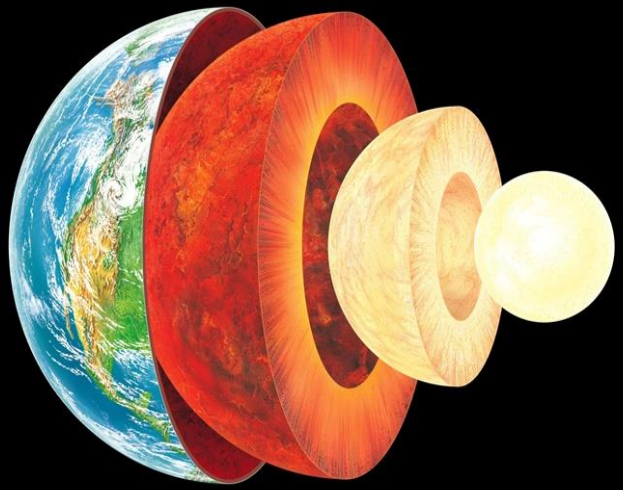
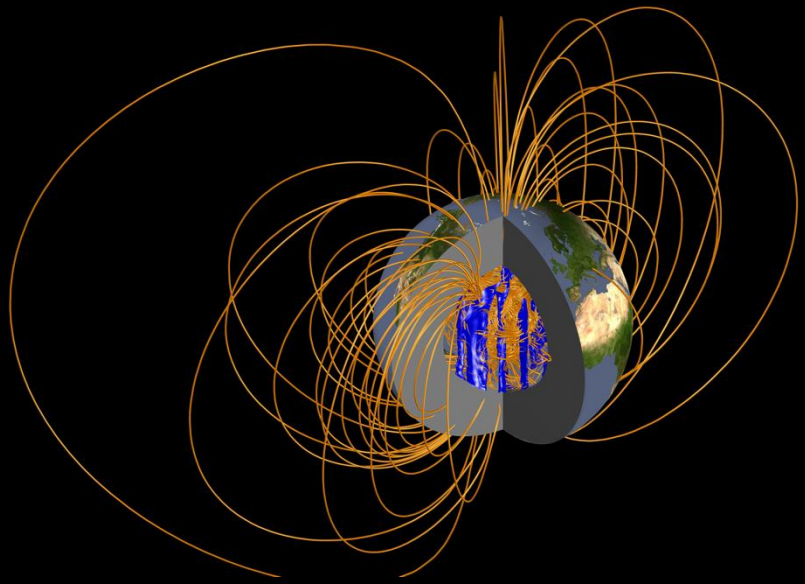
Parameter	Sun (Ossendrijver 2003)	Earth (Nataf & Schaeffer 2013, Davidson 2013, Tobias 2021)	Importance for modelling
$Rm=U\,L\,/\eta$	10^{9}-10^{10}	10^{2}-10^{3}	Impossible to reach solar regime!!!!
$Re=U\,L\,/\nu$	10^{12}-10^{13}	10^{8}-10^{9}	Inertia & Reynolds stresses are important in Sun
$Pm=\nu\,/\eta$	10^{-6}	10^{-5}-$5x10^{-7}$	Dissipation scale of velocity is much smaller than that of magnetic field in both
$Pr=\nu\,/\kappa$	10^{-7}-10^{-6}	~1 (if thermal) >>1 (if compositional)	Leads to very different dynamics of convection and instabilities in stable layers (e.g. Pr Ri important (Garaud 2021, GSF instabilities)
$Ro=U/2\Omega L$	0.1-1	10^{-5}-10^{-6}	Earth is much more rotationally constrained QG dynamics, magnetostrophic balance/dynamics on certain scales.
Mag Energy/Kin Energy	< 1	>> 1	

Towards Asymptotic Models for planetary dynamo modelling





Steve Tobias (University of Leeds)

DAMTP, Fluids January 24th 2025

European Research Council
Established by the European Commission

SIMONS FOUNDATION

Collaborators

Asymptotic Geodynamo

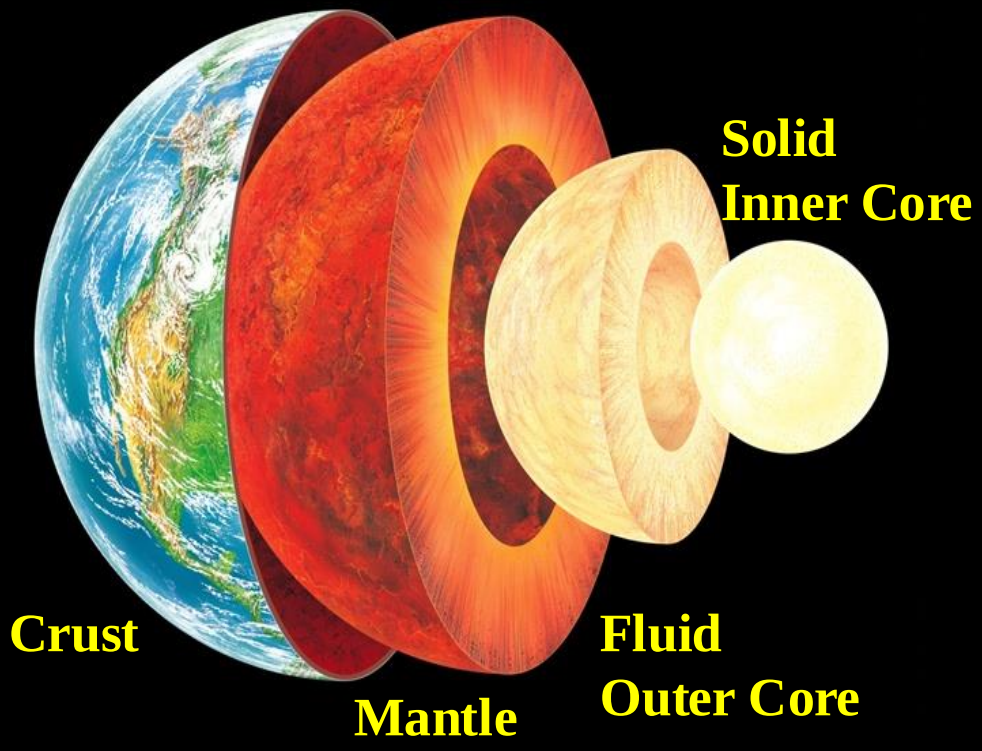
Keith Julien (University of Colorado)

Michael Calkins (University of Colorado)
Jon Aurnou (UCLA)
Meredith Plumley (University of Colorado)





Conditions inside the Earth



The fluid outer core of the Earth is a rapidly-rotating, magnetised, turbulent environment.

- Probably driven by thermal or compositional convection.
- Dynamo generates field
  - long timescale process
  - There are also dynamics on shorter timescales (e.g. torsional oscillations, length of day variations, MAC waves)

- Timescales and balances play a key role in determining dynamics

Dimensional Equations

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + 2\rho(\mathbf{\Omega}\times\mathbf{u}) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} + \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}$$

Inertia **Coriolis** **Pressure** **Buoyancy** **Viscous** **Lorentz**

Momentum

$$\left(\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T\right) = \quad \kappa\nabla^2 T$$

Temperature

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{B}$$

Induction

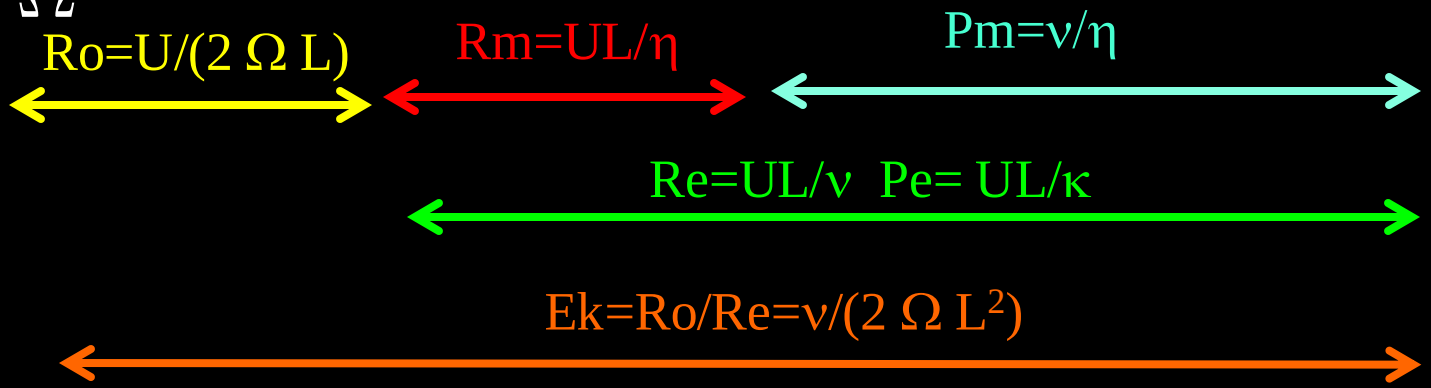
$$\nabla\cdot\mathbf{B} = \nabla\cdot\mathbf{u} = 0$$

Solenoidal/Incompressible

Timescales

- There is a large separation in timescales between different processes.

$$\frac{1}{\Omega} << t_{\mathrm{conv}} << \tau_{\mathrm{mag\ diff}} << T_{\mathrm{th/vis\ diff}}$$



  - Rotation: 1 day
  - Convection (global scale): ~10^2 years
  - Magnetic Diffusion (global scale): ~10^4 – 10^5 years
  - Thermal/Viscous Diffusion (global scale): ~ 10^9 years.
- Other timescales associated with waves and magnetic tension (6-100 years)

Balances

- The dominant balance in planets depends on scale (more later)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)+2\rho(\mathbf{\Omega}\times\mathbf{u})=-\nabla p+\rho\mathbf{g}+\mu\nabla^2\mathbf{u}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}$$

- In general pressure, Coriolis and Lorentz force contribute significantly to the leading order balance

$$2\rho(\mathbf{\Omega}\times\mathbf{u})\approx-\nabla p$$ Geostrophic Balance

$$2\rho(\mathbf{\Omega}\times\mathbf{u})\approx-\nabla p+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}$$ Magnetostrophic Balanc

See e.g. Nataf & Schaeffer (2015) (Earth) and Soderlund et al (2015), Aurnou & King (2017) (Planets) for a discussion of which balance occurs where.

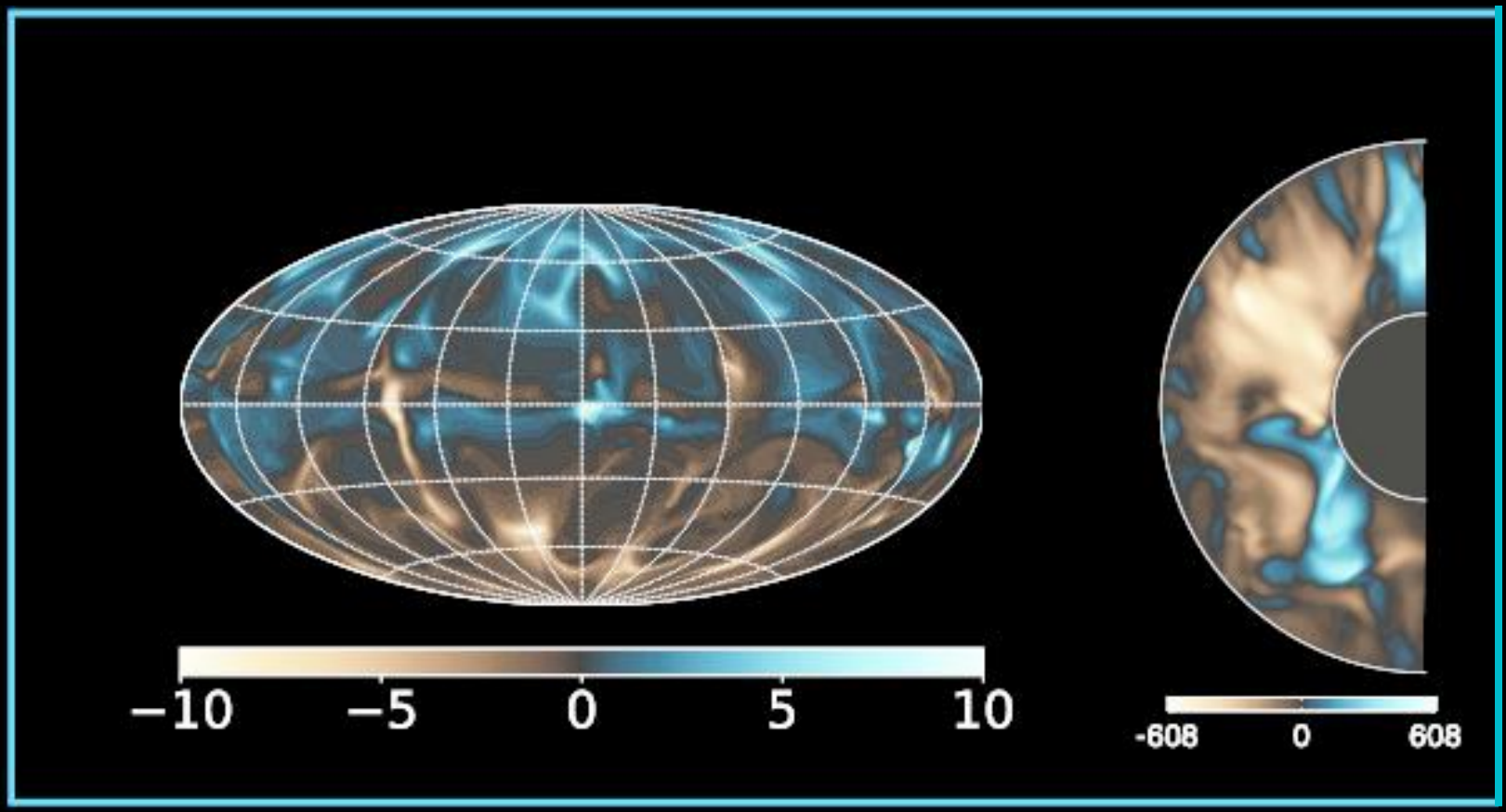
  - Flow driven by buoyancy

What do we know from numerical simulations?

- Unfortunately even heroic numerical calculations are way too viscous (Ekman number 10^{-7} instead of 10^{-15}) and some effects are viscously controlled (see e.g. Davidson 2014, Davidson & Ranjan 2015)
- And inertia tends to play too much of a role at large scales
- The induction equation *can* be solved at the correct parameters (unlike for the solar dynamo!)
- But the magnetic Prandtl number is way too big as well (may not be a problem given that other parameters are not correct Dormy (2016))

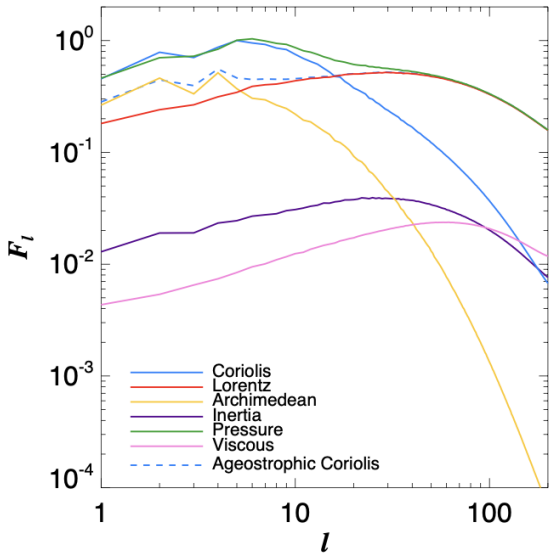
- Other approaches needed…
  - Asymptotics
  - Wave models (e.g. Davidson 2018)

Computational Dynamos



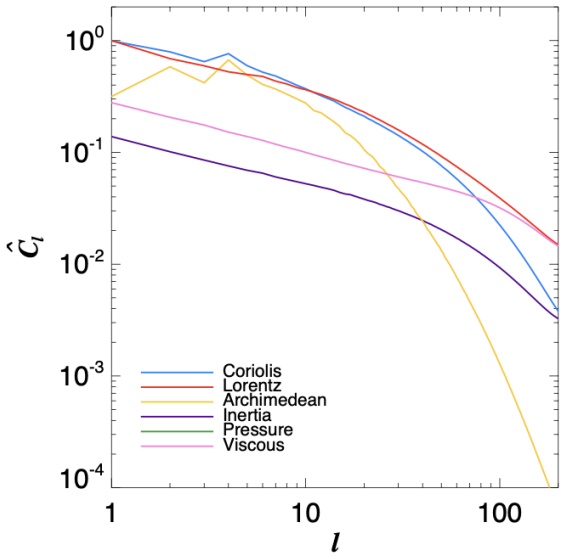
Schaeffer et al. 2017, Aubert et al. 2017, Aubert 2019, Teed & Dormy 2003

Computational Dynamos: You heard this yesterday!



Perform curl

Remove gradient parts



Teed & Dormy (2023)

Non-dimensional equations

(non-dimensionalised on typical velocity U and length l)

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \frac{1}{Ro}\hat{\boldsymbol{z}}\times\boldsymbol{u} = -Eu\,\nabla p + M\boldsymbol{B}\cdot\nabla\boldsymbol{B} + \Gamma\,\theta\hat{\boldsymbol{z}} + \frac{1}{Re}\nabla^2\boldsymbol{u},$$

Momentum

$$\partial_t\theta + \nabla\cdot(\boldsymbol{u}\theta) = \frac{1}{Pe}\nabla^2\theta,$$

Temperature

$$\partial_t\boldsymbol{B} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \frac{1}{Rm}\nabla^2\boldsymbol{B},$$

Induction

$$\nabla\cdot\boldsymbol{u} = 0, \quad \nabla\cdot\boldsymbol{B} = 0,$$

Solenoidal/Incompressible

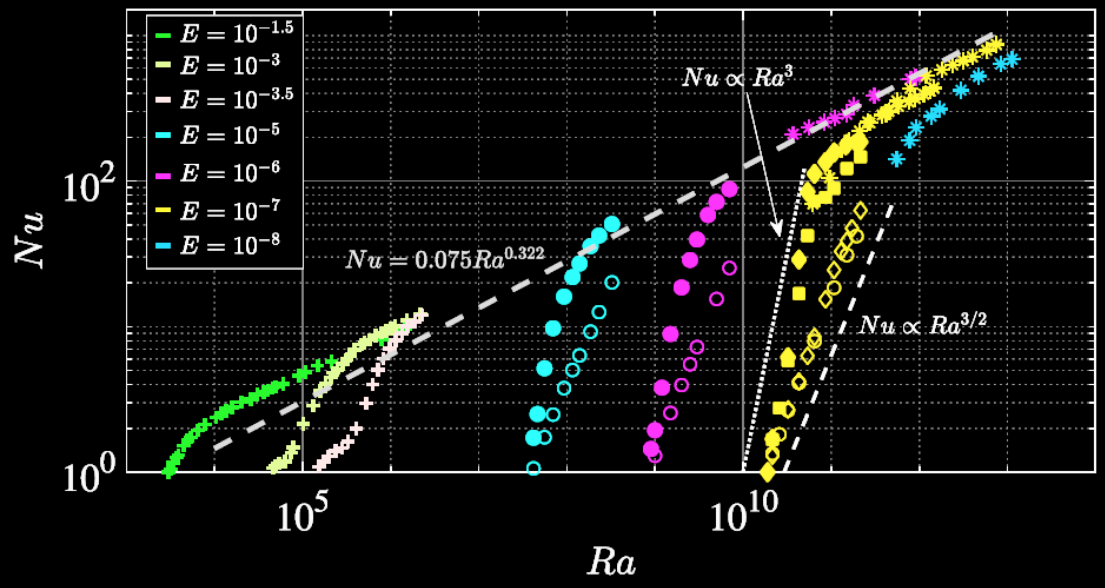
$$Ro = \frac{\mathscr{U}}{2\Omega\ell}, \quad \Gamma = \frac{g\alpha\Delta T\ell}{\mathscr{U}^2}, \quad Re = \frac{\mathscr{U}\ell}{\nu}, \quad Pe = \frac{\mathscr{U}\ell}{\kappa}, \quad Eu = \frac{\mathscr{P}}{\rho\mathscr{U}^2},$$

$$M = \frac{\mathscr{B}^2}{\rho\mu\mathscr{U}^2}, \quad Rm = \frac{\mathscr{U}\ell}{\eta}.$$

If ℓ is a local scale

Ro << 1 Γ >>1 Re, Pe >> 1 or ~ 1 M >> 1. Rm ~1 (or Rm << 1)

Addition of Rotation inhibits convection and prefers a small perpendicular lengthscale



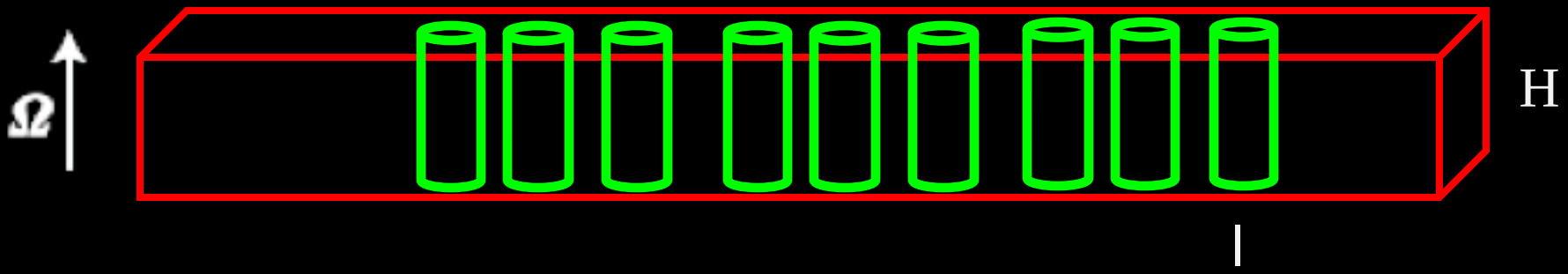
Rotation inhibits convection

- Taylor Proudman theorem
- Lab expts (+ signs)
- Numerical Expts (circles diamonds)
- Filled (no slip) open (stress-free)

See e.g. Chandrasekhar
Susanne's talk…

$$Ra_c \sim E^{-4/3} \qquad k_c \sim E^{-1/3}$$

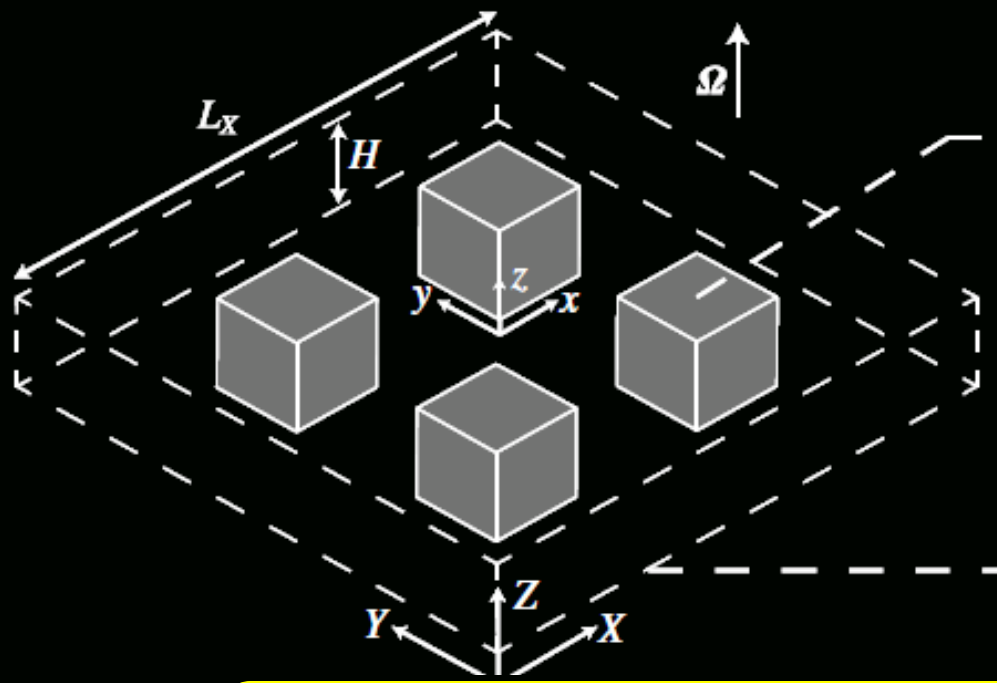
Asymptotic Models: not a new idea

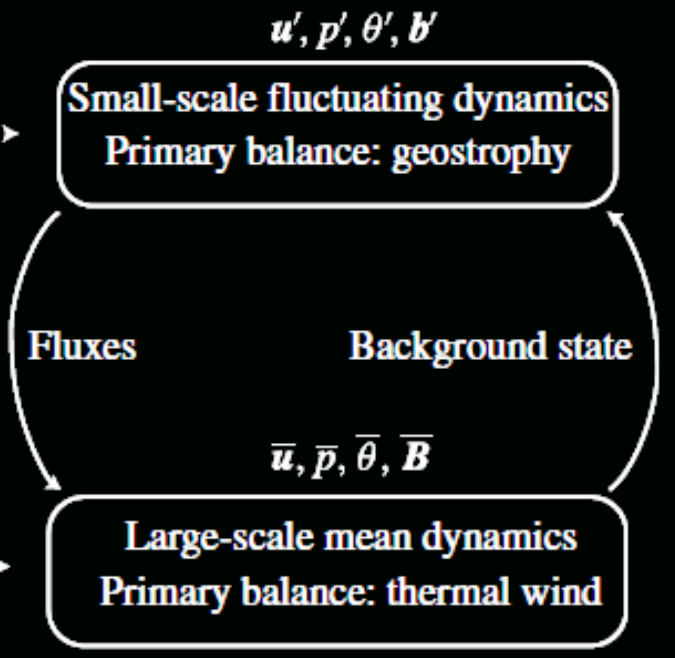


- Childress & Soward (1972), Soward (1974), Fautrelle & Childress (1982):
  - Laminar dynamo – near onset (weakly nonlinear, Re small)
  - Small parameter is Ekman number; Ek (based on depth of fluid layer)
  - Variables expanded in powers of $Ek^{1/6}$; spatially anisotropic. $l \ll H$
  - Single timescale
    - Applies to O(1) Pr, Pm
  - dynamical equation for large-scale horizontal magnetic field
    - driven by the mean electromotive force (emf) generated by the small-scale fluctuating velocity and magnetic fields
  - Three balances, in the governing equations based on the relative strength of the magnetic field
    - weak field (S74) → stable periodic nonlinear dynamos exist
    - intermediate field (FC82) → may be dynamically unstable to stronger fields
    - strong field

Reduced Model: Geostrophic Dynamo

Calkins et al. JFM, (2015)





$$A_X = A_Y = \frac{L_X}{\ell}, \quad A_Z = \frac{H}{\ell}, \quad A_\tau = \frac{t}{\tau}, \quad A_T = \frac{t}{T}.$$

$$\begin{aligned}
u(x, X, t, \tau, T) &= \overline{u}(X, \tau, T) + u'(x, X, t, \tau, T), \\
p(x, X, t, \tau, T) &= \overline{p}(X, \tau, T) + p'(x, X, t, \tau, T), \\
\theta(x, X, t, \tau, T) &= \overline{\theta}(X, \tau, T) + \theta'(x, X, t, \tau, T), \\
B(x, X, t, \tau, T) &= \overline{B}(X, \tau, T) + b'(x, X, t, \tau, T),
\end{aligned}$$

Mean Fluctuations

$$\partial_t \to \partial_t + \frac{1}{A_\tau}\partial_\tau + \frac{1}{A_T}\partial_T,$$

$$\nabla \to \nabla + \overline{\nabla},$$

$$\overline{\nabla} = \left(\frac{1}{A_X}\partial_X, \frac{1}{A_X}\partial_Y, \frac{1}{A_Z}\partial_Z\right)$$

Reduced Model: Geostrophic Dynamo

Calkins et al. JFM, (2015)

$$\left(\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\overline{\boldsymbol{u}}+\overline{\nabla}\cdot\left(\overline{\boldsymbol{u}}(\overline{\boldsymbol{u}})^{\mathrm{T}}\right)+\overline{\nabla}\cdot\overline{(\boldsymbol{u}'(\boldsymbol{u}')^{\mathrm{T}})}+\frac{1}{Ro}\hat{\boldsymbol{z}}\times\overline{\boldsymbol{u}}$$
$$=-Eu\overline{\nabla}\overline{p}+M\overline{\boldsymbol{F}}_L+\Gamma\,\overline{\theta}\hat{\boldsymbol{z}}+\frac{1}{Re}\overline{\nabla}^2\overline{\boldsymbol{u}},$$
$$\left(\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\overline{\theta}+\overline{\nabla}\cdot\left(\overline{\theta}\,\overline{\boldsymbol{u}}\right)+\overline{\nabla}\cdot\overline{(\theta'\boldsymbol{u}')}=\frac{1}{Pe}\overline{\nabla}^2\overline{\theta},$$
$$\left(\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\overline{\boldsymbol{B}}=\overline{\nabla}\times(\overline{\boldsymbol{u}}\times\overline{\boldsymbol{B}})+\overline{\nabla}\times\overline{(\boldsymbol{u}'\times\boldsymbol{b}')}+\frac{1}{Rm}\overline{\nabla}^2\overline{\boldsymbol{B}},$$
$$\overline{\nabla}\cdot\overline{\boldsymbol{u}}=0,\quad\overline{\nabla}\cdot\overline{\boldsymbol{B}}=0,$$

Mean Equations

$$\left(\partial_t+\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\boldsymbol{u}'+\nabla\cdot(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})+\overline{\nabla}\cdot(\boldsymbol{u}'\overline{\boldsymbol{u}}^{\mathrm{T}})$$
$$+\overline{\nabla}\cdot(\boldsymbol{u}(\boldsymbol{u}')^{\mathrm{T}})-\overline{\nabla}\cdot\overline{(\boldsymbol{u}'(\boldsymbol{u}')^{\mathrm{T}})}+\frac{1}{Ro}\hat{\boldsymbol{z}}\times\boldsymbol{u}'$$
$$=-Eu\,\left(\overline{\nabla}+\nabla\right)p'+M\boldsymbol{F}_L'+\Gamma\,\theta'\hat{\boldsymbol{z}}+\frac{1}{Re}(\nabla^2+\overline{\nabla}^2)\boldsymbol{u}',$$
$$\left(\partial_t+\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\theta'+\boldsymbol{u}'\cdot\overline{\nabla}\,\overline{\theta}+\overline{\boldsymbol{u}}\cdot\nabla\theta'+\overline{\nabla}\cdot\left(\boldsymbol{u}'\theta'\right)-\overline{\nabla}\cdot\overline{(\boldsymbol{u}'\theta')}$$
$$=\frac{1}{Pe}\left(\nabla^2+\overline{\nabla}^2\right)\theta',$$
$$\left(\partial_t+\frac{1}{A_\tau}\partial_\tau+\frac{1}{A_T}\partial_T\right)\boldsymbol{b}'=\overline{\nabla}\times(\overline{\boldsymbol{u}}\times\boldsymbol{b}')+\overline{\nabla}\times(\boldsymbol{u}'\times\overline{\boldsymbol{B}})+\overline{\nabla}\times(\boldsymbol{u}'\times\boldsymbol{b}')$$
$$-\overline{\nabla}\times\overline{(\boldsymbol{u}'\times\boldsymbol{b}')}+\nabla\times(\overline{\boldsymbol{u}}\times\boldsymbol{b}')+\nabla\times(\boldsymbol{u}'\times\overline{\boldsymbol{B}})$$
$$+\nabla\times(\boldsymbol{u}'\times\boldsymbol{b}')+\frac{1}{Rm}\nabla^2+\overline{\nabla}^2\boldsymbol{b}',$$
$$\left(\overline{\nabla}+\nabla\right)\cdot\boldsymbol{u}'=0,\quad\left(\overline{\nabla}+\nabla\right)\cdot\boldsymbol{b}'=0,$$

Fluctuating Equations

Scalings

$$\epsilon \sim Ro \sim Ek^{1/3}$$

$$A_\tau \sim \epsilon^{-3/2}; A_T \sim \epsilon^{-2}$$

$$A_X \sim \epsilon^{-3/2}; A_Z \sim \epsilon^{-1}$$

$$M \sim \epsilon^{-1/2}; Rm \sim \epsilon^{1/2}; Re \sim O(1)$$

$$Pm \sim O(\epsilon^{1/2}); Pr \sim O(1)$$

Note: Distinguished limit: 1 >> Pm >> Ek cf Dormy (2014)

Expand:

$$\boldsymbol{u} = \overline{\boldsymbol{u}}_0 + \boldsymbol{u}'_0 + \epsilon^{1/2}\left(\overline{\boldsymbol{u}}_{1/2} + \boldsymbol{u}'_{1/2}\right) + \epsilon\left(\overline{\boldsymbol{u}}_1 + \boldsymbol{u}'_1\right) + \cdots$$

etc, equate orders, apply solvability conditions …

Scaling Comparison

Parameter	Present work	Soward	Fautrelle & Childress
Eu	ϵ^{-2}	ϵ^{-2}	ϵ^{-2}
Re	$O(1)$	$\epsilon^{1/2}$	$\epsilon^{1/2}$
M	$\epsilon^{-1/2}, O(1)$	$\epsilon^{3/2}$	$\epsilon^{1/2}$
Γ	$\tilde{\Gamma}\epsilon^{-1}$	$\tilde{\Gamma}\epsilon^{-1}$	$\tilde{\Gamma}\epsilon^{-1}$
Pe	$O(1)$	$\epsilon^{1/2}$	$\epsilon^{1/2}$
Rm	$\epsilon^{1/2}, O(1)$	$\epsilon^{1/2}$	$\epsilon^{1/2}$
Pr	$O(1)$	$O(1)$	$O(1)$
Pm	$\epsilon^{1/2}, O(1)$	$O(1)$	$O(1)$
A_X	$\epsilon^{-3/2}$	—	—
A_Z	ϵ^{-1}	ϵ^{-1}	ϵ^{-1}
A_τ	$\epsilon^{-3/2}$	$\epsilon^{-3/2}$	$\epsilon^{-3/2}$
A_T	ϵ^{-2}	—	—

All numbers based on small convective scale

Reduced Equations Set

$$\overline{\boldsymbol{u}}_{1/2} = -\overline{\nabla} \times \overline{\Psi}_{1/2}\hat{\boldsymbol{z}},$$

$$\partial_Z \overline{\Psi}_{1/2} = \frac{\widetilde{Ra}}{Pr}\overline{\theta}_0,$$

Thermal wind

$$\partial_T \overline{\theta}_0 + \overline{\boldsymbol{u}}_{1/2} \cdot \nabla_X \overline{\theta}_0 + \partial_Z \overline{\left(w_0' \theta_1'\right)} = \frac{1}{Pr}\partial_Z^2 \overline{\theta}_0,$$

Secularity condition

$$\partial_\tau \overline{\boldsymbol{B}}_0^{\perp} = \hat{\boldsymbol{z}} \times \partial_Z \overline{\left(\boldsymbol{u}_0' \times \langle \boldsymbol{b}_{1/2}' \rangle\right)} + \frac{1}{\widetilde{Pm}}\partial_Z^2 \overline{\boldsymbol{B}}_0^{\perp},$$

$O(\varepsilon^{3/2})$ hor induction

$$\partial_\tau \overline{B}_{1/2}^{(z)} = \hat{\boldsymbol{y}} \cdot \partial_X \overline{\left(\boldsymbol{u}_0' \times \langle \boldsymbol{b}_{1/2}' \rangle\right)} - \hat{\boldsymbol{x}} \cdot \partial_Y \overline{\left(\boldsymbol{u}_0' \times \langle \boldsymbol{b}_{1/2}' \rangle\right)} + \frac{1}{\widetilde{Pm}}\partial_Z^2 \overline{B}_{1/2}^{(z)},$$

Leading order ver induction

$$\partial_X \overline{B}_0^{(x)} + \partial_Y \overline{B}_0^{(y)} + \partial_Z \overline{B}_{1/2}^{(z)} = 0,$$

$$\partial_t \nabla_\perp^2 \psi_0' + \mathrm{J}(\psi_0', \nabla_\perp^2 \psi_0') - \partial_Z w_0' = \hat{\boldsymbol{z}} \cdot \nabla \times \left(\overline{\boldsymbol{B}}_0 \cdot \nabla_\perp \langle \boldsymbol{b}_{1/2}' \rangle\right) + \nabla_\perp^4 \psi_0',$$

$$\partial_t w_0' + \mathrm{J}(\psi_0', w_0') + \partial_Z \psi_0' = \frac{\widetilde{Ra}}{Pr}\langle \theta_1' \rangle + \overline{\boldsymbol{B}}_0 \cdot \nabla_\perp \langle b_{1/2}'^{(z)} \rangle + \nabla_\perp^2 w_0',$$

Prognostic momentum equations

$$\partial_t \langle \theta_1' \rangle + \mathrm{J}(\psi_0', \langle \theta_1' \rangle) + w_0' \partial_Z \overline{\theta}_0 = \frac{1}{Pr}\nabla_\perp^2 \langle \theta_1' \rangle,$$

$O(\varepsilon)$ fluctuating temp

$$0 = \overline{\boldsymbol{B}}_0^{\perp} \cdot \nabla_\perp \boldsymbol{u}_0' + \frac{1}{\widetilde{Pm}}\nabla_\perp^2 \langle \boldsymbol{b}_{1/2}' \rangle,$$

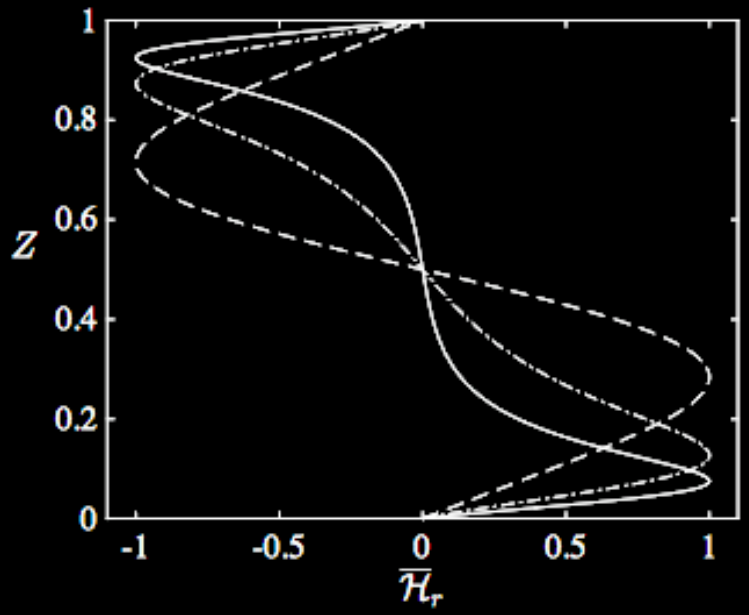
Fluctuating induction

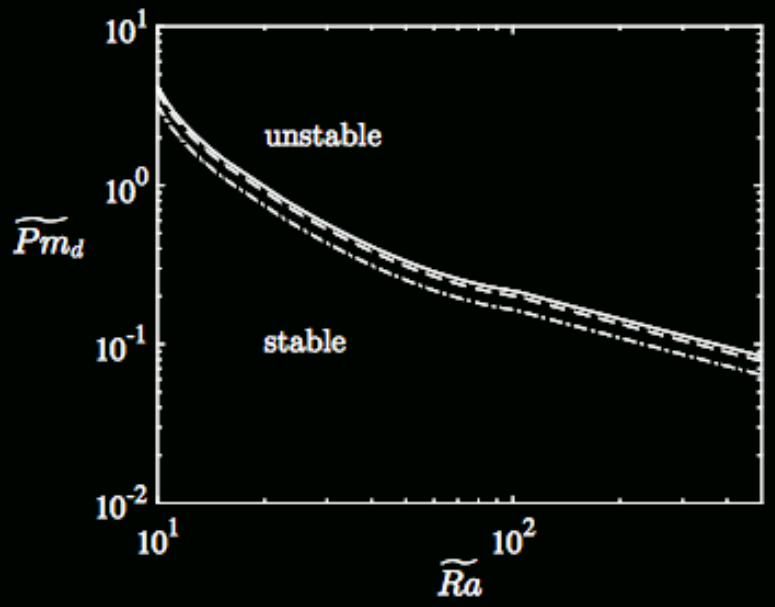
$$\partial_x \langle b_{1/2}'^{(x)} \rangle + \partial_y \langle b_{1/2}'^{(y)} \rangle = 0.$$

Kinematic Dynamo Action

- Can such rapidly-rotating small-scale flows be dynamos?
  - Ignore back-reaction of Lorentz force and calculate dynamo threshold (kinematic problem)
  - Impose planform/single mode solutions…
    - Squares, hexagons, triangles patchwork quilt etc
  - Consistently solve for flow, optimise Nu.
  - Examine dynamo threshold

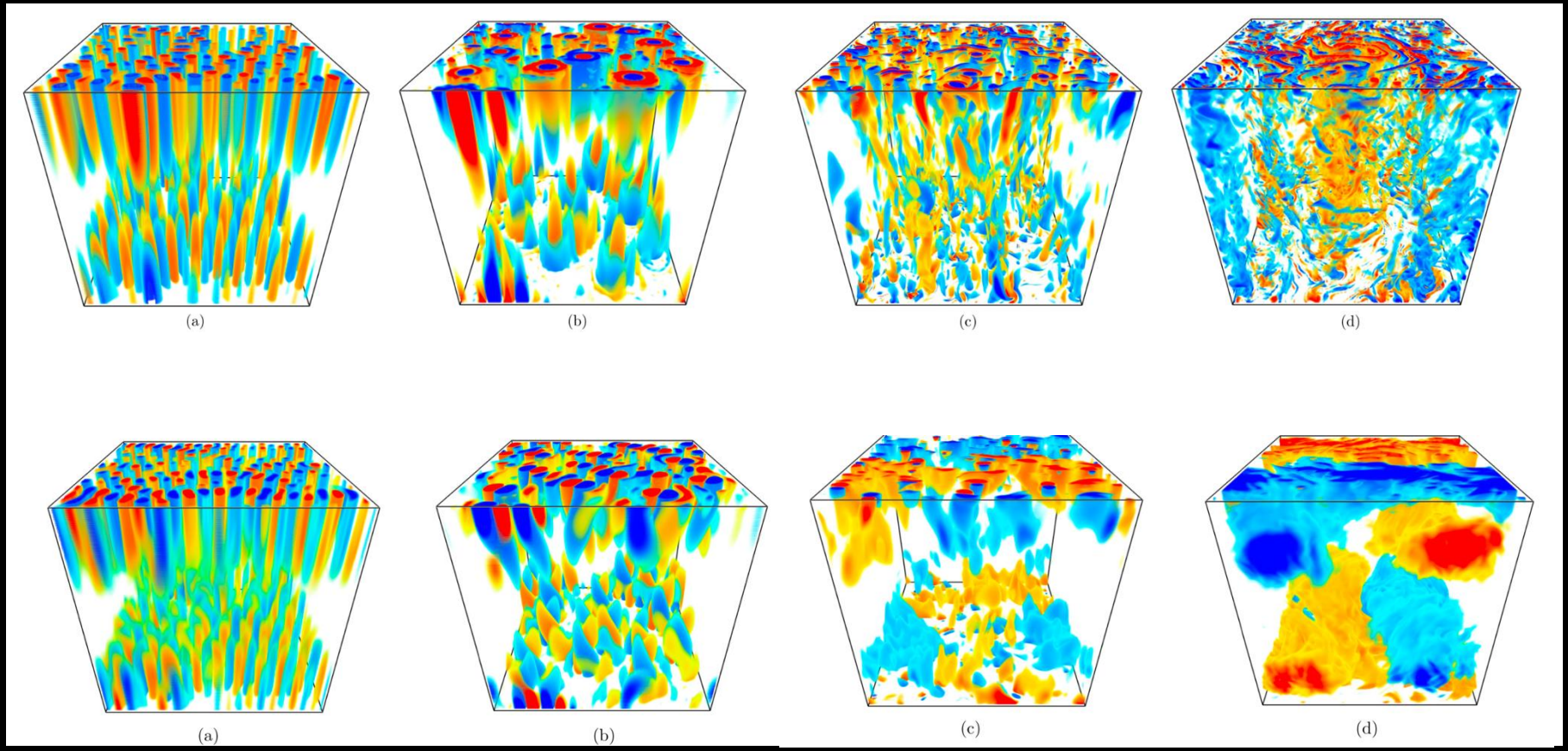
Calkins et al PRE (2016a,b)





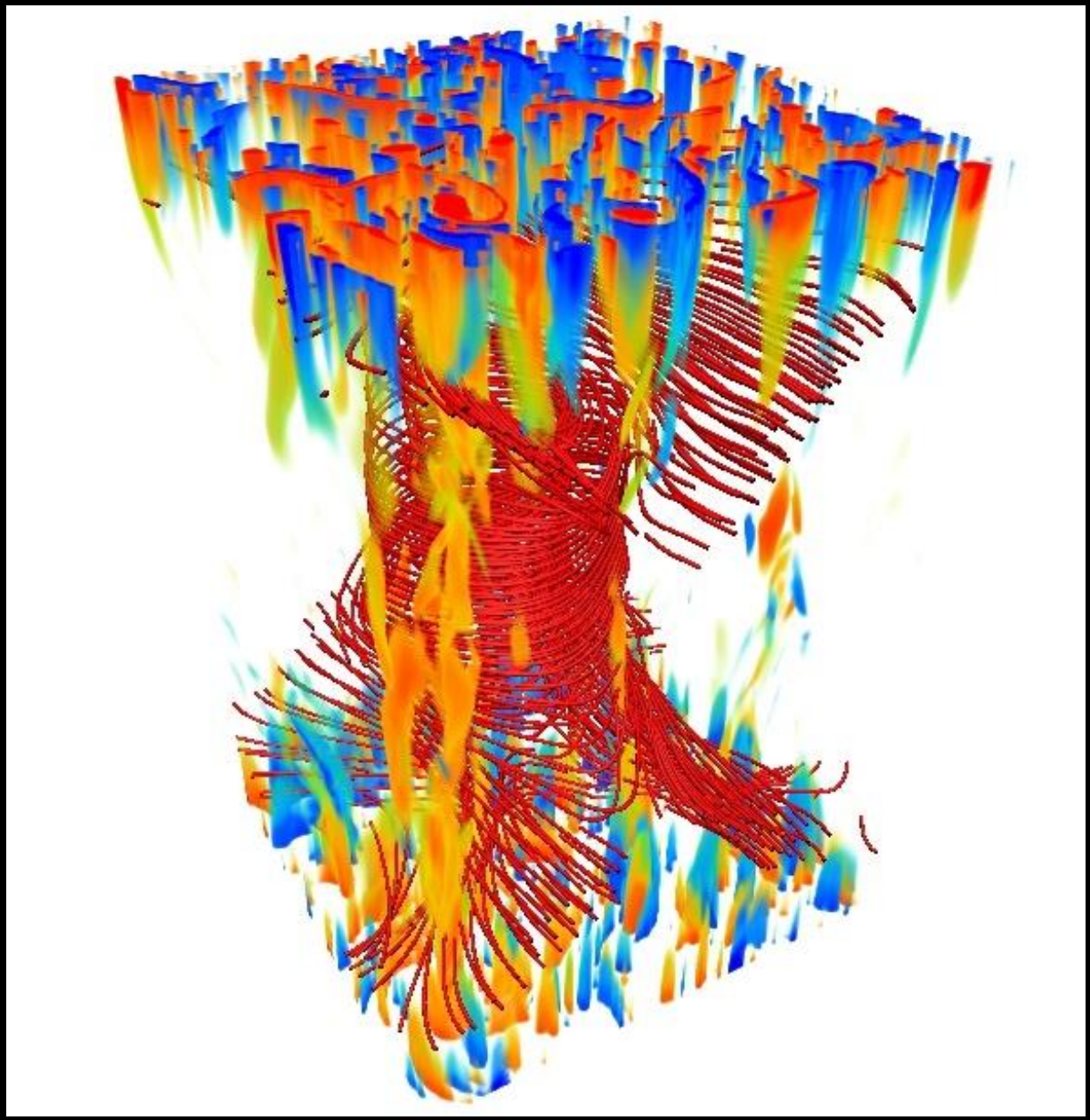
Kinematic Dynamo Action

- Or solve for the turbulent flow Calkins et al PRE (2016a,b)



- These are exceptionally good dynamos
  - Despite the flow being at very low "classical Rm"
- Geodynamo has no problem recovering kinematically from a reversal (or anything else!)

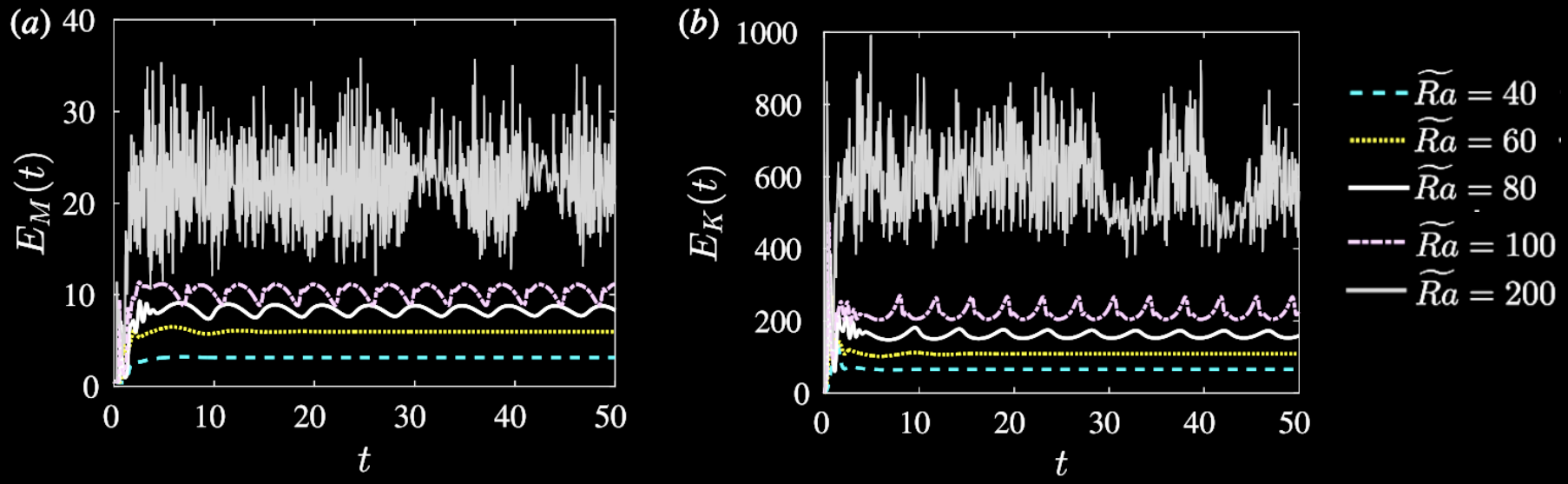
Kinematic Dynamo Action



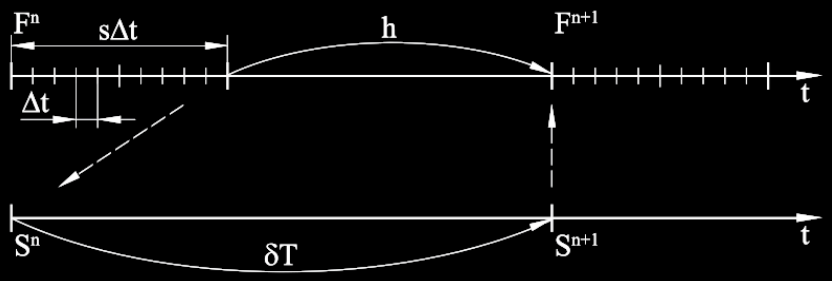
Nonlinear Asymptotic Dynamo Action

- Single mode solutions

Plumley et al JPP (2018)



- Saturate with ME>>KE
  - Note figures show unscaled energies!
- Multi-scale (in time) integration methods may be needed to integrate these equations
  - e.g. Parallel in time
  - HMM-type



Tretiak et al CPC (2022)

Interim Summary: Asymptotic Dynamo Model

- Can be constructed using the method of mulitiple scales
  - 2 spatial scale and 3 temporal scales
  - "Correct" balances
- The key asymptotic parameter is the ratio of the horizontal to vertical lengthscale.
  - Naturally small if Rossby number is small.
  - Can be tied to the Ekman number (not necessary though).
- Small-scale flows have no problems being large-scale dynamos!
  - No lower limits can be put on field strength to get dynamo started again!
- Multi-scale timestepping methods can be used for nonlinear single-mode solutions.

- Full equations need to be integrated!
- What about tying to other lengthscales?
- How do you put these into a sphere?

Summary of Course

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}$$

Stretching **Diffusion**

- The solar dynamo is hard to solve because the induction equation is hard to solve at high Rm…(and turbulence!)

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + 2\rho(\mathbf{\Omega} \times \mathbf{u}) = -\nabla p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{u} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

- The geodynamo is hard because the momentum equation is hard to solve in the rapidly rotating limit…
  - Balanced flows and fast waves (cf David's talk)

If you have enjoyed this talk…

Living Rev. Sol. Phys. (2017) 14:4
https://doi.org/10.1007/s41116-017-0007-8

REVIEW ARTICLE

Magnetism, dynamo action and the solar-stellar connection

Allan Sacha Brun[1] · Matthew K. Browning[2]

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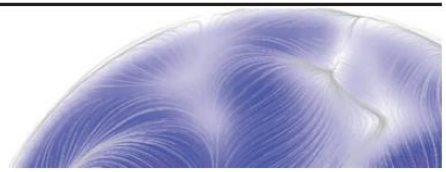
LECTURE NOTES
Dynamo theories

François Rincon[1,2,†]

[1]Université de Toulouse, UPS-OMP, IRAP: Toulouse, France
[2]CNRS, IRAP, 14 avenue Edouard Belin, F-31400 Toulouse, France

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The turbulent dynamo

S.M. Tobias†

Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK

