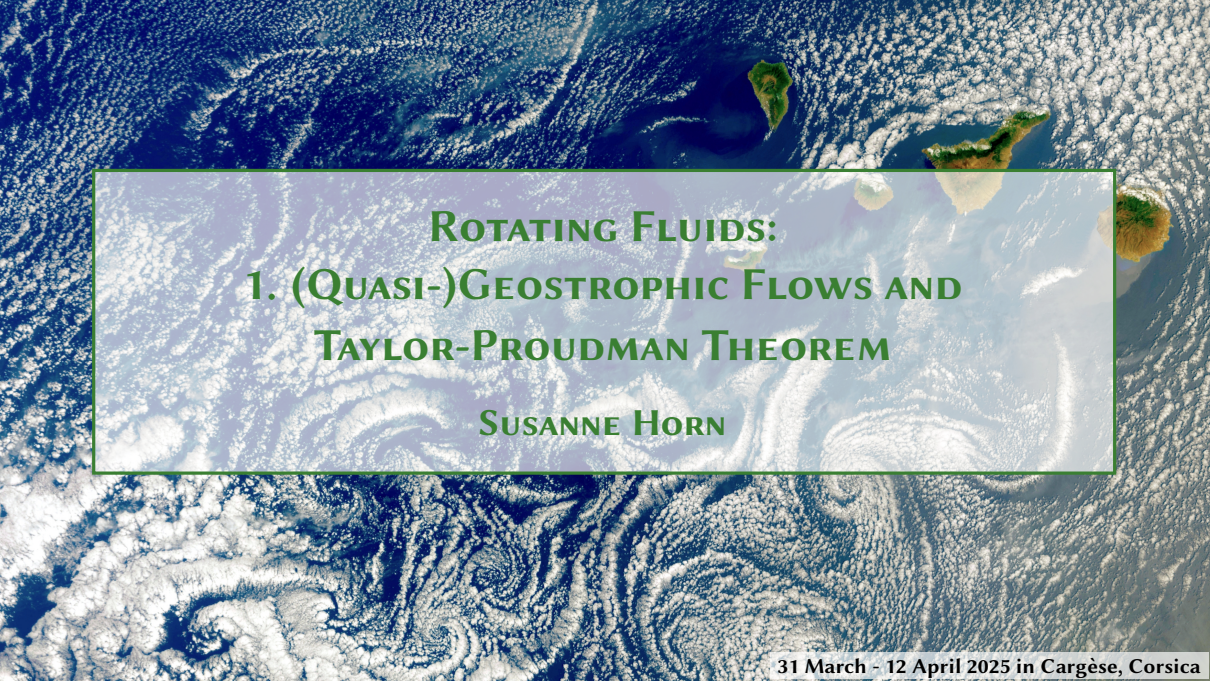
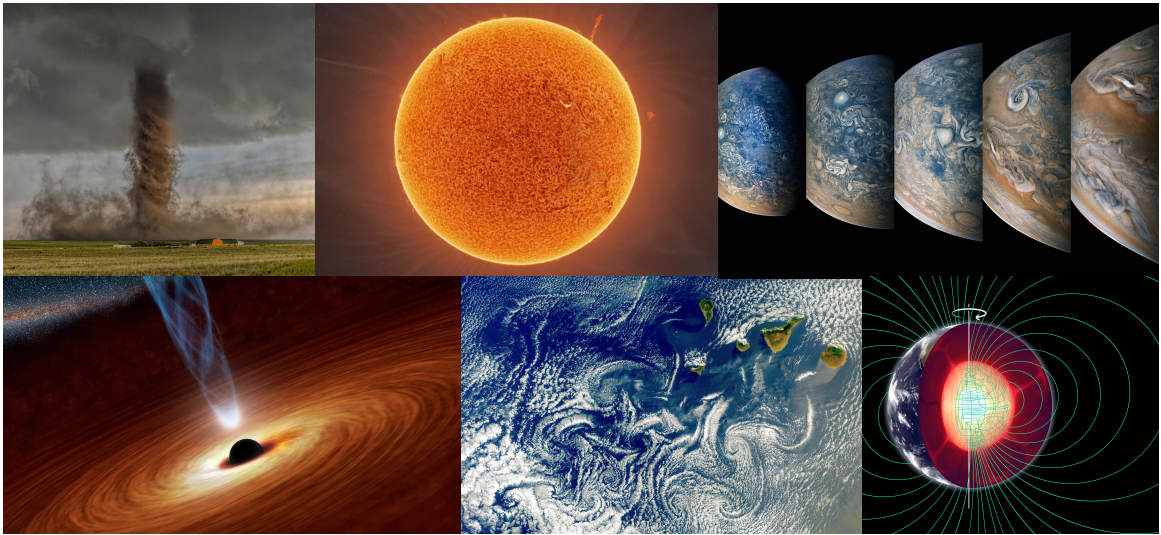


Rotating Fluids: 1. (Quasi-)Geostrophic Flows and Taylor-Proudman Theorem

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Rotating Flows in Nature



Rotating flows are everywhere: planetary interiors, oceans, atmospheres, stars, accretion disks, galaxies, …

Plan for this Course

1. (Quasi-)Geostrophic Flows and Taylor-Proudman Theorem
2. Boundary Layers
3. Potential Vorticity and Rossby Waves
4. Rotating Convection
5. Geostrophic Turbulence

Questions? ⇝ e-mail: susanne.horn@coventry.ac.uk

Books on Rotating Flows

- Greenspan, Harvey Philip. *The theory of rotating fluids.* (1969)
- Cushman-Roisin, Benoit and Beckers, Jean-Marie. *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects.* (2011)
- Friedlander, Susan. *An introduction to the mathematical theory of geophysical fluid dynamics.* (1980)
- Davidson, Peter Alan. *Turbulence in rotating, stratified and electrically conducting fluids.* (2013)
- Chandrasekhar, Subrahmanyan. *Hydrodynamic and hydromagnetic stability.* (1961)
- Glatzmaier, Gary. *Introduction to modeling convection in planets and stars: Magnetic field, density stratification, rotation.* (2013)
- Zhang, Keke and Liao, Xinhao. *Theory and Modeling of Rotating Fluids: Convection, Inertial Waves and Precession.* (2017)
- Tritton, David J.. *Physical fluid dynamics.* (1977)

1. (Quasi-)Geostrophic Flows and Taylor-Proudman Theorem

- importance of rotation
- basic set of governing equations
- geostrophic flows and Taylor-Proudman theorem

When Does Rotation Matter for a Flow?

- rotation introduces two accelerations:
  - Coriolis force
  - centrifugal force
- simple estimates*: fluid feels ambient rotation $\Omega = \dfrac{2\pi \text{ radians}}{\text{time of one revolution}}$ if
  - fluid motions evolving on a time scale longer than time of one rotation:

$$\omega = \frac{\text{time of one revolution}}{\text{motion time scale}} = \frac{2\pi/\Omega}{T} = \frac{2\pi}{\Omega T} \lesssim 1$$

  - fluid parcel travelling at speed U covers distance L in time greater than time of one rotation:

$$\varepsilon = \frac{\text{time of one revolution}}{\text{time taken by parcel to cover distance } L \text{ at speed } U} = \frac{2\pi/\Omega}{L/U} = \frac{2\pi U}{\Omega L} \lesssim 1$$

*related to Rossby numbers, i.e. $\mathrm{Ro}_T = 1/(\Omega T)$ and $\mathrm{Ro} = U/(\Omega L)$

When Does Rotation Matter for a Flow?

- on Earth, ambient rotation is $\Omega = \frac{2\pi}{24\mathrm{h}} + \frac{2\pi}{365.24\mathrm{d}} = 7.29 \times 10^{-5}\,\mathrm{s}^{-1}$

$\rightsquigarrow$ rotation matters on timescales longer than 24 hours ($\omega \lesssim 1$)

$\rightsquigarrow$ rotation matters for flows covering distances L with speeds U

length scale L	velocity scale U
1 m	$\lesssim 0.012\,\mathrm{mm/s}$
10 m	$\lesssim 0.12\,\mathrm{mm/s}$
100 m	$\lesssim 1.2\,\mathrm{mm/s}$
1 km	$\lesssim 1.2\,\mathrm{cm/s}$
10 km	$\lesssim 12\,\mathrm{cm/s}$
100 km	$\lesssim 1.2\,\mathrm{m/s} = 4.32\,\mathrm{km/h}$
1000 km	$\lesssim 12\,\mathrm{m/s} = 43.2\,\mathrm{km/h}$
6371 km (Earth's radius)	$\lesssim 74\,\mathrm{m/s} = 266.4\,\mathrm{km/h}$

When Does Rotation Matter for a Flow? - Examples



Jupiter's Great Red Spot:

- on Jupiter, length of day: 9.9259 h, length of year: 4330.595 d ($\sim$11.9 y)

$\rightsquigarrow \Omega = \dfrac{2\pi}{9.9259\text{h}} = 1.76 \times 10^{-4}\,\text{s}^{-1}$

- GRS existed for several hundred years

$\rightsquigarrow T \sim 100\,\text{y} \Rightarrow \omega = \dfrac{2\pi}{\Omega T} = 2.7 \times 10^{-4}$

$\rightsquigarrow$ GRS feels rotation

- semimajor axis: 14.400 km, semiminor axis: 7.500 km $\rightsquigarrow L \sim 10.000\,\text{km}$
- wind speeds: $U_{wind} = 110\,\text{m/s}$, zonal drift: $U_{drift} = 3\,\text{m/s}$

$\rightsquigarrow \varepsilon_{wind} = \dfrac{2\pi U_{wind}}{\Omega L} = 0.39,\ \varepsilon_{drift} = \dfrac{2\pi U_{wind}}{\Omega L} = 0.01$

$\rightsquigarrow$ the drift is likely more influenced by rotation than the vortex itself

$\rightsquigarrow$ **selection of scales strongly depend on the physical process we want to investigate**

Governing Equations

- rotating frame of reference; continuum, Newtonian fluid
- mass conservation*:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \Leftrightarrow \qquad -\frac{1}{\rho} D_t \rho = \nabla \cdot \mathbf{u}$$

- momentum conservation (Navier-Stokes equations)†:

$$\rho D_t \mathbf{u} = \underbrace{-\nabla P}_{\text{pressure gradient}} + \underbrace{\nabla \cdot \boldsymbol{\tau}}_{\text{viscous}} \underbrace{- \rho \mathbf{g}}_{\text{gravitational}} + \underbrace{2\rho \mathbf{u} \times \boldsymbol{\Omega}}_{\text{Coriolis}} + \underbrace{\rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})}_{\text{centrifugal}}$$

$\rho D_t \mathbf{u}$: inertial

$\rightsquigarrow$ assume $\boldsymbol{\Omega} = \Omega \hat{\mathbf{e}}_z$ and $\mathbf{g} = g \hat{\mathbf{e}}_z$:

$$\rho D_t \mathbf{u} = -\nabla P + \nabla \cdot \boldsymbol{\tau} - \rho g \hat{\mathbf{e}}_z + 2\rho \Omega \mathbf{u} \times \hat{\mathbf{e}}_z + \rho \Omega^2 \mathbf{r}$$

- energy conservation (temperature equation):

$$\rho c_p D_t T = \underbrace{\nabla \cdot (\Lambda \nabla T)}_{\text{conduction}} + \underbrace{\rho \Phi}_{\text{viscous dissipation}} + \underbrace{\varepsilon}_{\text{other sources/sinks}}$$

- equation of state: $\rho = \rho(T, p, \ldots)$

*material derivative: $D_t = \partial_t + \mathbf{u} \cdot \nabla$

†all terms are force densities

Governing Equations - Oberbeck-Boussinesq Approximation*

- Taylor expansion of ρ at $T = T_m$

$$\begin{aligned} \rho &= \rho(T_m) + \partial_T \rho(T_m)\,(T - T_m) + \mathcal{O}\big((T - T_0)^2\big) \Big| \quad \rho(T_m) \equiv \rho_m,\ \alpha \equiv -\tfrac{1}{\rho_m}\partial_T \rho \\ &= \rho_m(1 - \alpha(T - T_m)) \end{aligned}$$

$\rightsquigarrow$ neglect higher order term when ρ appears as multiplier

$\rightsquigarrow$ keep higher order term when gradients are involved (buoyancy terms); absorb mean part into reduced pressure:

$$-\rho g_m \hat{\mathbf{e}}_z + \rho_m \Omega^2 \hat{\mathbf{e}}_r = \nabla(-\rho_m g z + \tfrac{1}{2}\rho_m \Omega^2 r^2)$$

$\rightsquigarrow$ reduced pressure: $p \equiv \frac{1}{\rho_m} P + gz - \frac{1}{2}\Omega^2 r^2$

- all other material properties (ν, κ, …) are independent of pressure and temperature
- energy dissipation due to viscosity is neglected



Anton Oberbeck (1846–1900)



Joseph Boussinesq (1842–1929)

*Boussinesq, J.V., Théorie analytique de la chaleur, vol. 2 (Gauthier-Villars Paris, 1903); Oberbeck, A., Ueber die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen, *Annalen der Physik* (1879); Gray, D.D. and Giorgini, A. The validity of the Boussinesq approximation for liquids and gases, *Int. J. Heat Mass Transf.* 1976); Marques, F. et al., Centrifugal effects in rotating convection: axisymmetric states and three-dimensional instabilities, *J. Fluid Mech.* (2007)

Governing Equations - Oberbeck-Boussinesq Approximation

- incompressible (solenoidal) continuity equation:

$$\nabla \cdot \mathbf{u} = 0$$

- Navier–Stokes equations:

$$\mathrm{D_t}\mathbf{u} = \underbrace{-\nabla p}_{\text{pressure gradient}} + \overbrace{\nu\nabla^2\mathbf{u}}^{\text{viscous}} + \underbrace{2\Omega\mathbf{u}\times\hat{\boldsymbol{e}}_z}_{\text{Coriolis}} + \overbrace{g\alpha(T-T_m)\hat{\boldsymbol{e}}_z}^{\text{gravitational buoyancy}} - \underbrace{\Omega^2 r\alpha(T-T_m)\hat{\boldsymbol{e}}_r}_{\text{centrifugal buoyancy}}$$

- temperature equation:

$$\mathrm{D_t}T = \kappa\nabla^2 T$$

(Quasi-)Geostrophic Flows

Non-dimensionalisation

- dimensional equations:

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0 \\ D_t \mathbf{u} &= -\nabla p + \nu \nabla^2 \mathbf{u} + 2\Omega \mathbf{u} \times \hat{\mathbf{e}}_z \end{aligned}$$

- non-dimensionalisation: replace every variable by scaled counterpart $\tilde{\chi} = \dfrac{\chi}{\chi_{ref}}$:

$$x_{ref} = H, \qquad t_{ref} = \frac{1}{2\Omega},$$
$$u_{ref} = U \quad (\text{e.g. } U = u_{ff} = \sqrt{\alpha g \Delta H}), \qquad p_{ref} = \rho U 2\Omega H$$

- dimensionless form:

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}} &= 0 \\ \partial_t \tilde{\mathbf{u}} + Ro(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} &= -\nabla \tilde{p} + Ek \nabla^2 \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \times \hat{\mathbf{e}}_z \end{aligned}$$

$\rightsquigarrow$ **in rapidly rotating flows $Ro \ll 1$ and $Ek \ll 1$**

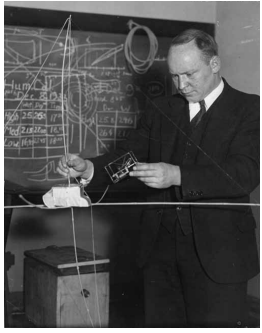
The Rossby Number and the Ekman Number

- control parameters are ratios of relevant time scales:
  - Coriolis time scale: $\tau_\Omega = \dfrac{1}{2\Omega}$
  - viscous time scale: $\tau_\nu = \dfrac{H^2}{\nu}$
  - inertial time scale: $\tau_i = \dfrac{H}{U}$

$\rightsquigarrow$

Rossby number: $\mathrm{Ro} = \dfrac{\tau_\Omega}{\tau_i} = \dfrac{U}{2\Omega H}$

Ekman number: $\mathrm{Ek} = \dfrac{\tau_\Omega}{\tau_\nu} = \dfrac{\nu}{2\Omega H^2}$



Carl-Gustaf Arvid Rossby
(1898–1957)



Vagn Walfrid Ekman
(1874–1954)

Geostrophic* Balance

- in rapidly rotating, steady flows $\mathrm{Ro} \ll 1$ and $\mathrm{Ek} \ll 1$, we have quasi-geostrophic flows:

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}} &= 0 \\ -\nabla \tilde{\mathrm{p}} + \tilde{\mathbf{u}} \times \hat{\boldsymbol{e}}_z &= \mathcal{O}(\mathrm{Ro},\, \mathrm{Ek}) \end{aligned}$$

- geostrophic balance = Coriolis force and pressure gradient balance:

$$\tilde{\mathbf{u}} \times \hat{\boldsymbol{e}}_z = \nabla \tilde{\mathrm{p}}$$

*from Greek, γη = Earth and στροφη = turning

Taylor-Proudman Theorem †

- take curl of (non-dimensional) geostrophic balance*:

$$\begin{aligned} & & \nabla \times (\mathbf{u} \times \hat{\mathbf{e}}_z) &= \nabla \times \nabla p \\ \Rightarrow \quad & \mathbf{u}(\nabla \cdot \hat{\mathbf{e}}_z) - \hat{\mathbf{e}}_z(\nabla \cdot \mathbf{u}) + (\hat{\mathbf{e}}_z \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\hat{\mathbf{e}}_z &&= 0 \\ \Rightarrow \quad & \partial_z \mathbf{u} &&= 0 \end{aligned}$$

- also, using continuity equation:

$$\partial_z u_z = 0 \quad \Rightarrow \quad \partial_x u_x + \partial_y u_y = \nabla_\perp \cdot \mathbf{u}_\perp = 0$$

⇝ no motion varying in the direction of $\mathbf{\Omega}$ (here z) is allowed

⇝ all particles on the same vertical height z move in concert

⇝ with flat top and bottom, $\partial_z u_z = 0 \Rightarrow u_z = 0$ everywhere, flow is strictly 2-D

*vector identity: $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

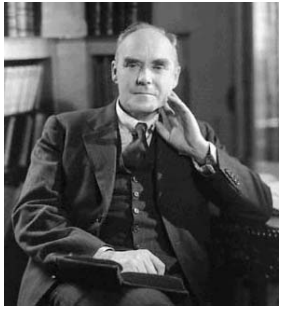
†Taylor, G.I. Experiments with rotating fluids, *Proc. Roy. Soc. A* (1921), Proudman, J., On the Motion of Solids in a Liquid Possessing Vorticity, *Proc. Roy. Soc. A* (1916)

Taylor-Proudman Theorem *

Taylor-Proudman theorem:

$$\partial_z \mathbf{u} = 0$$

$$\partial_z u_z = 0,\ \nabla_\perp \cdot \mathbf{u}_\perp = 0$$



Geoffrey Ingram Taylor
(1886–1975)



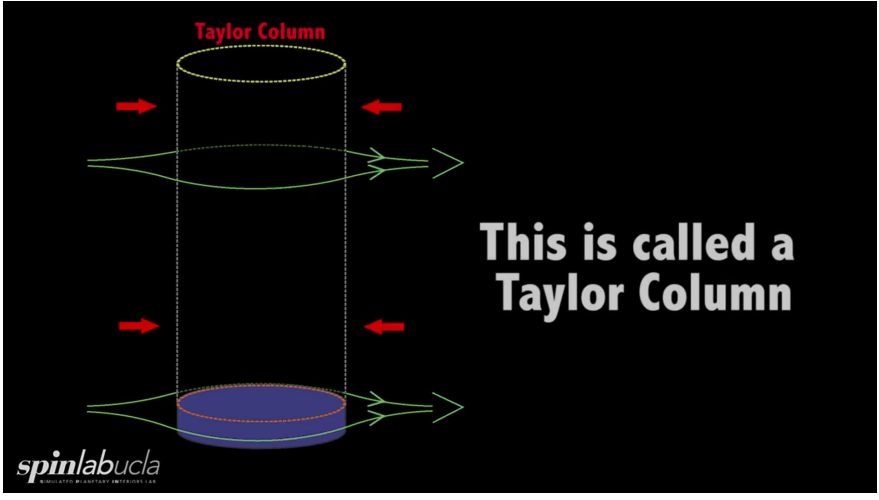
Joseph Proudman
(1888–1975)

- homogeneous geostrophic flow over an irregular bottom creates **Taylor columns**
- geostrophic flows follow topographic (depth) contours (isobaths) aligned with pressure contours

⇝ known as **geostrophic contours**

*Taylor, G.I. Experiments with rotating fluids, *Proc. Roy. Soc. A* (1921), Proudman, J., On the Motion of Solids in a Liquid Possessing Vorticity, *Proc. Roy. Soc. A* (1916)

Taylor Columns*



*SpinLab UCLA, https://youtu.be/7GGfsW7gOLI

Taylor Columns in Nature

- Canary Islands: Taylor columns create von Kármán vortex streets

