Rotating Fluids: 3. Potential Vorticity and Rossby Waves

Susanne Horn

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3. Potential Vorticity and Rossby Waves

- Vorticity Equation
- Ertel's Theorem and Potential Vorticity
- Atmospheric Rossby Waves
- Core Flow Rossby Waves

Vorticity Equation (Oberbeck-Boussinesq Approximation)

- vorticity is defined as $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{u}$, it holds $\boldsymbol{\nabla} \cdot \boldsymbol{\omega} = 0$

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$\rightsquigarrow$ vorticity equation is derived by taking curl of Navier-Stokes equations (Oberbeck-Boussinesq approximation):

$$\boldsymbol{\nabla} \times \left[D_t \mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + 2\mathbf{u} \times \boldsymbol{\Omega} + g\alpha(T - T_m)\hat{\mathbf{e}}_z - \Omega^2 r\alpha(T - T_m)\hat{\mathbf{e}}_r \right]$$

$$\Leftrightarrow \quad D_t \boldsymbol{\omega} = (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \boldsymbol{\nabla}\mathbf{u} + \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega} - g\alpha \hat{\mathbf{e}}_z \times \boldsymbol{\nabla} T + \Omega^2 r\alpha \hat{\mathbf{e}}_r \times \boldsymbol{\nabla} T$$

vortex stretching/tilting; viscous torque; baroclinic torques

Ertel's Theorem[‡]

- initially derived for inviscid, compressible, rotating fluid; generalisation by Truesdell[*]
- earlier version by Rossby, later joint publication[†]

⇝ conservation law that includes the nonlinear inertial term

⇝ better applicable to rotating turbulent flows than TP



Hans Ertel
(1904–1971)

[*] Truesdell, C.A.T., *On Ertel's Vorticity Theorem*, *Zeitschrift für angewandte Mathematik und Physik* (1951)

[†] Ertel, H., Rossby, C.-G. A, A New Conservation Theorem of Hydrodynamics, *Geofisica pura e applicata* (1949)

[‡] Ertel, H., Ein neuer hydrodynamischer Wirbelsatz, *Meteorol. Zeitschrift* (1942); Schubert, W. et al. English translations of twenty-one of Ertel's papers on geophysical fluid dynamics, *Meteorol. Z.* (2004)

Ertel's Equation and Ertel's Theorem

- for scalar quantity λ that is materially conserved, $D_t\lambda = 0$ holds

Ertel's equation: $$D_t\left(\boldsymbol{\nabla}\lambda \cdot \frac{\boldsymbol{\omega}_a}{\rho}\right) = \boldsymbol{\nabla}\lambda \cdot \frac{\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}p}{\rho^3}$$

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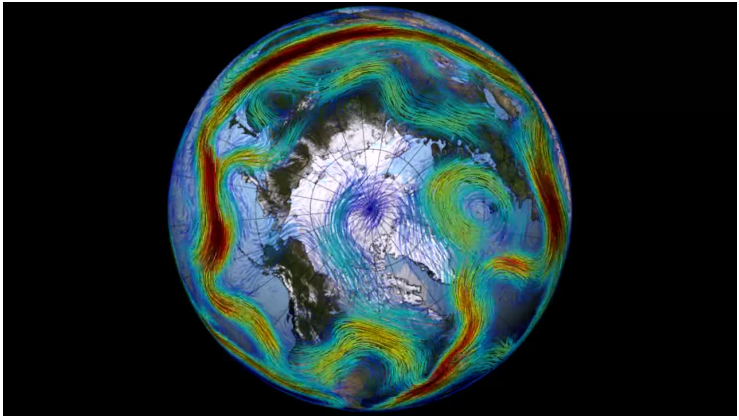
$$\text{Ertel's equation:} \quad D_t\left(\nabla\lambda \cdot \frac{\boldsymbol{\omega}_a}{\rho}\right) = \nabla\lambda \cdot \frac{\nabla\rho \times \nabla p}{\rho^3}$$

- the flow is barotropic $\nabla\rho \times \nabla p = 0$ or $\lambda = \lambda(\rho, p)$, holds the following

$$\text{Ertel's theorem:} \quad D_t\left(\nabla\lambda \cdot \frac{\boldsymbol{\omega}_a}{\rho}\right) = 0$$

where $\Pi = \frac{\nabla\lambda}{\rho} \cdot (\boldsymbol{\omega} + 2\boldsymbol{\Omega})$ is the **potential vorticity**

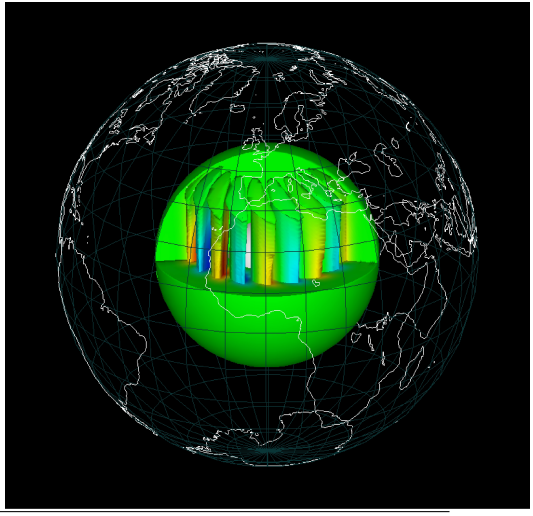
Atmospheric Rossby Waves*



- (barotropic) Rossby waves are slow inertial waves
- in the Earth's atmosphere, Rossby waves are meandering winds, they are related to the large-scale jet stream
- propagate westwards (from east to west)

*Video: NASA's Goddard Space Flight Center

Core Rossby Waves †



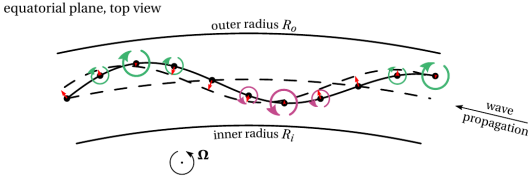


Figure 2. Sketch of the general propagating behaviour of (incompressible) Rossby waves, adapted from ref. [51]. Visualised is part of the equatorial plane of a rotating spherical shell. Initially, a series of fluid columns is at rest and aligned along the mid-radius. If these columns are displaced in a sinusoidal manner, then the vortices that move outwards (indicated by the green arrows) must reduce in length to conserve mass, thus, they get squashed and acquire retrograde vorticity to conserve potential vorticity. Similarly, the vortices that move inwards (indicated by the purple arrows) must extend in length, they get stretched and acquire prograde vorticity. The effect on the neighbouring columns is that the columns to the west are pushed away from the rotation axis and the columns to the east are pushed towards the rotation axis (red arrows). This results in the phase of the initial sinusoidal pattern propagating westwards (prograde) which is known as the Rossby wave.

†Dormy, E. https://www.math.ens.psl.eu/~dormy/IMAGES/, Pothérat, A., Horn, S., Seven decades of exploring planetary interiors with rotating convection experiments, *Comptes rendus physique* (2024), Busse, F.H, Simitev, R., Convection in Rotating Spherical Fluid Shells (2009)

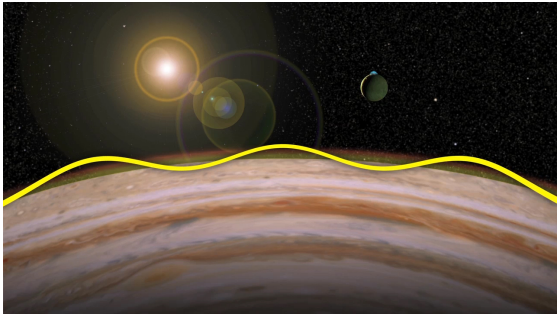
Rossby Waves on Jupiter‡



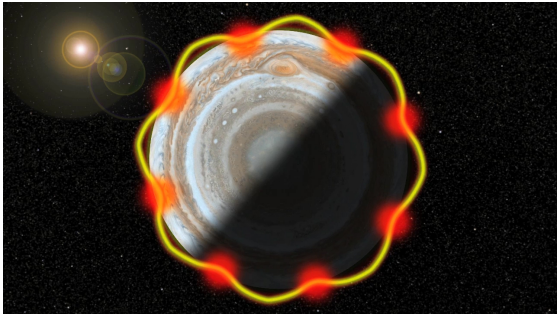
During its flyby in 2000, Cassini spotted a line of hot spots about 7 degrees north of the equator.

‡Choi, D.S., et al., Meteorology of Jupiter's equatorial hot spots and plumes from Cassini, *Icarus* (2013); Images: nasa.gov

Rossby Waves on Jupiter§



Hot spots are formed by a Rossby wave.



Typically, 8 to 10 hot spots are found.

§Choi, D.S., et al., Meteorology of Jupiter's equatorial hot spots and plumes from Cassini, *Icarus* (2013); Images: nasa.gov