and $z = 1$

- vertical velocity mode shape is with amplitude $A \neq 0$:

$$W = A \sin(\underbrace{n\pi z}_{\text{vertical wavenumber}})$$

- insert to obtain characteristic equation:

$$Ra = \frac{\left(n^2\pi^2 + k^2\right)^3 + n^2\pi^2 E k^{-2}}{k^2}$$

- lowest value of Ra is obtained for $n = 1$

Linear Stability Analysis

Stationary Onset (Viscous Mode)

- characteristic equation for $n = 1$

$$Ra = \frac{1}{k^2}\left((\pi^2 + k^2)^3 + \frac{\pi^2}{Ek^2}\right)$$

- for $Ek \to \infty$ (no rotation):

$$Ra = \frac{\left(\pi^2 + k^2\right)^3}{k^2}$$

$\rightsquigarrow$ minium critical Rayleigh number for $\partial_k Ra = \dfrac{2(2k^2 - \pi^2)(k^2 + \pi^2)^2}{k^3} = 0$

$\rightsquigarrow$ critical horizontal wavenumber $k_c = \pi/\sqrt{2}$, wave length $\lambda_c = 2\pi/k_c = 2\sqrt{2}$

$\rightsquigarrow$ $Ra_c = \dfrac{27\pi^4}{4} = 657.5$

Linear Stability Analysis

Stationary Onset (Viscous Mode)

- in rotating case: $\mathrm{Ra} = \frac{1}{k^2}\left((\pi^2 + k^2)^3 + \frac{\pi^2}{\mathrm{Ek}^2}\right)$
- substitute $k^2 = \pi^2 x$, $\mathrm{Ek} = \frac{\mathrm{Ek}_1}{\pi^2}$ $\quad\rightsquigarrow\quad$ $\mathrm{Ra} = \pi^4 \frac{1}{x}\left[(1+x)^3 + \frac{1}{\mathrm{Ek}_1^2}\right]$
- minimal critical Rayleigh number is achieved for $\partial_x \mathrm{Ra} = 0$, solve polynomial

$$x^3 - \frac{3}{2}x^2 - \frac{1}{2\mathrm{Ek}_1^2} - \frac{1}{2} = 0.$$

- take a series expansion for $\mathrm{Ek} \to 0$ and only keep leading order term
- this is the usual solution for stationary onset:

$$k_{s,c} = \frac{\pi^{1/3}}{2^{1/6}}\mathrm{Ek}^{-1/3}, \quad \mathrm{Ra}_{s,c} = \frac{3\pi^{4/3}}{2^{2/3}}\mathrm{Ek}^{-4/3}$$

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)‡

- take full perturbation equations again, but keeping σ:

$$\left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right) Z = \frac{1}{Ek} d_z W \tag{7}$$

$$\left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right)(d_{zz}^2 - k^2) W = -\frac{1}{Ek} d_z Z - Ra\, k^2 \Theta \tag{8}$$

$$\left(\sigma - d_{zz}^2 + k^2\right)\Theta = W \tag{9}$$

- same game as before, apply $\left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right)\left(\sigma - d_{zz}^2 + k^2\right)$ to elimate Θ and Z from eq. (8)

$$\left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right)^2 \left(\sigma - d_{zz}^2 + k^2\right)(d_{zz}^2 - k^2) W = -\frac{1}{Ek^2}\left(\sigma - d_{zz}^2 + k^2\right) d_{zz}^2 W - Ra\, k^2 \left(\frac{\sigma}{Pr} - d_{zz}^2 + k^2\right) W$$

- set $W = A \sin(\pi z)$ with $n = 1$
- substitute $k^2 = \pi^2 x, \quad Ek = \frac{Ek_1}{\pi^2}, \quad Ra = R_1 \pi^4, \quad \sigma = i\pi^2 \sigma_1$

‡Chandrasekhar, S., Elbert D.D The instability of a layer of fluid heated below and subject to Coriolis forces. II, *Proc. Roy. Soc. A* (1955), Zhang, K., Roberts, P.H. Thermal inertial waves in a rotating fluid layer: exact and asymptotic solutions. *Phys. Fluids* (1997)

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)

- characteristic equation takes form

$$R_1 = \frac{1}{x}(1 + x + iPr\sigma_1)\left[(1 + x)(1 + x + i\sigma_1) + \frac{1}{Ek_1^2(1 + x + i\sigma_1)}\right]$$

- obviously, for $\sigma_1 = \sigma = 0$, we obtain the stationary equation
- for arbitrary σ_1, R_1 will be complex, yet physically it needs to be real

$\rightsquigarrow$ relationship between real and imaginay part of σ_1

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)

- determine criteria for when convection is stationary and when oscillatory [§]
- express R_1 more clearly in real and imaginary part:

$$R_1 = \frac{1+x}{x}\left\{(1+x)^2 - Pr\sigma_1^2 + \frac{1}{Ek^2(1+x)}\frac{(1+x)^2 + Pr\sigma_1^2}{(1+x)^2+\sigma_1^2} + i\sigma_1\left[(1+x)(1+Pr) - \frac{1-Pr}{Ek_1^2((1+x)^2+\sigma_1^2)}\right]\right\}$$

- onset via purely oscillatory convection, so assume $\sigma_1 \in \mathbb{R}$

$\rightsquigarrow$ imaginary part must vanish, such that R_1 is real, i.e. double characteristic value problem

$$R_1 = \frac{1+x}{x}\left\{(1+x)^2 - Pr\sigma_1^2 + \frac{1}{Ek^2(1+x)}\frac{(1+x)^2 + Pr\sigma_1^2}{(1+x)^2+\sigma_1^2}\right\} \tag{10}$$

$$(1+x)(1+Pr) = \frac{1-Pr}{Ek_1^2((1+x)^2+\sigma_1^2)} \tag{11}$$

[§] that obtained criterion is necessary and sufficient is discussed in detail in Chandrasekhar's *Hydrodynamic and hydromagnetic stability* (1961)

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)

- solve eq. (11) for σ_1^2, we have

$$\sigma_1^2 = \frac{1}{Ek_1^2(1+x)}\frac{1-Pr}{1+Pr} - (1+x^2)$$

- after some algebra one can use eq. (11) to show that the last term in eq. (10) is

$$\frac{1}{Ek^2(1+x)}\frac{(1+x)^2+Pr\sigma_1^2}{(1+x)^2+\sigma_1^2} = \frac{1}{Ek_1^2(1+x)} - (1+Pr)\sigma_1^2$$

- and eq. (10) can be written as

$$R_1 = \frac{1}{x}\left((1+x)^3 + \frac{1}{Ek_1^2} - (1+x)(1+2Pr)\sigma_1^2\right)$$

- then insert σ_1^2 to obtain

$$R_1 = 2(1+Pr)\frac{1}{x}\left[(1+x)^3 + \frac{Pr}{Ek_1^2(1+Pr)^2}\right]$$

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)

- two equations that must be satisfied for oscillatory convection:

$$\begin{aligned} \sigma_1^2 &= \frac{1}{Ek_1^2(1+x)}\frac{1-Pr}{1+Pr} - (1+x)^2 \\ R_1 &= 2(1+Pr)\frac{1}{x}\left[(1+x)^3 + \frac{Pr^2}{Ek_1^2(1+Pr)^2}\right] \end{aligned}$$

$\rightsquigarrow$ note: $\dfrac{1}{Ek_1^2(1+x)}\dfrac{1-Pr}{1+Pr} > 1 \Rightarrow Pr < 1$ so that $\sigma_1^2 > 0$

$\rightsquigarrow$ further if $Pr < 1$, then oscillatory convection only if: $\dfrac{1}{Ek_1^2(1+x)}\dfrac{1-Pr}{1+Pr} > (1+x)^2$

$\rightsquigarrow$ oscillatory convection occurs only for $x < x_*$ with x_* such that

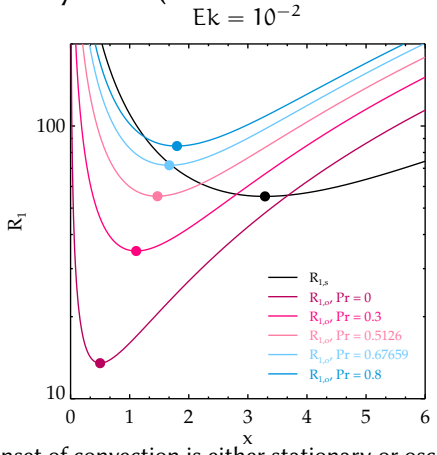
$$(1+x_*)^3 = \frac{1}{Ek_1^2}\frac{1-Pr}{1+Pr}$$

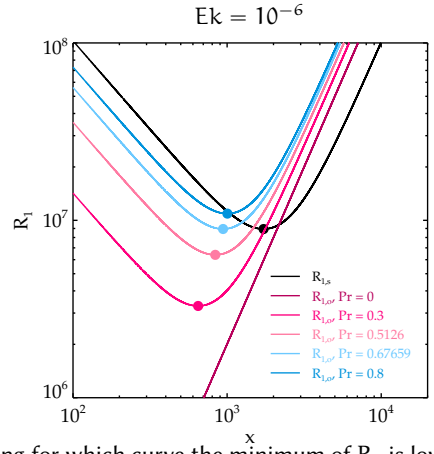
$\rightsquigarrow$ if $x < x_*$, $\sigma_1^2 = 0$, then $R_1 = \dfrac{1}{x_*}\left((1+x_*)^3 + \dfrac{1}{Ek_1^2}\right)$

$\rightsquigarrow$ solution for stationary convection

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)





- onset of convection is either stationary or oscillatory, depending for which curve the minimum of R_1 is lower
- there is a value of Pr such that the critical Rayleigh number for oscillatory onset $R_{1,o}$ approaches that of stationary onset $R_{1,s}$

Linear Stability Analysis

Oscillatory Onset (Thermal Inertial Mode)

- for $\mathrm{Ek} \to 0$, the critical Rayleigh number for oscillatory onset $\mathrm{R}_{1,\mathrm{o}}$ approaches that of stationary onset $\mathrm{R}_{1,\mathrm{s}}$, with

$$\mathrm{R}_{1,\mathrm{s}} \to 3\left(\frac{1}{2}\mathrm{Ek}_1^2\right)^{2/3}, \quad \mathrm{R}_{1,\mathrm{o}} \to 2(1+\mathrm{Pr})\left[3\left(\frac{1}{2}\frac{\mathrm{Pr}^2}{(1+\mathrm{Pr})^2\mathrm{Ek}_1^2}\right)^{2/3}\right]$$

- for $\mathrm{R}_{1,\mathrm{o}} \to \mathrm{R}_{1,\mathrm{s}}$ and $\mathrm{Ek}_1 \to 0$:

$$2(1+\mathrm{Pr})\left(\frac{\mathrm{Pr}^2}{(1+\mathrm{Pr})^2}\right)^{2/3} = 1 \quad \Leftrightarrow \quad 2\frac{\mathrm{Pr}^{4/3}}{(1+\mathrm{Pr})^{1/3}} = 1 \qquad \Rightarrow \mathrm{Pr} = 0.67659$$

$\rightsquigarrow$ oscillatory convection for $\mathrm{Ek}^2/\mathrm{Pr}^2 \to 0$

$$\mathrm{k}_{\mathrm{c,o}} = \left(\frac{\mathrm{Pr}}{1+\mathrm{Pr}}\right)\left(\frac{\pi^2}{2\mathrm{Ek}^2}\right)^{2/3}, \quad \mathrm{Ra}_{\mathrm{c,o}} = \frac{3}{2}\frac{(2\pi\mathrm{Pr})^{4/3}}{(1+\mathrm{Pr})^{1/3}}\mathrm{Ek}^{-4/3}, \quad \omega_{\mathrm{c,o}} = (2\pi)^{2/3}\frac{(2-3\mathrm{Pr}^2)^{\frac{1}{2}}}{\mathrm{Pr}^{1/3}(1+\mathrm{Pr})^{2/3}}\mathrm{Ek}^{1/3}$$

Linear Stability Analysis

Wall Mode Onset*

- preferred onset in confined containers via wall modes

$$\mathrm{Ra}_{c,w} = \pi^2 (6\sqrt{3})^{1/2} \mathrm{Ek}^{-1}, \qquad \tilde{\omega}_{c,w} = (2\pi)^2 \sqrt{3} (2+\sqrt{3})^{1/2} \frac{\mathrm{Ek}}{\mathrm{Pr}}$$

⇝ independent of Pr

⇝ retrograde precessing wave with frequency

*Zhong et al, *Phys. Rev. Lett.* (1991), Ecke et al., *Europhys. Lett.* (1992); Kuo & Cross, *Phys. Rev. E* (1993); Herrmann & Busse, *J. Fluid Mech.* (1993); Goldstein et al., *J. Fluid Mech.* (1993); Zhang & Liao, *J. Fluid Mech.* (2009)

Relevance for Nonlinear Problems

Rayleigh-Bénard Convection in Cylindrical Geometries

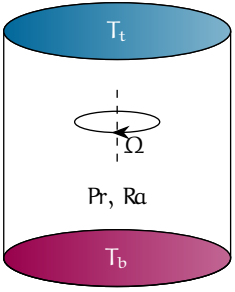
- incompressible Navier–Stokes equations in Oberbeck–Boussinesq approximation

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ D_t \mathbf{u} &= \sqrt{\frac{Pr}{Ra\,\gamma^3}}\,\nabla^2 \mathbf{u} - \nabla p + \sqrt{\frac{\gamma}{Ro^2}}\,\mathbf{u} \times \hat{\mathbf{e}}_z + T\hat{\mathbf{e}}_z \\ D_t T &= \sqrt{\frac{1}{Ra\,Pr\,\gamma^3}}\,\nabla^2 T \end{aligned}$$

- control parameters:

$$Ra = \frac{\alpha g \Delta H^3}{\kappa \nu}, \qquad Pr = \frac{\nu}{\kappa}, \qquad \Gamma = \frac{D}{H}$$

$$Ro = \frac{\sqrt{\alpha g \Delta H}}{2\Omega H}, \qquad Ek = \frac{\nu}{2\Omega H^2} = \sqrt{\frac{Ro^2\,Pr}{Ra}}$$

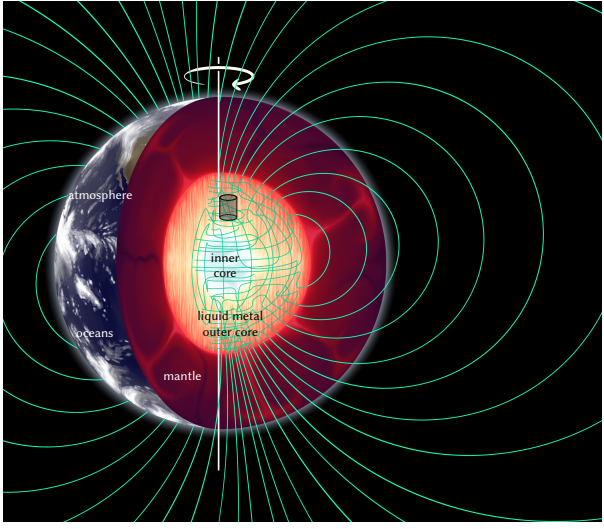


- isothermal heating and cooling plates, adiabatic sidewalls
- no-slip conditions on all walls

Relevance for Nonlinear Problems

Planetary Core Convection*

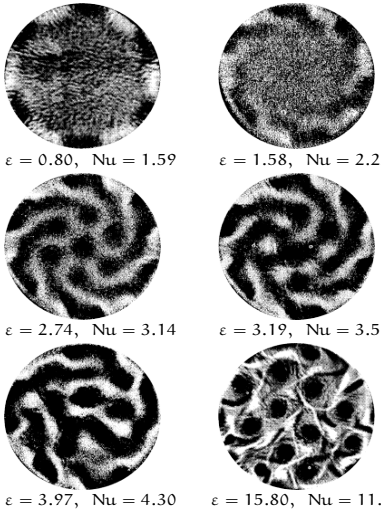
- fundamental fluid physics described by the idealised system of Rayleigh-Bénard convection
- in polar region: gravity and rotation axis are aligned
- core fluid is liquid iron $\rightsquigarrow \mathrm{Pr} \approx 0.025$



*picture courtesy of Jon Cheng

Shadowgraphs in Water, $\mathrm{Pr} = 6.4$, $\mathrm{Ek} = 2.3 \times 10^{-4}$, $\Gamma = 2$

Experiments*

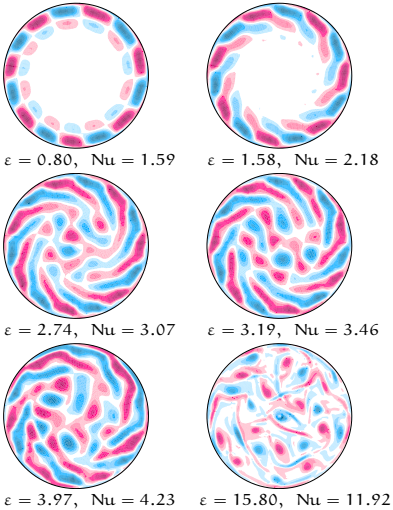


$\varepsilon = 0.80$, $\mathrm{Nu} = 1.59$ $\varepsilon = 1.58$, $\mathrm{Nu} = 2.23$

$\varepsilon = 2.74$, $\mathrm{Nu} = 3.14$ $\varepsilon = 3.19$, $\mathrm{Nu} = 3.59$

$\varepsilon = 3.97$, $\mathrm{Nu} = 4.30$ $\varepsilon = 15.80$, $\mathrm{Nu} = 11.76$

DNS



$\varepsilon = 0.80$, $\mathrm{Nu} = 1.59$ $\varepsilon = 1.58$, $\mathrm{Nu} = 2.18$

$\varepsilon = 2.74$, $\mathrm{Nu} = 3.07$ $\varepsilon = 3.19$, $\mathrm{Nu} = 3.46$

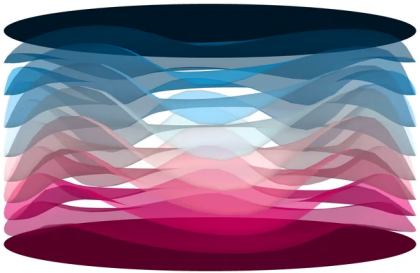
$\varepsilon = 3.97$, $\mathrm{Nu} = 4.23$ $\varepsilon = 15.80$, $\mathrm{Nu} = 11.92$

*Zhong et al., *Phys. Rev. Lett.* (1991); $\varepsilon = (\mathrm{Ra} - \mathrm{Ra}_{c,w})/\mathrm{Ra}_{c,w}$

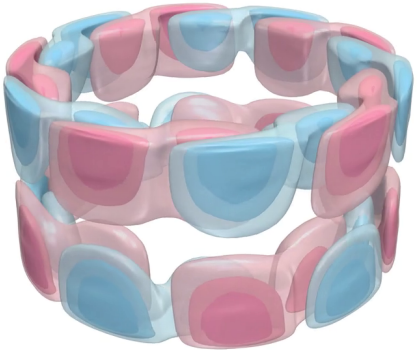
Wall Mode Convection in Water

$\mathbf{Ra/Ra_s = 0.26,\ Ra/Ra_w = 1.04,\ Ro = 0.026}$

$$Ra = 1.55 \times 10^5,\ Ek = 2.3 \times 10^{-4},\ Pr = 6.4$$



T



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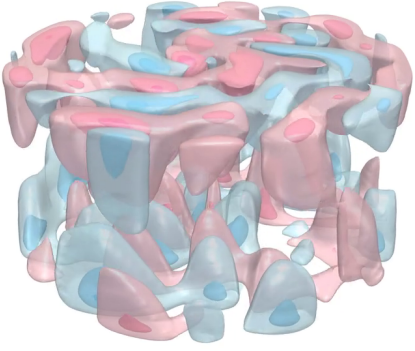
Columnar Convection in Water

$\mathrm{Ra}/\mathrm{Ra_s} = 4.29$, $\mathrm{Ra}/\mathrm{Ra}_w = 17.47$, $\mathrm{Ro} = 0.149$

$Ra = 2.60 \times 10^6$, $Ek = 2.3 \times 10^{-4}$, $Pr = 6.4$



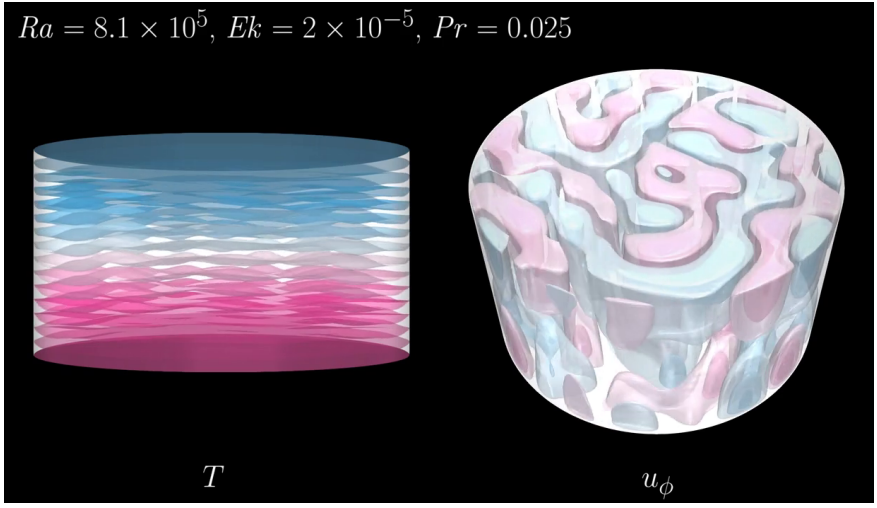
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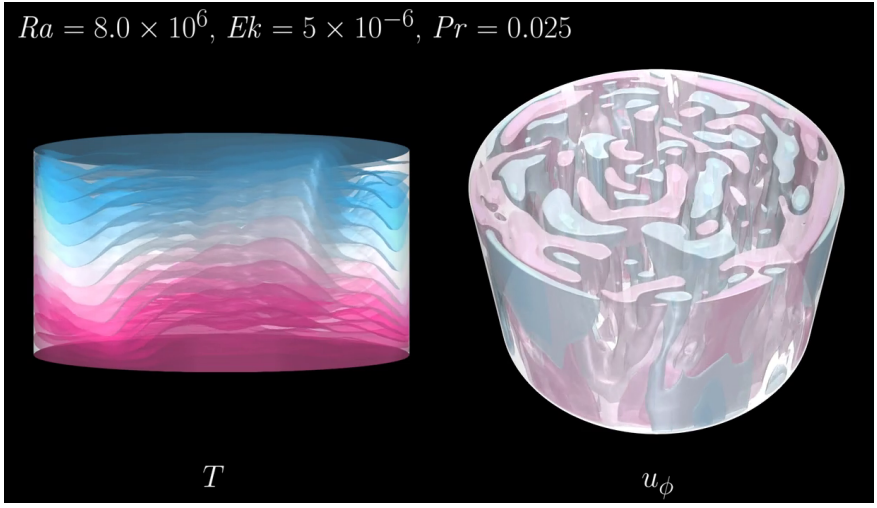
Oscillatory Convection in Liquid Metal

$\mathbf{Ra/Ra_s = 0.05,\ Ra/Ra_w = 0.49,\ Ra/Ra_o = 1.29,\ Ro = 0.115}$



Wall Mode Convection in Liquid Metal

$\mathbf{Ra/Ra_s = 0.08,\ Ra/Ra_w = 1.23,\ Ra/Ra_o = 2.30,\ Ro = 0.089}$



Governing Equations in Coriolis-Centrifugal Convection (C^3)

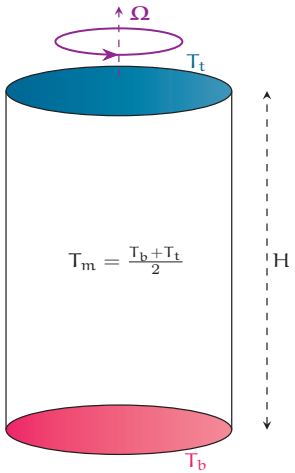
- incompressible Navier–Stokes equations in Oberbeck–Boussinesq approximation

$$\nabla \cdot \mathbf{u} = 0$$

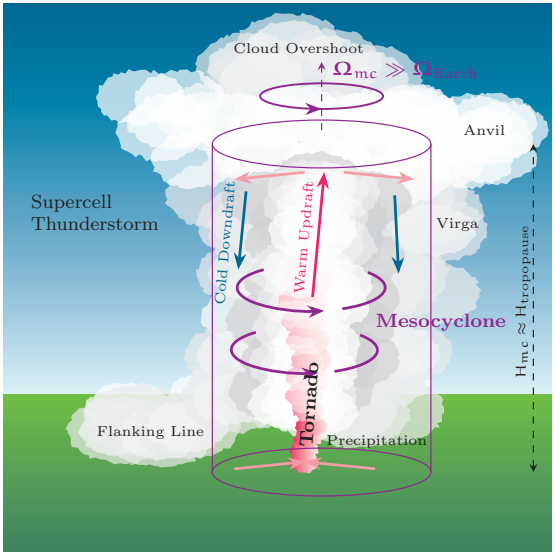
$$D_t \mathbf{u} = \underbrace{\nu \nabla^2 \mathbf{u}}_{\text{viscous}} - \underbrace{\nabla p}_{\text{pressure gradient}} + \underbrace{2\Omega \mathbf{u} \times \hat{\mathbf{e}}_z}_{\text{Coriolis}} - \underbrace{\Omega^2 r \alpha (T - T_m)\hat{\mathbf{e}}_r}_{\text{centrifugal buoyancy } \mathbf{b}_\perp} + \underbrace{g\alpha(T - T_m)\hat{\mathbf{e}}_z}_{\text{gravitational buoyancy } \mathbf{b}_\parallel}$$

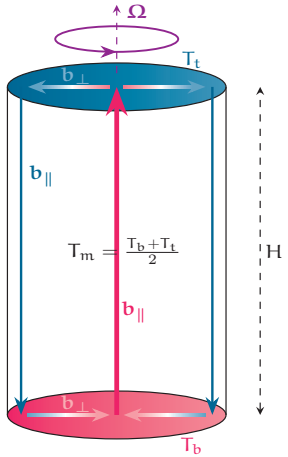
$$D_t T = \kappa \nabla^2 T$$

- gravitational buoyancy force $\mathbf{b}_\parallel$
  - warm (less dense) fluid moves up
  - cold (denser) fluid moves down
- centrifugal buoyancy force $\mathbf{b}_\perp$
  - ⇝ warm (less dense) fluid moves radially towards the centre
  - ⇝ cold (denser) fluid moves radially away from the centre



Idealised Models: Tornadoes in Isolation from their Parent Storm





Supercell Thunderstorm with Tornado Type I*



*http://theweatherlab.tumblr.com/

Tornado Type I - Supercell Tornado (USA)



Governing Equations in Coriolis-Centrifugal Convection (C^3)

- scale all variables with a reference scale
  - length scale: radius R
  - velocity scale: buoyancy velocity $\sqrt{\alpha g \Delta R}$
  - temperature scale: temperature difference $\Delta = T_b - T_t$

    time scale: $R/\sqrt{\alpha g \Delta R}$

    pressure scale: $\rho g \alpha R \Delta$

equations become non-dimensional:

$$
\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
D_t \mathbf{u} &= \sqrt{\frac{Pr}{Ra\,\gamma^3}}\,\nabla^2 \mathbf{u} - \nabla p + \sqrt{\frac{\gamma}{Ro_\parallel^2}}\,\mathbf{u} \times \hat{\mathbf{e}}_z - Fr\,T\,r\,\hat{\mathbf{e}}_r + T\hat{\mathbf{e}}_z \\
D_t T &= \sqrt{\frac{1}{Ra\,Pr\,\gamma^3}}\,\nabla^2 T
\end{aligned}
$$

Control Parameters = Ratios of Characteristic Time Scales

- Prandtl number:

$$\mathrm{Pr} = \frac{\nu}{\kappa} = \frac{\tau_\kappa}{\tau_\nu}$$

- Rayleigh number:

$$\mathrm{Ra} = \frac{\alpha g \Delta H^3}{\kappa \nu} = \frac{\tau_\kappa \, \tau_\nu}{\tau_{\mathrm{ff}}^2}$$

- gravitational Rossby number:

$$\mathrm{Ro}_\parallel = \frac{\sqrt{\alpha g \Delta H}}{2\Omega H} = \frac{\tau_\Omega}{\tau_{\mathrm{ff}}}$$

- Froude number (aspect ratio $\gamma = \frac{R}{H}$):

$$\mathrm{Fr} = \frac{\Omega^2 R}{g} = \frac{\tau_{\mathrm{ff}}^2}{\tau_{\mathrm{cb}}^2}\gamma$$

- thermal diffusion time:

$$\tau_\kappa = \frac{H^2}{\kappa}$$

- viscous diffusion time:

$$\tau_\nu = \frac{H^2}{\nu}$$

- Coriolis time scale:

$$\tau_\Omega = \frac{1}{2\Omega}$$

- gravitational buoyancy (free-fall) time scale:

$$\tau_{\mathrm{ff}} = \frac{H}{\sqrt{\alpha \Delta g H}}$$

- centrifugal buoyancy time scale:

$$\tau_{\mathrm{cb}} = \frac{R}{\sqrt{\alpha \Delta \Omega^2 R^2}}$$

Direct Numerical Simulations (DNS) †

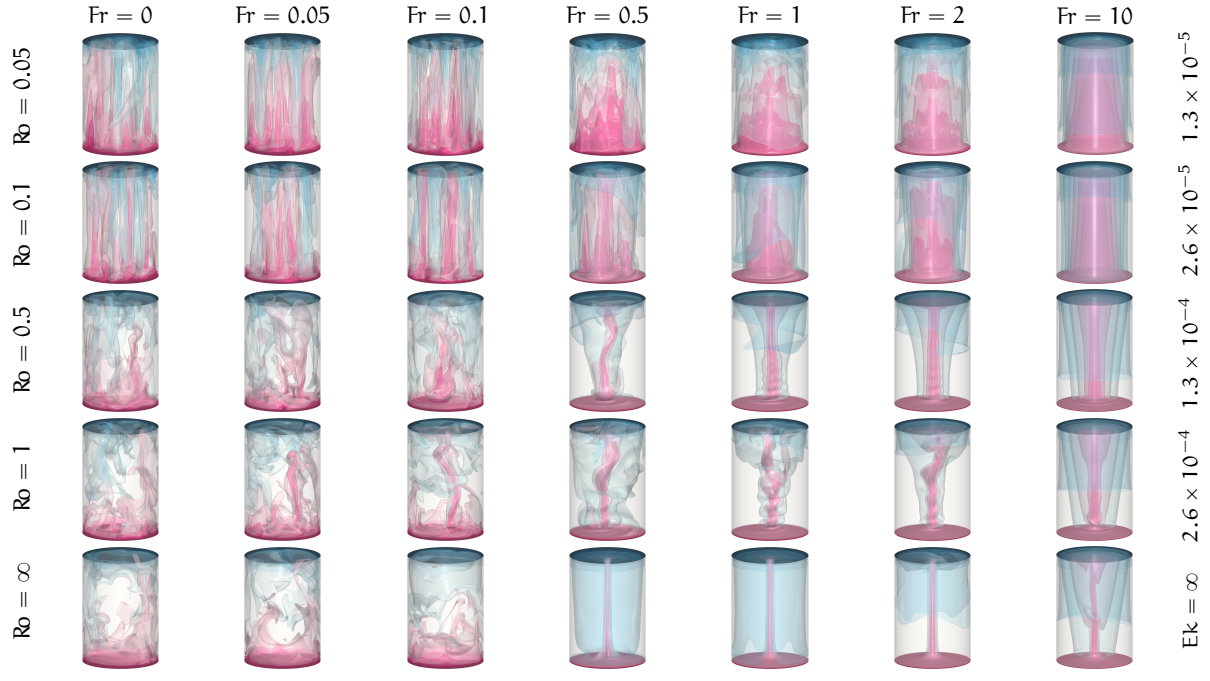
- solve discretised equations using finite volume code GOLDFISH

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ \mathrm{D_t}\mathbf{u} &= \sqrt{\frac{\mathrm{Pr}}{\mathrm{Ra}\,\gamma^3}}\,\nabla^2\mathbf{u} - \nabla \mathrm{p} + \sqrt{\frac{\gamma}{\mathrm{Ro}_\parallel^2}}\,\mathbf{u} \times \hat{\mathbf{e}}_z - \mathrm{Fr}\,\mathrm{T}\,\mathrm{r}\,\hat{\mathbf{e}}_\mathrm{r} + \mathrm{T}\hat{\mathbf{e}}_z \\ \mathrm{D_t T} &= \sqrt{\frac{1}{\mathrm{Ra}\,\mathrm{Pr}\,\gamma^3}}\,\nabla^2 \mathrm{T} \end{aligned}$$

- advantage over laboratory experiments:
  - decouple $\mathrm{Ro}_\parallel = \frac{\sqrt{\alpha g \Delta \mathrm{H}}}{2\Omega \mathrm{H}}$ and $\mathrm{Fr} = \frac{\Omega^2 \mathrm{R}}{g}$
  - set them to ∞ or 0, respectively
- simulated parameter space of 165 DNS with $\sim 10^8$ volume cells

$$\mathrm{Pr} = 6.52,\ \gamma = 0.365,\ \mathrm{Ra} \in \{10^7, 10^8, 10^9\},\ 0.0125 \leqslant \mathrm{Ro}_\parallel \leqslant \infty,\ 0 \leqslant \mathrm{Fr} \leqslant 10$$

† Horn & Aurnou, Regimes of Coriolis-Centrifugal Convection, *Phys. Rev. Lett.* (2018); Horn & Aurnou, Rotating convection with centrifugal buoyancy: Numerical predictions for laboratory experiments, *Phys. Rev. Fluids* (2019); Horn & Aurnou, Tornado-like vortices in the quasi-cyclostrophic regime of Coriolis-centrifugal convection, *J. Turbul.* (2021)


Fr = 0
Fr = 0.05
Fr = 0.1
Fr = 0.5
Fr = 1
Fr = 2
Fr = 10
Ro = 0.05
Ro = 0.1
Ro = 0.5
Ro = 1
Ro = ∞
1.3 × 10⁻⁵
2.6 × 10⁻⁵
1.3 × 10⁻⁴
2.6 × 10⁻⁴
Ek = ∞

Regimes of Coriolis-Centrifugal Convection (C^3)[‡]

- transition from 3D to Coriolis dominated (quasi-geostrophic) flow

$$\text{gravitational Rossby number } \mathrm{Ro}_\parallel = \frac{\tau_\Omega}{\tau_{\mathrm{ff}}} = \frac{\sqrt{\alpha g \Delta H}}{2\Omega H}$$

  relatively well constrained: $6 \lesssim \mathrm{Pr}^{3/4}\mathrm{Ra}^{1/4}\widetilde{\mathrm{Ro}_\parallel}^{3/2} \lesssim 20$

- transition to centrifugally (quasi-cyclostrophic) dominated flow

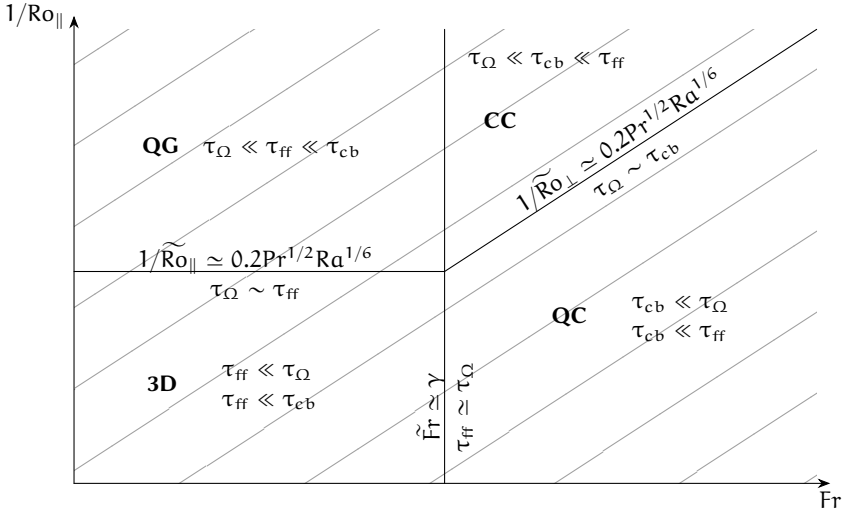
$$\text{centrifugal Rossby number } \mathrm{Ro}_\perp = \frac{\tau_\Omega}{\tau_{\mathrm{cb}}} = \frac{\sqrt{\alpha\Delta}}{2} = \mathrm{Ro}_\parallel\sqrt{\frac{\mathrm{Fr}}{\gamma}}$$

- transition to centrifugal domination defined by

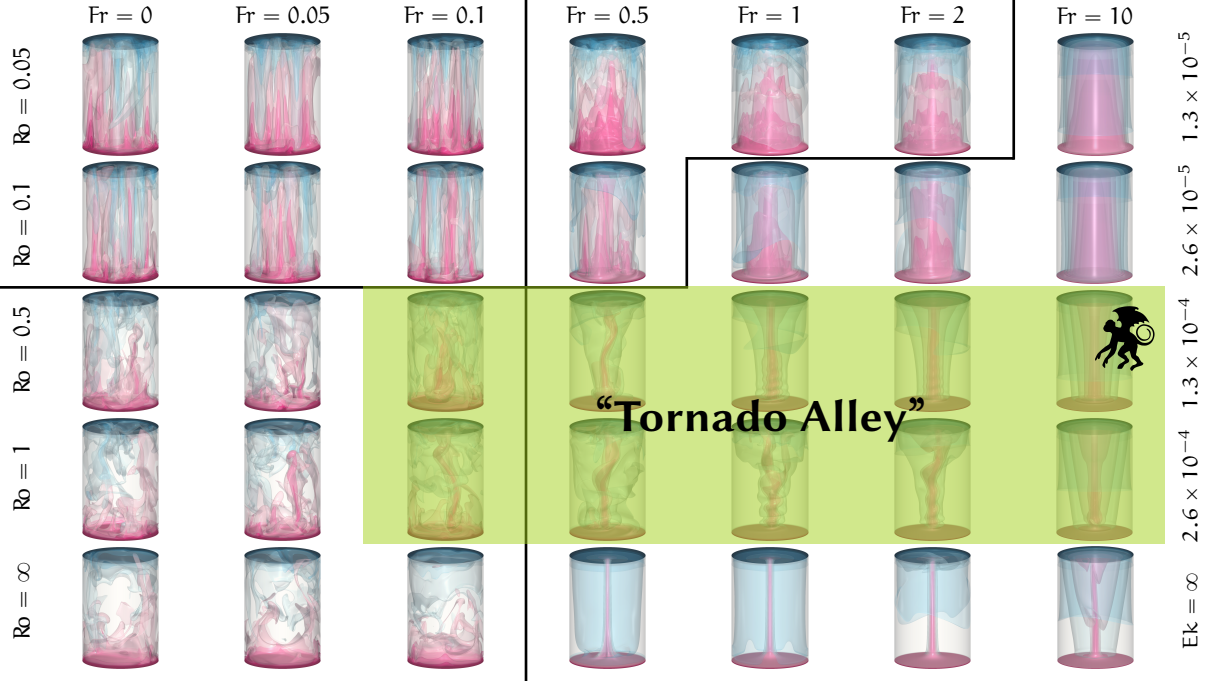
$$\mathrm{Ro}_\parallel \simeq \mathrm{Ro}_\perp \quad \Longleftrightarrow \quad \boxed{\mathrm{Fr}_{\mathrm{cr}} = \frac{\Omega^2 R}{g} \simeq \gamma \iff \widetilde{H} \simeq \frac{g}{\Omega^2}}$$

[‡] Horn & Aurnou, *Phys. Rev. Lett.* (2018); Horn & Aurnou, *Phys. Rev. Fluids* (2019)

Regimes of Coriolis-Centrifugal Convection (C^3)*



*Horn & Aurnou, *Phys. Rev. Lett.* (2018); Horn & Aurnou, *Phys. Rev. Fluids* (2019)


Fr = 0
Fr = 0.05
Fr = 0.1
Fr = 0.5
Fr = 1
Fr = 2
Fr = 10
Ro = 0.05
Ro = 0.1
Ro = 0.5
Ro = 1
Ro = ∞
1.3 × 10⁻⁵
2.6 × 10⁻⁵
1.3 × 10⁻⁴
2.6 × 10⁻⁴
Ek = ∞
“Tornado Alley”

Tornado-Like Vortices in the Quasi-Cyclostrophic Regime

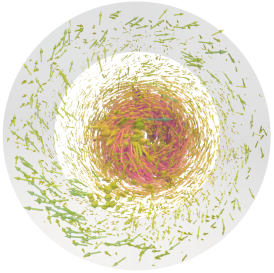








$\mathrm{Ra} = 10^8,\ \mathrm{Ro}_{\parallel} = 1,\ \mathrm{Fr} = 1$



- bulging deformation $\rightsquigarrow$ helical upward flow due to thermal wind
- multiple vortices, sometimes with single convergence
- pressure profile, ring of high surface pressure surrounding the core

Predictions for Tornadogenesis*

Non-Dimensional Control Parameters Based on Observations†



- mesocyclone radius: $\mathrm{R}_{\mathrm{mc}} \sim 1.5\,\mathrm{km}$ to $4.5\,\mathrm{km}$
- mesocyclone height (up to tropopause): $\mathrm{H}_{\mathrm{mc}} \sim 12\,\mathrm{km}$

  $\gamma_{\mathrm{mc}} \sim 0.125$ to 0.375
- rotation rate: $\Omega_{\mathrm{mc}} \sim 1.6 \times 10^{-3}\,\mathrm{s}^{-1}$ to $4.7 \times 10^{-2}\,\mathrm{s}^{-1}$

  $\mathrm{Fr}_{\mathrm{mc}} \sim 0.83$
- superadiabatic temperature differences: $\Delta \sim 1\,\mathrm{K} - 30\,\mathrm{K}$

  $\mathrm{Ro}_{\parallel,\mathrm{mc}} > 0.1$, $\mathrm{Ro}_{\perp,\mathrm{mc}} \sim 0.2$

  $\mathrm{Ra}_{\mathrm{mc}} \sim 10^{18}$
- formation time scales:

$$\tau_{\mathrm{cb}} \simeq \sqrt{\frac{1}{\alpha \Delta_{\mathrm{mc}} \Omega_{\mathrm{mc}}^2}} = \sqrt{\frac{1}{\alpha \Delta_{\mathrm{mc}}} \frac{\mathrm{H}_{\mathrm{mc}}}{\mathrm{g}}} \sim 65\,\mathrm{s}$$

*Horn & Aurnou, *J. Turbul.* (2021)
†Rotunno, *Ann. Rev. Fluids* (2013)

Predictions for Tornadogenesis*



- tornado-like vortices form when $\mathrm{Ro}_{\perp,\mathrm{mc}} \gtrsim \widetilde{\mathrm{Ro}}_{\perp}$ and $\mathrm{Fr}_{\mathrm{mc}} \gtrsim \gamma_{\mathrm{mc}}$

$\rightsquigarrow$ dimensionally this implies

$$\widetilde{\Delta}_{\mathrm{mc}} \gtrsim 13\frac{2^{3/2}\kappa}{\alpha g^{1/4}\nu^{1/2}}\frac{1}{H_{\mathrm{mc}}^{3/4}} = \frac{31.7\,\mathrm{K\,m^{3/4}}}{H_{\mathrm{mc}}^{3/4}}$$

$$\widetilde{\Omega}_{\mathrm{mc}} \gtrsim \sqrt{\frac{g}{H_{\mathrm{mc}}}}$$

$\rightsquigarrow$ with $H_{\mathrm{mc}} = 12\,\mathrm{km}$, we obtain

$$\widetilde{\Delta}_{\mathrm{mc}} \gtrsim 0.03\,\mathrm{K}$$

$$\widetilde{\Omega}_{\mathrm{mc}} \gtrsim 3\times10^{-2}\,\mathrm{s^{-1}}$$

*Horn & Aurnou, *J. Turbul.* (2021)