Inviscid (anomalous) dissipation

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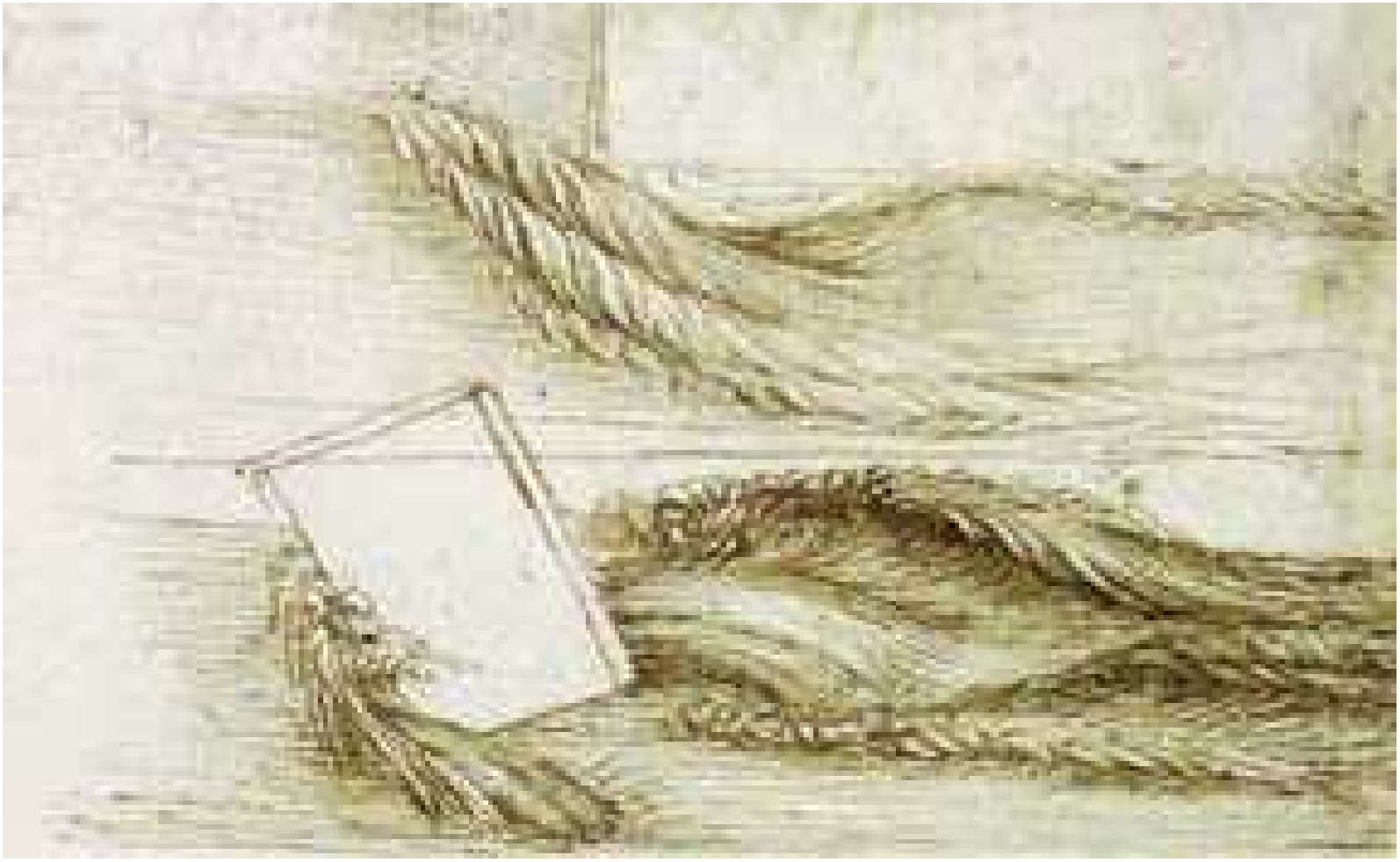
Cargese talk

April, 2025

Reynolds number $\sim 10^{10}$



Da Vinci draws water (Reynolds number = 10^6?)



Hokusai paints a wave



Euler versus Navier-Stokes

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} + \boldsymbol{\nabla} p = \nu \nabla^2 \mathbf{u},$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0.$$

For Euler,

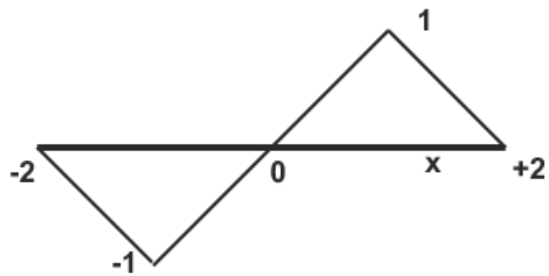
$$\frac{dE}{dt} = -\boldsymbol{\nabla} \cdot \mathbf{Q}, \quad E = \frac{1}{2} u^2, \; \mathbf{Q} = \mathbf{u}(p + \frac{1}{2} u^2), \; \nabla^2 p = \frac{\partial u_j}{\partial u_i} \frac{\partial u_i}{\partial u_j}.$$

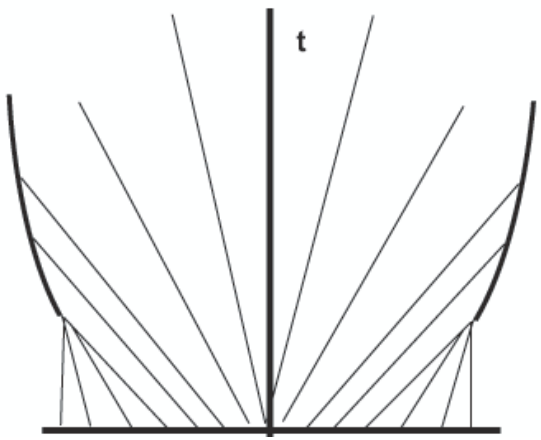
Some curious results

- A solution of Euler's equations need not be a limit of a Navier-Stokes flow as $\nu \longrightarrow 0$.
- To maintain turbulence as $\nu \longrightarrow 0$, you need to keep stirring (the dissipation anomaly).
- There exist unforced (weak) solutions of Euler's equations, with continuous velocity, whose total kinetic energy varies arbitrarily with time (De Lellis and Székelyhidi 2013).

An analog problem: The Burgers N-wave

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$





Remarkable results

- For our Burgers-Euler problem there exists a weak solution: an N-wave with shocks at $x = \xi_{\pm}(t) = \pm\sqrt{2(1+t)}$.
- The jump in u across each shock, $u_+ - u_-$, decays as $\sqrt{2/(1+t)}$. Once the N-wave is established we have $u = x/(1+t), \xi_- < x < \xi_+$. The total energy $E(t)$ decays as $\sqrt{2}/3(1+t)^{-3/2}$.
- Under $t, u \to -t, -u$, Burgers-Euler possesses another weak solution, which decreases entropy.
- The dissipation of energy in the Burgers-viscous N-wave may be computed through the shock using
  $u = \frac{u_- + u_+}{2} - \frac{u_- - u_+}{2} \tanh\left[\frac{x(u_- - u_+)}{4\nu}\right]$.
  $2\lim_{\nu\to 0} \nu \int_{-\infty}^{+\infty} u_x^2 dx = -\sqrt{2}/3(1+t)^{-3/2}$.

The dissipation inequality for Burgers-Euler

$$\frac{\partial E}{\partial t} + \frac{\partial Q}{\partial x} \leq 0, \quad E = \frac{1}{2}u^2,\ Q = \frac{1}{3}u^3.$$

This selects the physical shock in the weak solution.

Arnold-Euler equations in $\mathbb{R}^3$

$\mathcal{D}$=group volume preserving transformations (diffeomorphisms) on $\mathbb{R}^3$, i.e all of the smooth divergence-free velocity fields in $\mathbb{R}^3$. For any such field $\mathbf{u}$ the Lagrangian flow

$$\mathbf{x}(t, \mathbf{a}) : \partial_t \mathbf{x} = \mathbf{u}(t, \mathbf{x})$$

satisfies Euler's equations provided that

$$\partial_t^2 \mathbf{x} = -\boldsymbol{\nabla} p, \quad \det \partial_{\mathbf{a}} \mathbf{x} = 1$$

for some pressure field p. This flow is a geodesic on the manifold of measure=preserving diffeomorphisms, , with metric

$$\int \partial_t(\mathbf{x})^2 dt$$

Isometric embeddings in $\mathbb{R}^3$

Stir an inviscid fluid in a random way. We get turbulence and arbitrarily small scales. Such a flow is an isometric embedding of an Arnold-Euler flow in $\mathbb{R}^3$, but a complicated one.
Simpler problem: Embed the flat 2-torus in $\mathbb{R}^3$.



Explicit construction (Borelli et al. 2012)



Dissipation in a single bad region

Assume the Euler flow is smooth outside the Lagrangian ball B_δ of radius δ, observed at the origin, and that there is no source of energy from infinity.

$$dE/dt = -\int_{|\mathbf{x}|=\delta} \mathbf{Q}\cdot\mathbf{n}\,dS, \quad \mathbf{Q} = \mathbf{u}(\frac{1}{2}u^2+p), \quad E = \frac{1}{2}\int_{|\mathbf{x}|>\delta} u^2\,dV.$$

If $u \sim O(\delta^{1/3})$, $p \sim O(\delta^{2/3})$, then

$$\frac{1}{\mathrm{vol}(B_\delta)}\,dE/dt \sim O(1).$$

(Recall $|u(\mathbf{x}) - u(\mathbf{y})| \sim |\mathbf{x}-\mathbf{y}|^{1/3}$ in the Richardson-Kolmogorov cascade.) Can bad regions occur almost everywhere?

Onsager's conjecture

$\mathbf{u} \in C^{0,\alpha}$, a Hölder continuous vector field with exponent α, if

$$\sup_{\mathbf{x} \neq \mathbf{y}} \left[\frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^{\alpha}} \right] < \infty.$$

Theorem

Suppose $\mathbf{u} \in C^{0,\alpha}$, *is a weak solution of Euler's equations which is differentiable in time. Then, if* $\alpha > 1/3$, *total kinetic energy is conserved in time.*

Proof(Constantin, E, Titi)

The weak solution $\mathbf{u}$ satisfies

$$\int \mathbf{u} \cdot (\mathbf{v}_t + \mathbf{u} \cdot \nabla \mathbf{v}) d\mathbf{x} = 0. \tag{1}$$

for any smooth divergence-free test vector $\mathbf{v}$ at any time. Let

$$u^\epsilon(\mathbf{x}, t) = M[\mathbf{u}] = \epsilon^{-3} \int \phi((\mathbf{x} - \mathbf{y})/\epsilon)\mathbf{u}(\mathbf{y}, t) d\mathbf{y}, \quad \mathbf{v} = M[\mathbf{u}^\epsilon].$$

Then $\mathbf{u} \in C^{0,\alpha}$ implies $|\mathbf{u} - \mathbf{u}^\epsilon| \leq C\epsilon^\alpha$ and (1) yields

$$\Big| \frac{d}{dt} \int |\mathbf{u}^\epsilon|^2 d\mathbf{x} \Big| = 2 \Big| \int (u_i u_j)^\epsilon(\mathbf{x}, t) \frac{\partial u_i^\epsilon}{\partial x_j}(\mathbf{x}, t) d\mathbf{x} \Big| \leq C_1 \epsilon^{3\alpha - 1}.$$

Duchon and Robert(2000)

$$\frac{d}{dt}(\frac{1}{2}|\mathbf{u}^2|) + \boldsymbol{\nabla} \cdot \left[(\frac{1}{2}|\mathbf{u}|^2 + p)\mathbf{u}\right] = -D(\mathbf{u})$$

$$D(\mathbf{u}) = \lim_{\epsilon \longrightarrow 0} \frac{1}{4\epsilon^3} \int \boldsymbol{\nabla}\phi(\mathbf{r}/\epsilon) \cdot \nabla\mathbf{u}(\mathbf{r})|\nabla\mathbf{u}(\mathbf{r})|^2 d\mathbf{r}$$

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(x)$$

Turbulent la la land

- The "bad" structures, which for Euler flow replace the shock waves of Burgers-Euler, are understood and are known to be inviscid limits of Navier-Stokes flows.
- These structures explain the departures of the cascade from strict Kolmogorov scaling.
- The temporal evolution of these structures has a random element which we understand.
- Using statistical mechanics we may compute the free decay of inviscid turbulence as well as the energy needed to maintain it in a statistically steady state.

Turbulent embedding in one dimension

Our "fluid" will be a smooth closed curve $\mathcal{C}$ in $\mathbb{R}^3$, given by $\mathbf{f}(t), 0 \le t \le 2\pi$. "Energy" will equal line length. The short embedding is to $\rho\mathbf{f}(t) = \mathbf{f}_0(t),\ 0 < \rho < 1$. To restore the missing length and obtain an isometric embedding we set

$$\mathbf{f}(t) = \mathbf{f}_0(0) + \int_0^t |\mathbf{f}'(u)|\mathbf{H}(u, Nu)du, \quad N \gg 1$$

where $\mathbf{H}$ will introduce a helical structure:

$$\mathbf{H}(t,u) = (\alpha^2 + \beta)^{-1/2}\big[\alpha\mathbf{t} + \beta\big(\cos(u)\mathbf{n} + \sin(u)\mathbf{b}\big)\big]$$

where $\mathbf{t}(t) = \mathbf{f}_0'(t)/|\mathbf{f}_0'(t)|$ and $\mathbf{n}(t), \mathbf{b}(t)$ are the smooth normal and binormal unit vectors of $\mathbf{f}_0(t)$. Isometry is set by requiring

$$2\pi\mathbf{f}_0'(t) = \int_0^{2\pi} |\mathbf{f}'(t)|\mathbf{H}(t,u)du \ \Rightarrow \rho = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$$

$\mathbf{f}$ CAN BE AS CLOSE AS WE LIKE TO $\rho\mathbf{f}_0$

Let

$$\mathbf{h}(t) = |\mathbf{f}'(t)|\mathbf{H}(t,u)$$

and recall

$$\mathbf{H}(t,u) = (\alpha^2+\beta^2)^{-1/2}[\alpha\mathbf{t}(t) + \beta(\cos(u)\mathbf{n}(t) + \sin(u)\mathbf{b}(t)]$$

Then

$$\| \mathbf{f} - \mathbf{f}_0 \| = \sup_{0\le t\le 2\pi} |\mathbf{f}(t) - \mathbf{f}_0(t)| \le \frac{1}{N} \| \frac{\partial \mathbf{h}}{\partial t} \| .$$

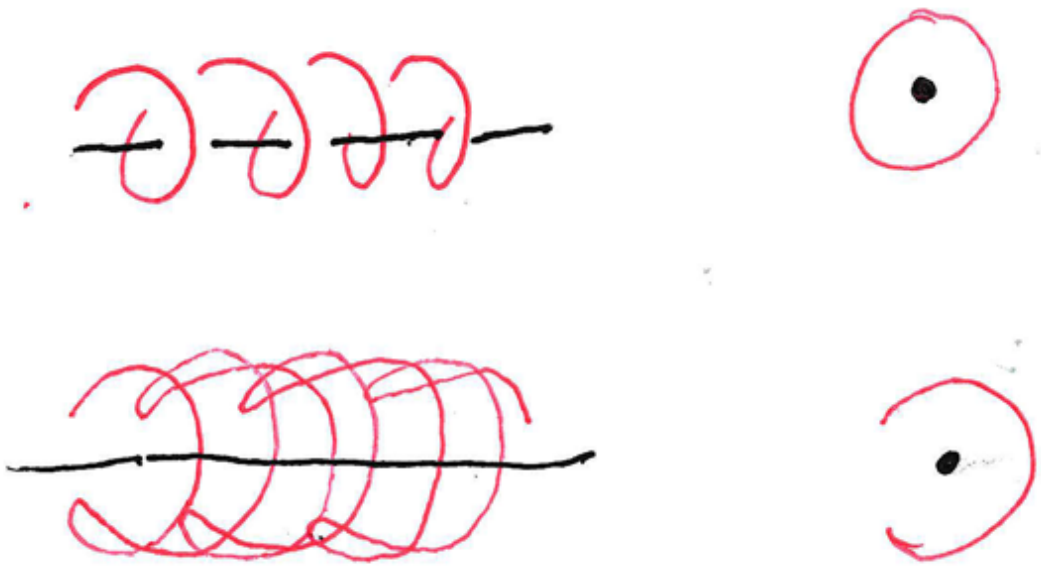
Note that $\mathbf{n}(t), \mathbf{b}(t)$ are unit vectors on $\mathbf{f}_0$. So $|\partial\mathbf{h}/\partial t|$ doesn't see the fast oscillations.

Iteration: Helices on helices

Start with the smooth closed curve $\mathcal{C}_0$ of length L. Short by the factor ρ and deform with helices which are a function of Nt, as for the previous $\mathbf{f}_0 \longrightarrow \mathbf{f}$. The result is $\mathcal{C}_1$ of length L having the first set of helical perturbations. Now short $\mathcal{C}_1$ by ρ and repeat with helical perturbations depending on $N^2 t$. The next iterations involves helices which vary as $N^3 t$, etc. At step M the curve lies within a ball of radius $O(\rho^M)$. This is a C^1 embedding although control is lost of all higher derivatives, and in the limit of large M the curve converges to a point.

Why is this called "convex integration"?

Integration means that given $\mathbf{f}_0'$ we find $\mathbf{f}_0$. Convex integration is intended to provide many nearby integrals as we add stuff like helices, giving many $\mathbf{f}'$ close to $\mathbf{f}_0'$. But we do not need to use helices. General idea: restrict circuits to an open set $\mathcal{C}$ whose convex hull contains $\mathbf{f}_0'$. THis vastly expands the scope of the convex integration process.



Dissipative Euler flows: The work of de Lellis and Székelyhidi

Given a smooth positive function $e(t)$, $t > 0$, there exists for $t > 0$ a weak solution of Euler's equations with total kinetic energy $e(t)$. !!! The (very technical) proof utilizes a version of convex integration. Seek a sequence $\mathbf{v}_q, p_q, \mathring{\mathbf{R}}_q$ satisfying the Euler-Reynolds system

$$\partial_t \mathbf{v}_q + \mathbf{v}_q \cdot \boldsymbol{\nabla} \mathbf{v}_q + \boldsymbol{\nabla} p_q = \boldsymbol{\nabla} \cdot \mathring{\mathbf{R}}_q, \ \boldsymbol{\nabla} \cdot \mathbf{v}_q = 0$$

such that $\mathring{\mathbf{R}}_q \to 0$, $\frac{1}{2} \int |\mathbf{v}_q|^2 d\mathbf{x} \to e(t)$. The convergence of $\mathbf{v}_q$ is to a Hölder-continuous function with Hölder exponent $\alpha < 1/3$.

Main ideas behind construction

Set $\mathbf{v}_{q+1} = \mathbf{v}_q + w_{q+1}$ where $\mathbf{v}_q$ has spatial and temporal oscillations $\sim \lambda_q$ while w_{q+1} oscillates lile $\lambda_{q+1} \gg \lambda_q$. Main structural element of w_{q+1} is linear combination of Beltrami waves $\mathbf{W_k}(\mathbf{x})$.

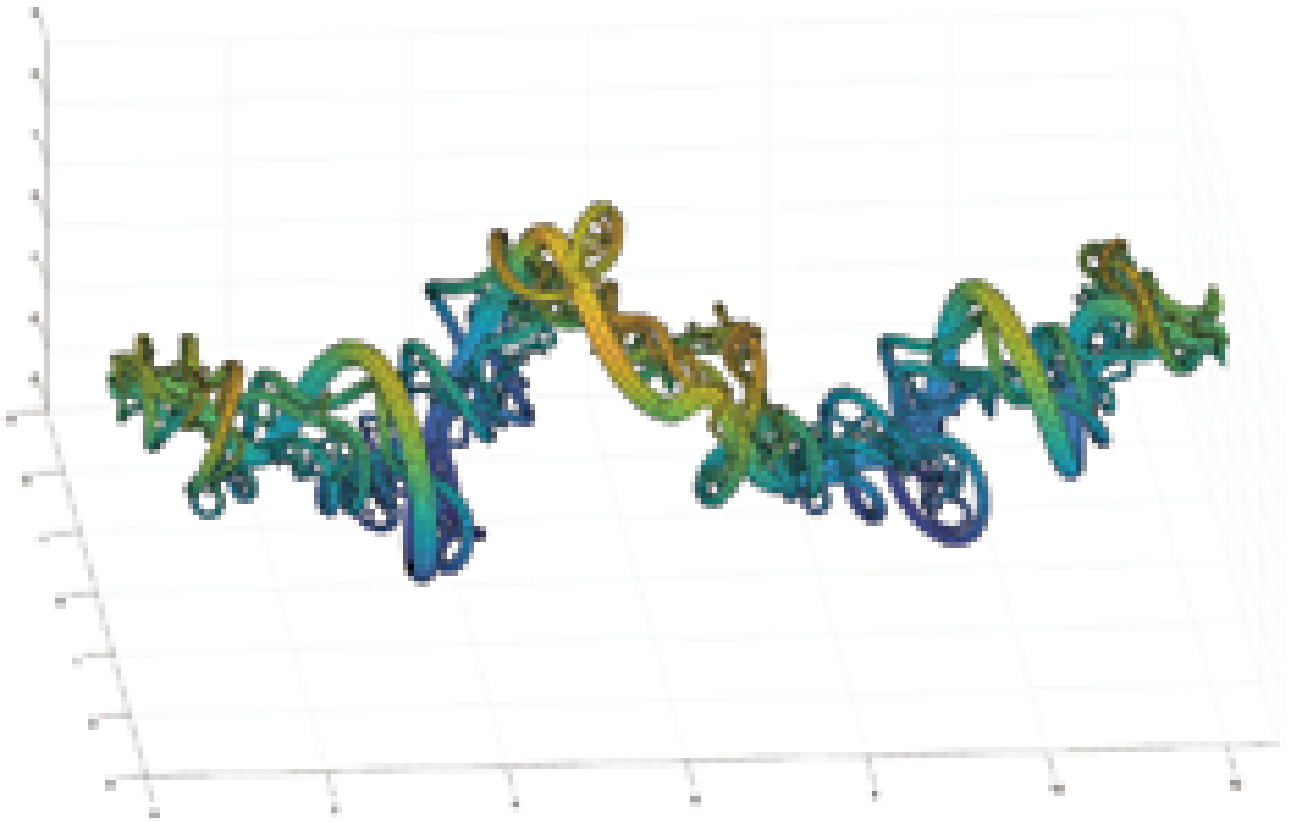
$$\mathbf{W_k} \times (\boldsymbol{\nabla} \times \mathbf{W_k}) = 0, \ \boldsymbol{\nabla} \cdot \mathbf{W_k} = 0$$

$$\mathbf{W_k} = \left(\mathbf{A_k} + i\frac{\mathbf{k}}{|\mathbf{k}|} \times \mathbf{A_k}\right) e^{\mathbf{k}\cdot\mathbf{x}\lambda_{q+1}}$$

With λ_q^{-1} a length scale, the velocity scale $\sim \lambda_q^{-1/3}$ and so the oscillations reflect Kolmogorov scaling. Main problem: the extremely large λ_q construct a sequence $\mathbf{v}_q$ with widely separated convolutions, very different from turbulence. Also not clear what physical structures of Beltrami waves are needed to insure decay of energy.

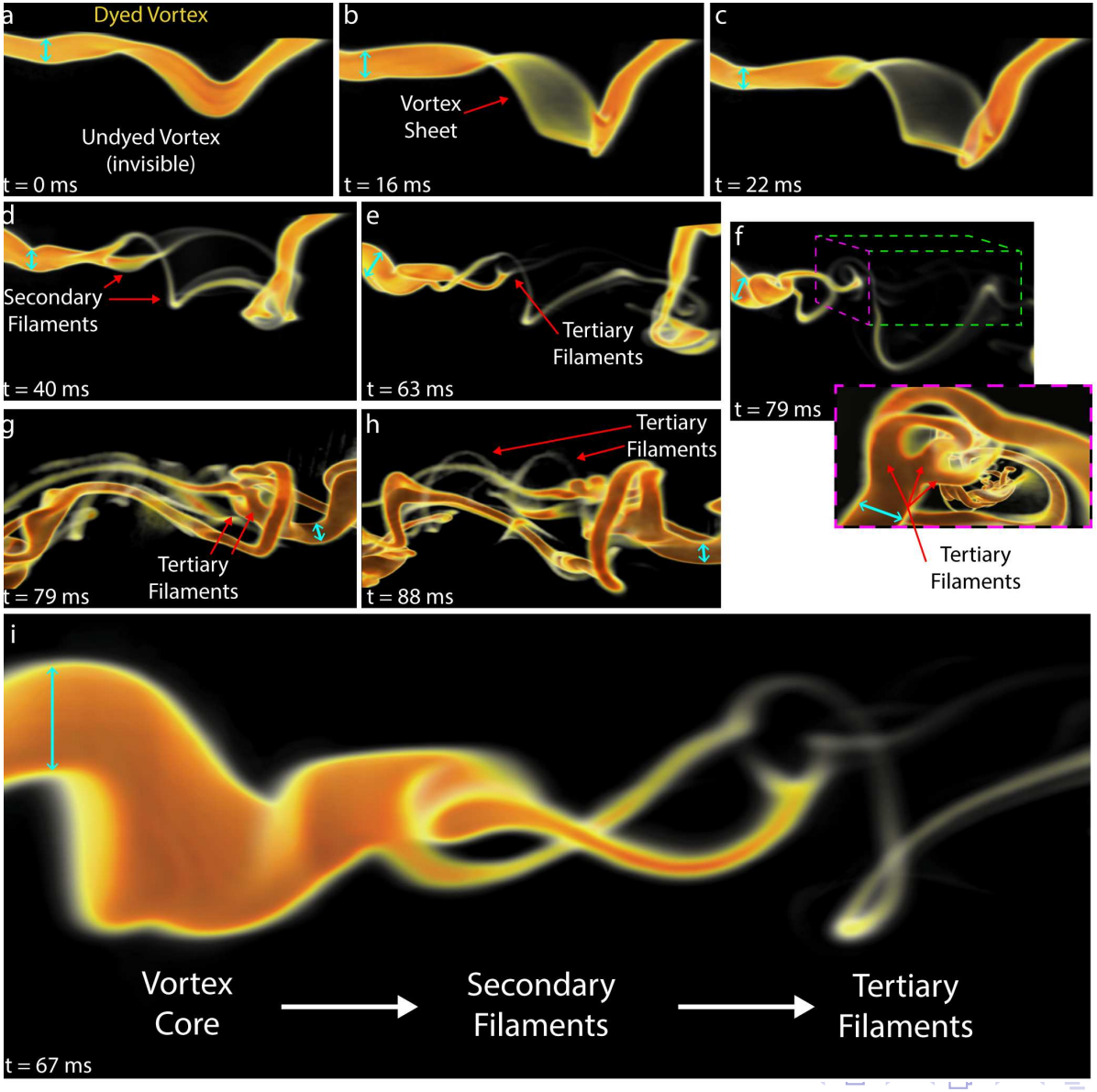
Kinematics: A toy model of an inviscid cascade

The idea: Vortical structures will break up into smaller structures, conserving vorticity volume and kinetic energy. Dissipation occurs as their size reaches zero. The fundamental element will be the helical vortex tube. (C. & A.D.G. FDR 2020)



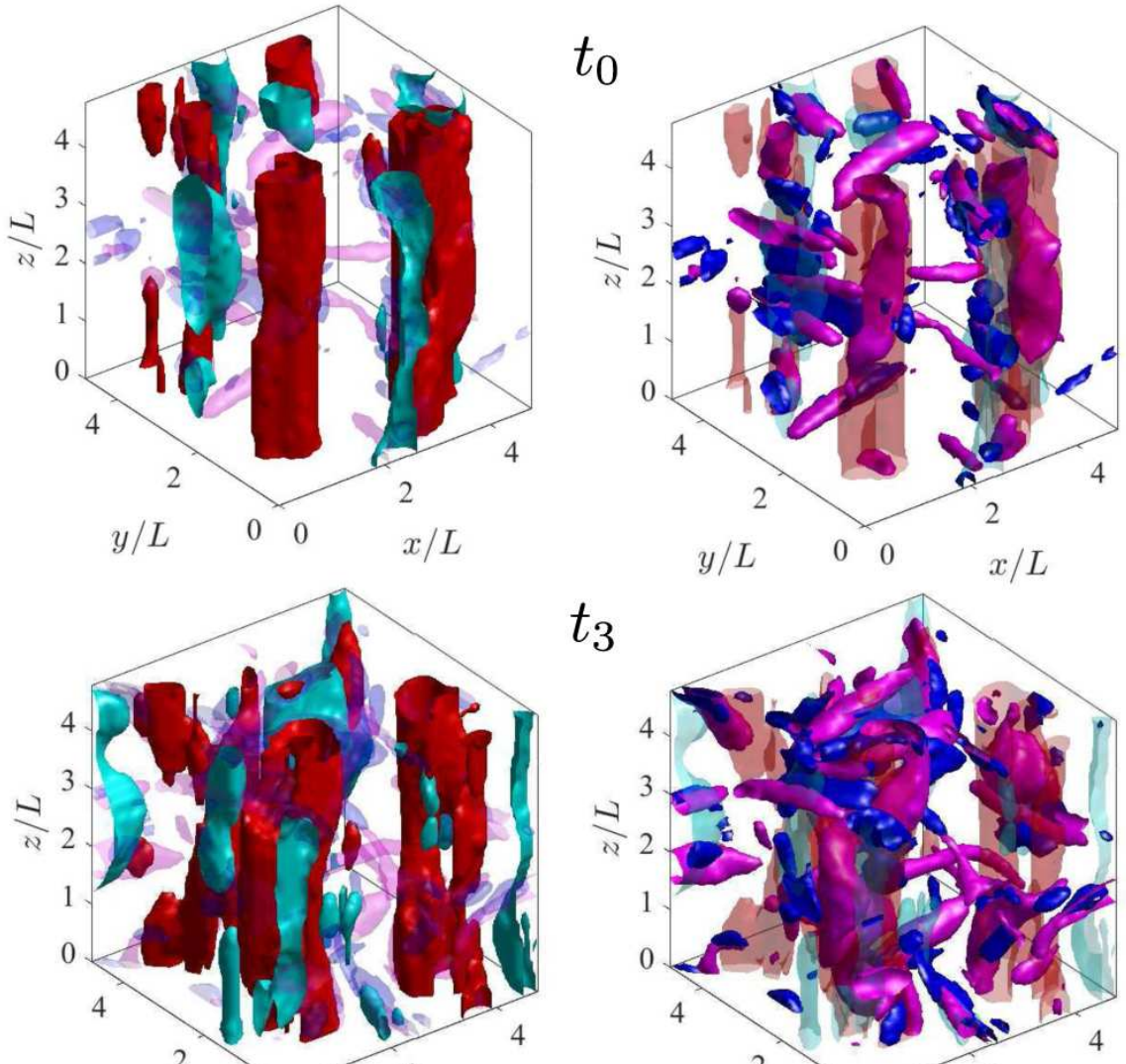
Cascade observed in vortex ring collision

McKeown et al. (2018)



Helical daughter filaments observed under time-periodic forcing

Lennaert van Vee, Vela-Martin and Kawahara (2018)



Vortex tubes in simulations of turbulence (She et al. 1990)

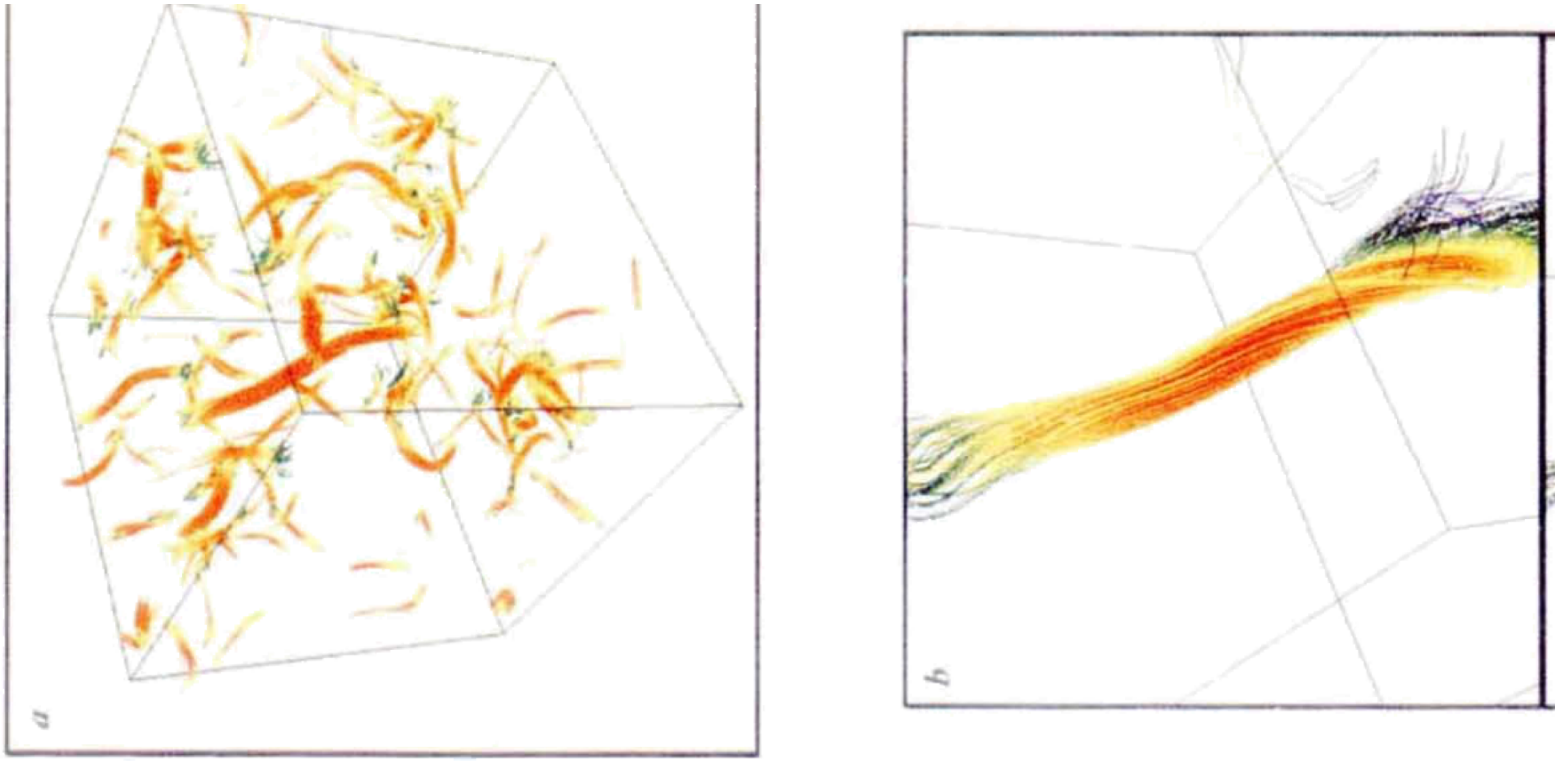
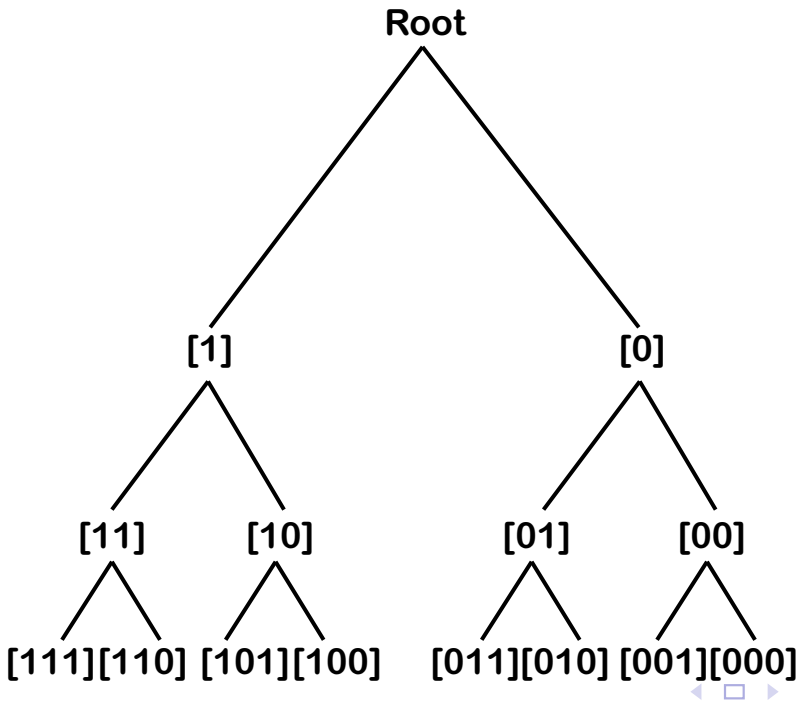


Figure:

The model: A binary cascade of helical vortex tubes

Start with a root vortex with unit flux. It will split into an H[0] daughter helix and an H[1] daughter helix with fluxes[β_0 and $1-\beta_0$, with stretching factors s_0, s_1. At step 2 helices H[00],H[01],H[10],H[11] are similarly produced. Scaling factors:
tube diameter:$\lambda_{0,1} = \sqrt{\beta_{0,1}/s_{0,1}}$,
velocity:$U_{0,1} = \sqrt{\beta_{0,1} s_{0,1}} = \lambda_{0,1}^{\alpha_{0,1}}$.



Test: Calculate accurately observed ζ_p in fully developed turbulence, $1 \leq p \leq 10$

$$\langle \delta u(r)/U \rangle^p \sim (r/L)^{\zeta_p}$$

$$\beta_0^{(p-\zeta_p)/2+1} s_0^{(p+\zeta_p)/2} + \beta_1^{(p-\zeta_p)/2+1} s_1^{(p+\zeta_p)/2} = 1$$

$$\beta_0 + \beta_1 = 1$$

Require

$$\zeta_3 = 1 : \beta_0^2 s_0^2 + \beta_1^2 s_1^2 = 1$$

Agreement with experiment

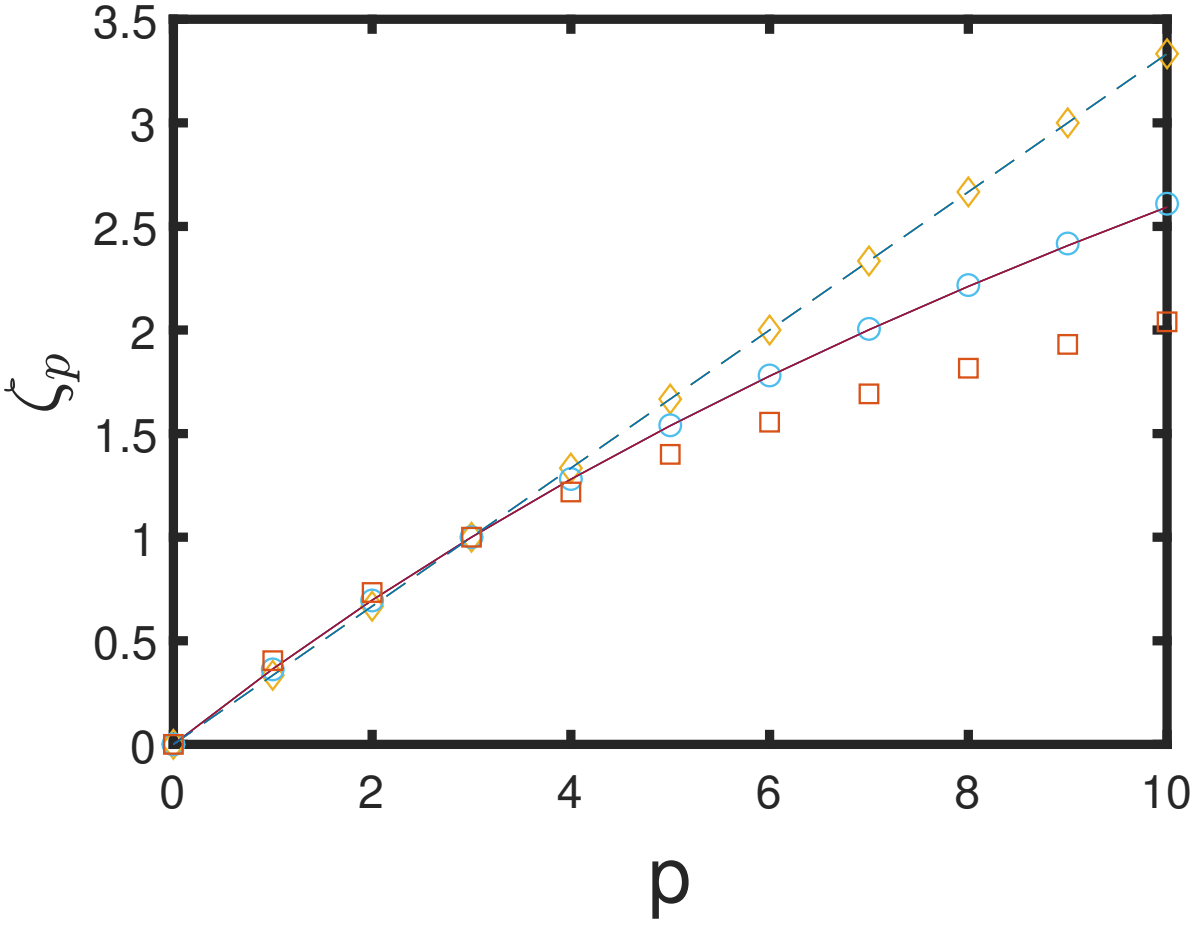


Figure:

Geometric, energetic, and scaling constraints leave few options

β_0	s_0	s_1	ζ_1	ζ_2	λ_0	λ_1	α_0	α_1	s_0/s_1^2	s_1/s_0^2
0.41	1.213	1.471	0.368	0.695	0.582	0.633	0.645	0.155	0.561	1.000
0.25	1.497	1.237	0.357	0.690	0.409	0.779	0.549	0.151	0.979	0.552
0.23	1.554	1.213	0.356	0.690	0.385	0.797	0.539	0.150	1.056	0.502

Dissipation in the model: Introduce time scale of splitting

Assume H[0] splits after a time scaled by inverse vorticity, $1/s_0$, and similarly H[1] splits after time $\Lambda/s_1 > 1/s_0$, $s_{0,1} > 1$. Then H[0000...] reaches zero thickness in time $\frac{1}{s_0} + \frac{1}{s_0^2} + \cdots = \frac{1}{s_0 - 1}$, and similarly for H[1 1 1 1 1 1...] in time $\frac{\Lambda}{s_1 - 1}$. All elements of the cascade disappear in the time window

$$\frac{1}{s_0 - 1} \leq t \leq \frac{\Lambda}{s_1 - 1}$$

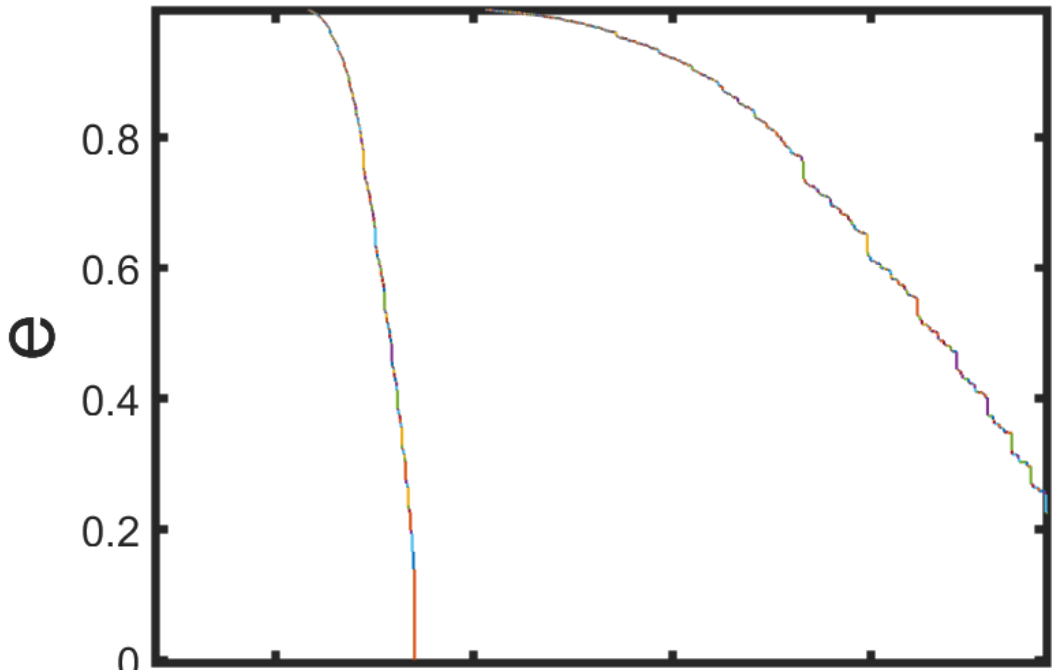
Let the operators $\mathcal{O}_{0,1}$ increment time by the splitting time. Note $\mathcal{O}_0\mathcal{O}_1$ increments by $\frac{\Lambda}{s_1} + \frac{1}{s_1 s_0}$, $\mathcal{O}_1\mathcal{O}_0$ by $\frac{1}{s_0} + \frac{\Lambda}{s_0 s_1}$. The operators commute iff the time window shrinks to a point.

Energy flow: here the filament is removed at the 11th step

We model the distribution of energy fractions over the cascade according to

$$(\mathrm{e}[0]+\mathrm{e}[1])^{-n}\binom{n}{k}\mathrm{e}[0]^{k}\mathrm{e}[1]^{n-k}.$$

Here $s_0 = 1.5$, $s_1 = 1.2735$, $e[0] = 0.14$, $e[1] = 0.71$. The left curve is for $\Lambda = 1$, the right for $\Lambda = 3$.



Summary

- An Euler theory of turbulence will involve a cascade of scales to a fractal weak solution.
- Existing example of inviscid decay in solutions of Euler's equations involve unrealistic, vast separations of scales.
- It is not known what constrains these models to non-increasing kinetic energy.
- Kinematic models suggest that the break-up of vortex tubes, plus modest vortex stretching, might be important in realistic dynamic models.