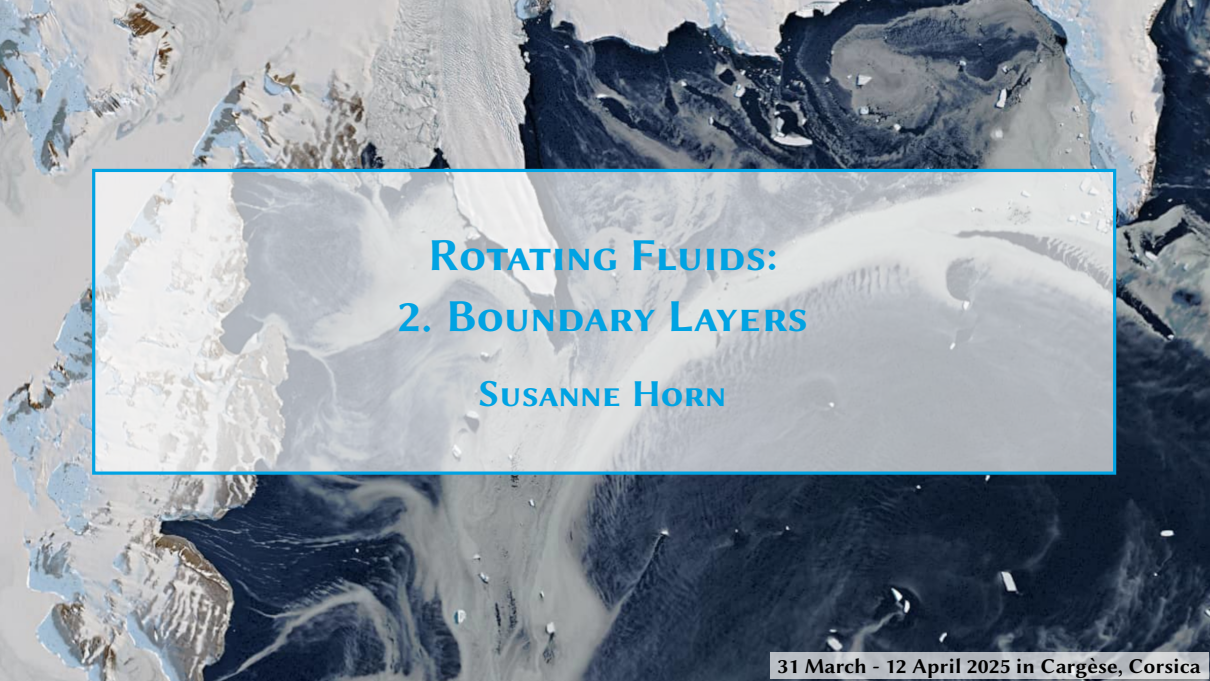


Rotating Fluids:
2. Boundary Layers

Susanne Horn

31 March - 12 April 2025 in Cargèse, Corsica

2. Boundary Layers

- Ekman layer
- Nansen's Fram Expedition
- Ekman Pumping/Suction
- Stewartson layers

Reminder: Governing Equations and Geostrophic Flows

- previous lecture, governing equations:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \mathrm{Ro}(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} &= -\boldsymbol{\nabla} p + \mathrm{Ek}\boldsymbol{\nabla}^2 \mathbf{u} + \mathbf{u} \times \hat{\mathbf{e}}_z \end{aligned}$$

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- in rapidly rotating flows $\mathrm{Ro} \ll 1$ and $\mathrm{Ek} \ll 1$, i.e. quasi-geostrophic flows:

$$-\boldsymbol{\nabla} p + \mathbf{u} \times \hat{\mathbf{e}}_z = \mathcal{O}(\mathrm{Ro}, \mathrm{Ek})$$

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$\rightsquigarrow$ however, this, is only valid in the bulk (far from the boundaries)

$\rightsquigarrow$ at the boundaries, viscosity matters (even when Ek is small!)

The Ekman Layer

- consider viscous flow, rotationg about z-axis, bounded by no-slip $x - y$ plane at $z = 0$
- assume geostrophic, uniform, unidirectional flow over (but far from the) boundary in x-direction $\mathbf{U} = (U_0, 0, 0)$
- look for BL flow solution uniform in x and y direction

$\rightsquigarrow$ in BL: $u_x \neq 0$, $u_y \neq 0$ and $\partial_x u_x = \partial_y u_x = \partial_x u_y = \partial_y u_y = 0$; $u_z = 0$

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Bulk Flow	**vs**	**Boundary Flow**
geostrophic		triple balance (BL approximation)
$0 = -\partial_x p_0$		$-u_y = -\partial_x p + Ek\partial^2_{zz} u_x$
$U_0 = -\partial_y p_0$		$u_x = -\partial_y p + Ek\partial^2_{zz} u_y$
$0 = -\partial_z p_0$		$0 = -\partial_z p$

The Ekman Layer

- combining equations and using $\xi = \dfrac{u_x + i\,u_y}{U_0}$

$\rightsquigarrow$ 2nd order ODE: $Ek\,d^2_{zz}\xi - i\,(\xi - 1) = 0$ with boundary conditions (BCs):
$\xi = 0 \quad \text{at} \quad z = 0, \qquad \xi = 1 \quad \text{as} \quad z \to \infty$

$\rightsquigarrow$ solution:

$$\xi = 1 - \exp\left(\left(\frac{i}{Ek}\right)^{1/2} z\right)$$

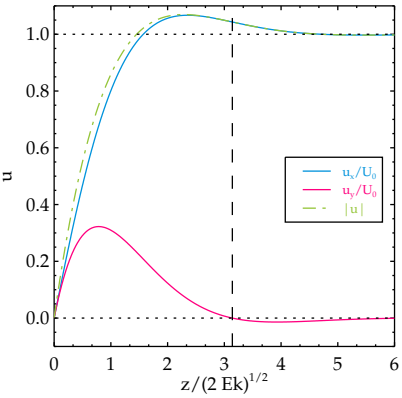
$\rightsquigarrow$ velocity components are:

$$\begin{aligned}\frac{u_x}{U_0} &= Re(\xi) = 1 - \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\cos\left(\frac{z}{\sqrt{2Ek}}\right)\\ \frac{u_y}{U_0} &= Im(\xi) = \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\sin\left(\frac{z}{\sqrt{2Ek}}\right)\end{aligned}$$

$\rightsquigarrow$ distance at which BL solution approaches interior solution is the Ekman layer thickness

$$\lambda_{Ek} = \sqrt{2\,Ek}$$

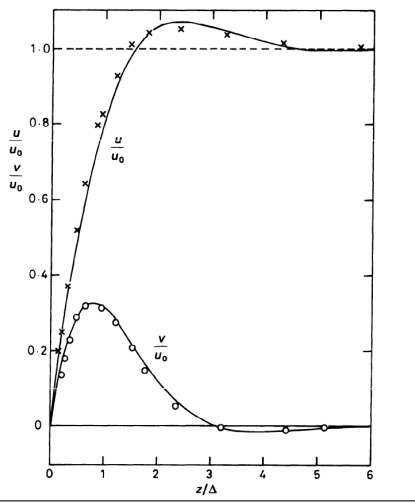
Ekman spiral*



- flow at boundary $z = 0$ is $u_x/u_y = 1$, i.e. 45° to the geostrophic flow U_0
- at $z/\sqrt{Ek} = \pi$ the flow is in the same direction, but 1.1 times faster

*Tritton, David J., Physical fluid dynamics. (1977); see also Caldwell, D.R., van Atta, C.W., Characteristics of Ekman boundary layer instabilities,*J. Fluid Mech.*(1970)

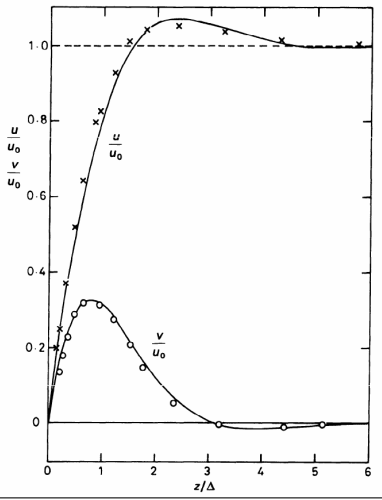
Ekman spiral*

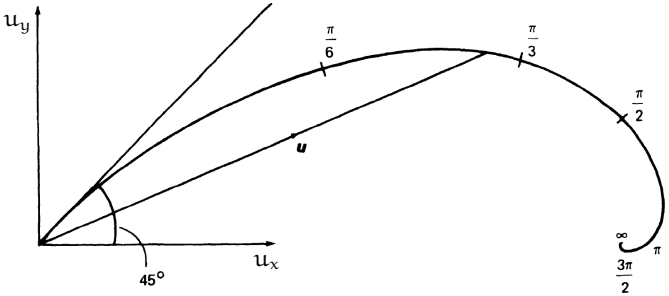


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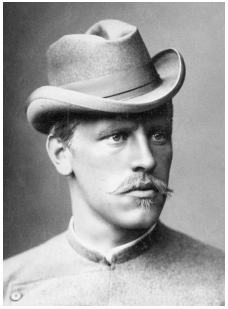


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Fridtjof Nansen's Fram Expedition to the North Pole

The Plan…



Fridtjof Nansen
(1861–1930)

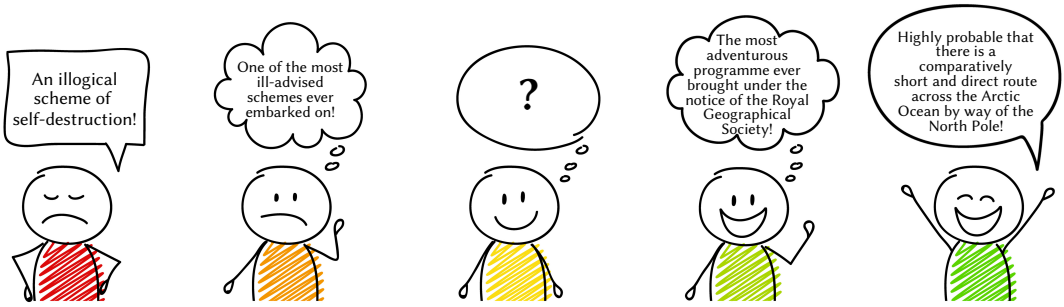




- Fram expedition (1893–1896), wanted to reach North Pole by using drift of the ice with the wind
- theory: *"current flows […] from the Siberian Arctic Sea to the east coast of Greenland,"* probably across the Pole

⇝ take enough provisions, get frozen into the ice and drift with it

Fridtjof Nansen's Fram Expedition to the North Pole



Fridtjof Nansen's Fram Expedition to the North Pole

What Actually Happened…



Routes taken during the 1893–1896 *Fram* expedition:
— *Fram's* route eastward from Vardø along the Siberian coast, turning north at the New Siberian Islands to enter the pack ice, July–September 1893
— *Fram's* drift in the ice from the New Siberian Islands north and west to Spitsbergen, September 1893 – August 1896
— Nansen and Johansen's march to Farthest North, 86°13.6'N, and subsequent retreat to Cape Flora in Franz Josef Land, March 1895 – June 1896
— Nansen and Johansen's return to Vardø from Cape Flora, August 1896
— *Fram's* voyage from Spitsbergen to Tromsø, August 1896

- ⇝ locked in ice for about 35 months, never reached it (drift from 1100 km(78°49'N) to 394 km(85°55'N, march up to 86°13.6'N) from the North Pole)
- ⇝ noticed that there was a consistent drift of 20° to 40° to the right of the wind
- ⇝ led his student (techically Vilhelm Bjerknes's student) Walfrid Ekman do the calculation

Ekman's Original Paper

ARKIV FÖR MATEMATIK, ASTRONOMI OCH FYSIK.
BAND 2. N:o 11.

On the Influence of the Earth's Rotation on Ocean-Currents.

By

V. WALFRID EKMAN.

With 1 plate and 10 figures in the text.

Communicated May 10th 1905 by O. PETTERSSON and V. BJERKNES.

Ekman's Original Paper

of friction is therefore eliminated from several questions. On studying the observations of wind and ice-drift taken during the drift of the »Fram», FRIDTJOF NANSEN found that the drift produced by a given wind did not, according to the general opinion, follow the wind's direction but deviated 20°—40° to the right[1]. He explained this deviation as an obvious consequence of the earth's rotation; and he concluded further that the water-layer immediately below the surface must have a somewhat greater deviation than the latter and so on, since every water-layer is put in motion by the layer immediately above, sweeping over it like a wind. It might therefore be assumed *a priori*, that the current would at some depth run even in the *opposite* direction to the surface-current; and there would consequently be a limit to the capability of the wind in generating currents.

tates four right angles. The direction and velocity of the

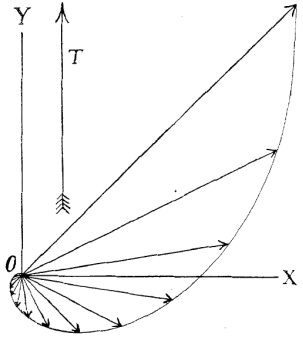
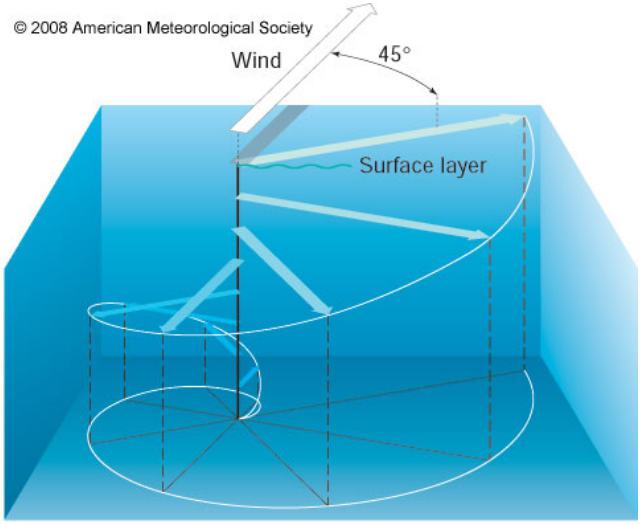


Fig. 1.

current at different depths are represented by the arrows in Fig. 1 above; the longest arrow refers to the surface, the next one to the depth $z = \pi/10\,a$, and the other ones to 2, 3, 4 etc. times this depth. These arrows, if conceived as drawn in position at their respective depths, would then appear in the form of a spiral staircase, the breadth of the steps decreasing rapidly downwards. This stair-case would actually become microscopical before the first winding is completed.

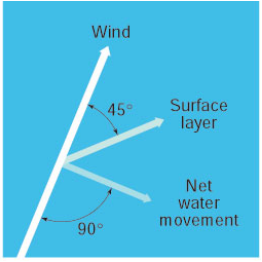
Surface Ekman layer



- Ekman layers occur wherever there is a horizontal frictional stress

⇝ wind over the ocean:

⇝ similar result, surface flow is 45° to the right of the wind (northern hemisphere), net transport is 90° to the right



Generalization to non-uniform currents

- assume geostrophic, spatially nonuniform flow (still far from the) boundary, $\mathbf{U} = (U_0, V_0, 0)$
- equations for bulk flow are:

$$-V_0 = -\partial_x p_0, \qquad U_0 = -\partial_y p_0$$

- dynamic pressure is the same at all depths, so $p_0(x, y) = p$, so BL equations are:

$$-(V_0 - u_y) = Ek\partial^2_{zz} u_x, \qquad U_0 - u_x = Ek\partial^2_{zz} u_y$$

- boundary conditions are now

$$u_x \to U_0 \quad \text{and} \quad u_y \to V_0 \quad \text{as } z \to \infty; \quad u_x = u_y = 0 \quad \text{at } z = 0$$

- solution is

$$\begin{aligned} u_x &= U_0 \left[1 - \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\cos\left(\frac{z}{\sqrt{2Ek}}\right)\right] - V_0 \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\sin\left(\frac{z}{\sqrt{2Ek}}\right) \\ u_y &= U_0 \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\sin\left(\frac{z}{\sqrt{2Ek}}\right) + V_0 \left[1 - \exp\left(-\frac{z}{\sqrt{2Ek}}\right)\cos\left(\frac{z}{\sqrt{2Ek}}\right)\right] \end{aligned}$$

Ekman Transport and Ekman Pumping/Suction

- net transport in x and y in BL structure:

$$U = \int_0^\infty (u_x - U_0)\,dz = -\frac{\sqrt{2Ek}}{2}(U_0 + V_0)$$

$$V = \int_0^\infty (u_y - V_0)\,dz = \frac{\sqrt{2Ek}}{2}(U_0 - V_0)$$

- now look at the divergence of it:

$$\partial_x U + \partial_y V = \int_0^\infty (\partial_x u_x + \partial_y u_y)\,dz = -\frac{\sqrt{2Ek}}{2}(\partial_x V_0 + \partial_y U_0) = -\frac{\sqrt{2Ek}}{2}\underbrace{\omega_{0,z}}_{\text{vorticity}}$$

$\rightsquigarrow$ boundary layer converges or diverges if interior flow has relative vorticity $\omega_{0,z}$

$\rightsquigarrow$ Ekman pumping/suction in the bulk is $W_0 = \frac{\sqrt{2Ek}}{2}\omega_{0,z}$

Stewartson Layers - Sidewall Boundary Layers†

On almost rigid rotations

By K. STEWARTSON
Department of Mathematics, University of Bristol

(*Received 4 May* 1957)

SUMMARY

In order to answer some of Proudman's questions (1956) concerning shear layers in rotating fluids, a study is made of the flow between two coaxial rotating discs, each having an arbitrary small angular velocity superposed on a finite constant angular velocity. It is found that, if the perturbation velocity is a smooth function of r, the distance from the axis, then the angular velocity of the main body of fluid is determined by balancing the outflow from the boundary layer on one disc with the inflow to the boundary layer on the other at the same value of r. At a discontinuity in the angular velocity of either disc a shear layer parallel to the axis occurs. If the angular velocity of the main body of the fluid is continuous, according to the theory given below the purpose of this shear layer is solely to transfer fluid from the boundary layer on one disc to the boundary layer of the other. It has a thickness $O(\nu^{1/3})$, where ν is the kinematic viscosity, and in it the induced angular velocity is $O(\nu^{1/6})$ of the perturbation angular velocity of the discs. On the other hand, if the angular velocity of the main body of fluid is discontinuous, according to the theory given below the thickness of the shear layer is $O(\nu^{1/4})$. A secondary circulation is also set up in which fluid drifts parallel to the axis in this shear layer and is returned in an inner shear layer of thickness $O(\nu^{1/3})$.

The theory is also applied to the motion of fluid inside a closed circular cylinder of finite length rotating about its axis almost as if solid.

J. Fluid Mech. (1966), *vol.* 26, *part* 1, *pp.* 131–144
Printed in Great Britain

On almost rigid rotations. Part 2

By K. STEWARTSON
Department of Mathematics, University College London

(Received 15 December 1965)

The dynamical properties of a fluid, occupying the space between two concentric rotating spheres, are considered, attention being focused on the case where the angular velocities of the spheres are only slightly different and the Reynolds number R of the flow is large. It is found that the flow properties differ inside and outside a cylinder $\mathscr{C}$, circumscribing the inner sphere and having its generators parallel to the axis of rotation. Outside $\mathscr{C}$ the fluid rotates as if rigid with the angular velocity of the outer sphere. Inside $\mathscr{C}$ the fluid rotates with an angular velocity intermediate to the angular velocities of the two spheres and determined by the condition that the flux of fluid into the boundary layer of the faster-rotating sphere is equal to the flux out of the boundary layer of the slower-rotating sphere at the same distance from the axis. The return of fluid is effected by a shear layer near $\mathscr{C}$ and we show that it has a complicated structure for it can be divided into three separate layers, two outer ones, of thickness $\sim R^{-\frac{1}{2}}$ and $\sim R^{-\frac{1}{4}}$, and an inner layer of thickness $\sim R^{-\frac{1}{3}}$.

†STEWARTSON, K., On almost rigid rotations., *J. Fluid Mech.* (1957), STEWARTSON, K., On almost rigid rotations. Part 2, *J. Fluid Mech.* (1966), KUNNEN, R.P.J. ET AL., The structure of sidewall boundary layers in confined rotating Rayleigh-Bénard convection, *J. Fluid Mech.* (2013)

Stewartson Layers - Sidewall Boundary Layers‡

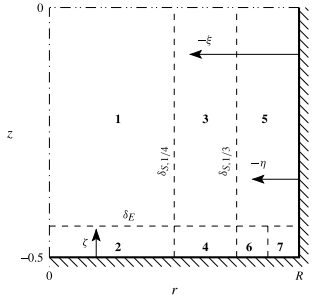


FIGURE 3. Sketch of the regions of the flow domain (bold numbers) with the bounding boundary-layer thicknesses δ_E, $\delta_{S,1/4}$ and $\delta_{S,1/3}$ (dashed lines; not to scale) and boundary-layer coordinates ζ, ξ and η to be introduced later. Only the lower half of the domain is included; the top half follows from symmetry.

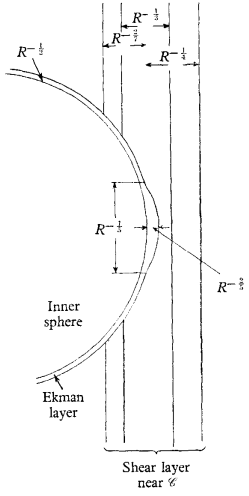


FIGURE 1. Schematic drawing (not to scale) of the intersection region of the Ekman layer near the inner sphere and the shear layer near $\mathscr{C}$.

‡STEWARTSON, K., On almost rigid rotations., *J. Fluid Mech.* (1957), STEWARTSON, K., On almost rigid rotations. Part 2, *J. Fluid Mech.* (1966), KUNNEN, R.P.J. ET AL., The structure of sidewall boundary layers in confined rotating Rayleigh-Bénard convection, *J. Fluid Mech.* (2013)