Mathematical Fluid Dynamics: Advanced Summer School 2025
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Response-function formulation of hydrodynamic and magnetohydrodynamic turbulence: Momentum transport and dynamos

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Contents

- Theoretical formulation
  - Non-linearity: Closure with response functions
  - Inhomogeneities, anisotropies, non-equilibrium properties: Multiple-scale analysis
- Momentum transport by inhomogeneous kinetic helicity
- Dynamos
  - Cross-interaction responses

Approaches to turbulence



群盲撫象　葛飾北斎 Hokusai

Turbulence

Nonlinear

Cascade

Dissipative

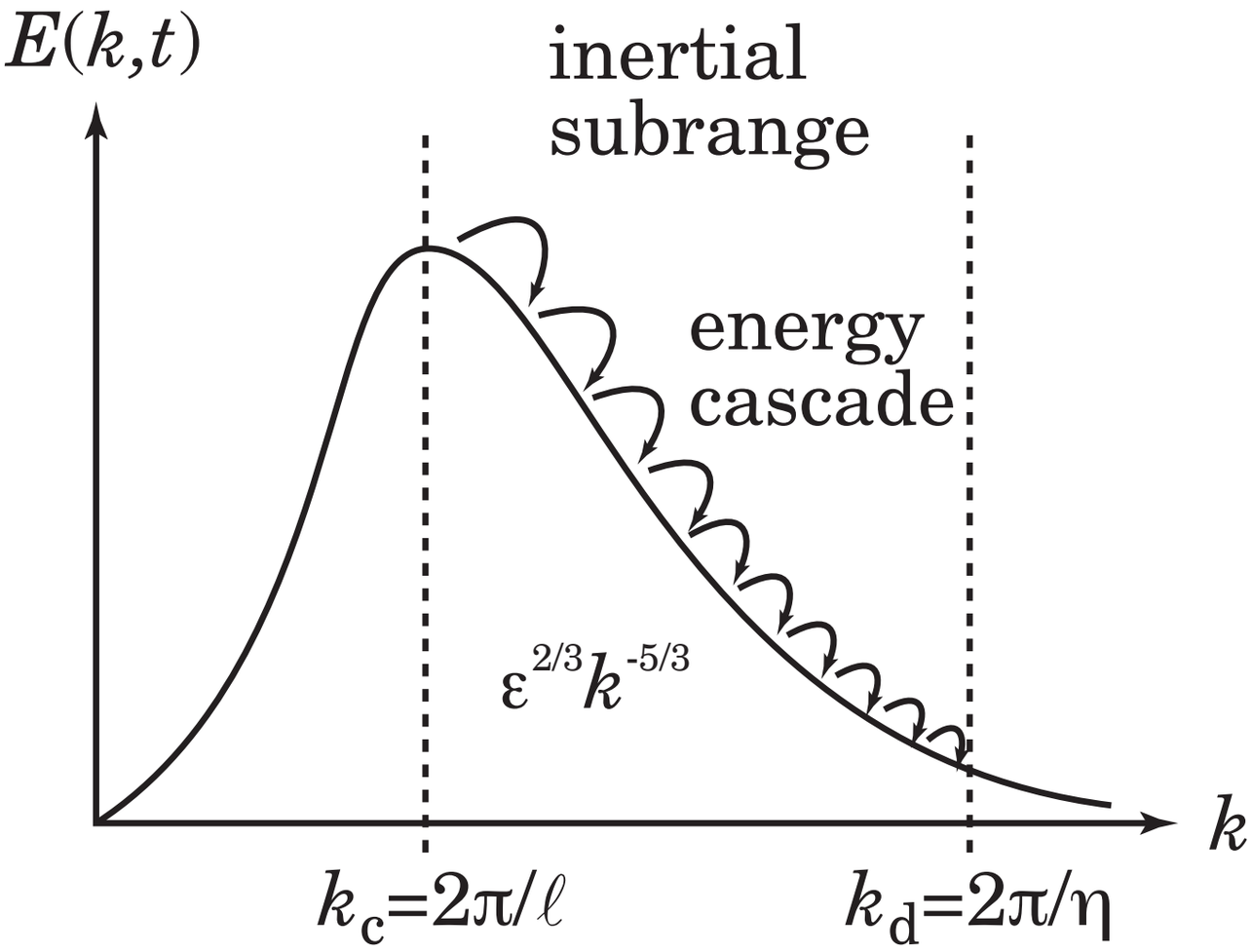
Anomalous transport

Local vs. non-local

Homogeneous
vs. inhomogeneous

Isotropic vs. anisotropic

Equilibrium vs. non-equilibrium

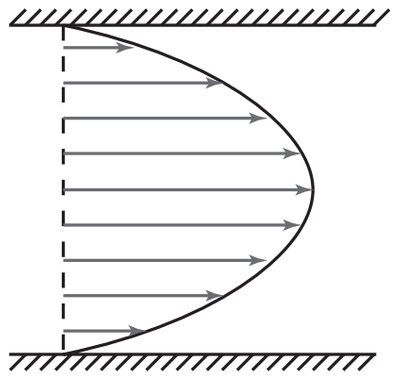


Simple picture of cascade

Turbulent transport

$$\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla\right)\mathbf{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u} \qquad \nabla\cdot\mathbf{u}=0$$

$$\begin{cases}\mathbf{u}=\mathbf{U}+\mathbf{u}', & \mathbf{U}=\langle\mathbf{u}\rangle\\ p=P+p', & P=\langle p\rangle\end{cases}$$



Laminar pipe flow

Equations of mean velocity

$$\frac{DU_\alpha}{Dt}\equiv\left(\frac{\partial}{\partial t}+U_a\frac{\partial}{\partial x_a}\right)U_\alpha=-\frac{\partial P}{\partial x_\alpha}-\frac{\partial}{\partial x_a}\langle u_a'u_\alpha'\rangle+\nu\frac{\partial^2 U_\alpha}{\partial x_a^2} \qquad \frac{\partial U_a}{\partial x_a}=0$$

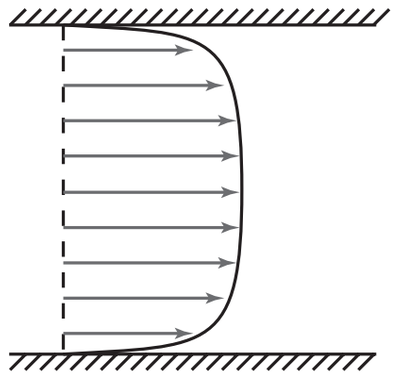
Reynolds stress: $\langle u_\alpha' u_\beta'\rangle=-\nu_{\mathrm{T}}\left(\frac{\partial U_\alpha}{\partial x_\beta}+\frac{\partial U_\beta}{\partial x_\alpha}\right)+\frac{2}{3}K\delta_{\alpha\beta}$ (Boussinesq, 1877)

ν_{T} : eddy viscosity (turbulent viscosity) Mixing-length theory $\nu_{\mathrm{T}}\sim u\ell$ (Prandtl, 1925)

→ $$\frac{\partial U_\alpha}{\partial t}+U_a\frac{\partial U_\alpha}{\partial x_a}=-\frac{\partial P}{\partial x_\alpha}+\frac{\partial}{\partial x_a}\left[(\nu+\nu_{\mathrm{T}})\left(\frac{\partial U_\alpha}{\partial x_a}+\frac{\partial U_a}{\partial x_\alpha}\right)\right]$$

Primary effect of turbulence

- Enhancing transport
- Spatial and temporal dependence



Turbulent pipe flow (Mean flow)

Equation of fluctuating velocity

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

turbulence–mean interaction

turbulence–turbulence interaction

→ Instability approach → Quasi-linear theory (+ non-linearity)

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial U_\alpha}{\partial x_a} - \frac{\partial p^{(\mathrm{R})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Linear in $\mathbf{u}'$ and $p'^{(\mathrm{R})}$, each (Fourier) mode evolves independently

→ Closure approach → Homogeneous turb. theory (+ multiple scale)

$$\frac{\partial u'_\alpha}{\partial t} + U_a \frac{\partial u'_\alpha}{\partial x_a} = -u'_a \frac{\partial u'_\alpha}{\partial x_a} + \frac{\partial}{\partial x_a} \langle u'_a u'_\alpha \rangle - \frac{\partial p'^{(\mathrm{S})}}{\partial x_\alpha} + \nu \frac{\partial^2 u'_\alpha}{\partial x_a^2}$$

Homogeneous turbulence, no dependence on large-scale inhomogeneity

Navier–Stokes equation in the wave-number space

Homogeneous and Isotropic Turbulence (HIT)

$$\frac{\partial \hat{u}_\alpha(\mathbf{k};t)}{\partial t} - ik_a \iint d\mathbf{p}d\mathbf{q}\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\hat{u}_a(\mathbf{p};t)\hat{u}_\alpha(\mathbf{q};t) = ik_\alpha \hat{p}(\mathbf{k};t) - \nu k^2 \hat{u}_\alpha(\mathbf{k};t)$$

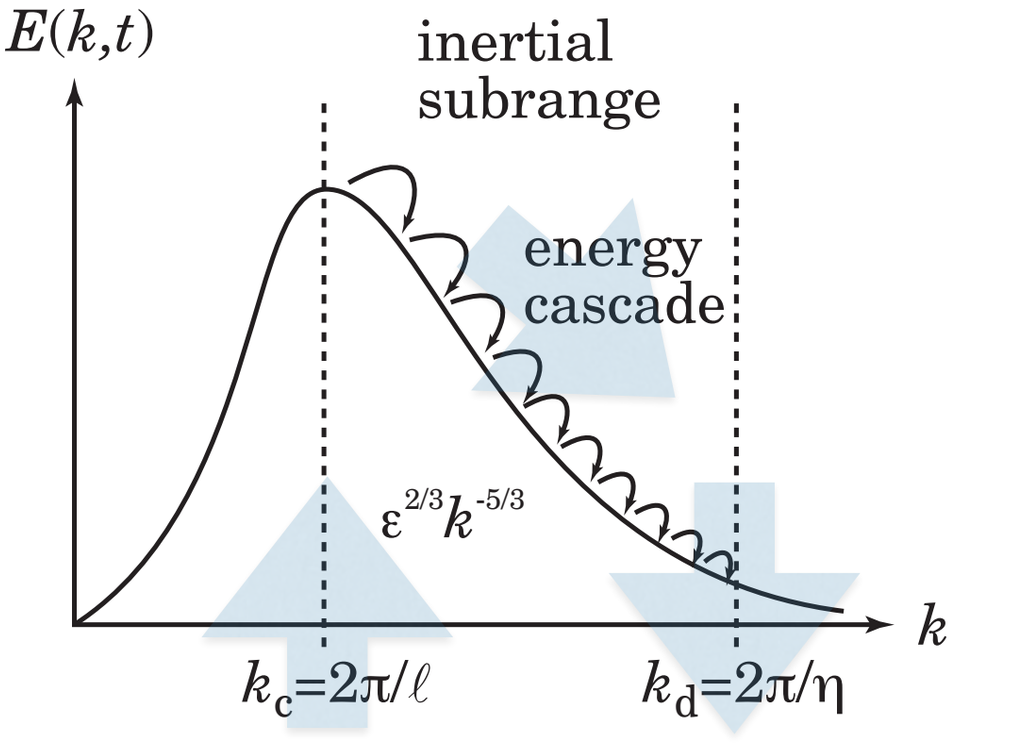
The dynamics of **k** mode is governed by its interaction with all other modes

All the scales, from the largest to smallest scales, have to be solved

Energy injection

Energy flux ε

Energy dissipation



Largest scale (integral scale)

$\ell = \ell\{K, \varepsilon\}$

Smallest scale (viscous scale)

$\eta = \eta\{\nu, \varepsilon\}$

Kolmogorov microscale $\eta = \eta\{\nu, \varepsilon\}$

$$u_\eta \sim \frac{\nu}{\eta} \qquad \left[Re = \frac{u_\eta \eta}{\nu} \sim 1 \right]$$

$$\eta \sim \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} \quad \leftarrow \quad \varepsilon \sim \nu \frac{u_\eta^2}{\eta^2} \sim \nu \frac{(\nu/\eta)^2}{\eta^2} \sim \frac{\nu^3}{\eta^4}$$

$$u_\eta = (\varepsilon\nu)^{1/4} \qquad u_\eta \sim \frac{\nu}{\eta} \sim \frac{\nu}{(\nu^3/\varepsilon)^{1/4}} \sim (\varepsilon\nu)^{1/4}$$

$$\tau_\eta = \left(\frac{\nu}{\varepsilon} \right)^{1/2} \qquad \tau_\eta \sim \frac{\eta}{u_\eta} \sim \frac{(\nu^3/\varepsilon)^{1/4}}{(\varepsilon\nu)^{1/4}} \sim \left(\frac{\nu}{\varepsilon} \right)^{1/2}$$

Integral scale $\ell = \ell\{K, \varepsilon\}$

$$\ell \sim \frac{u_\ell^3}{\varepsilon} \sim \frac{K^{3/2}}{\varepsilon} \quad \leftarrow \quad \varepsilon \sim \frac{u_\ell^2}{\ell/u_\ell} \sim \frac{u_\ell^3}{\ell} \qquad Re = \frac{u_\ell \ell}{\nu}$$

$$u_\ell \sim K^{1/2}$$

$$\tau_\ell \sim \frac{K}{\varepsilon} \qquad \varepsilon \sim \frac{K}{\tau_\ell}$$

Largest and smallest scales in turbulence

Largest scale (integral scale) $\ell = \ell\{K, \varepsilon\}$ $\quad \ell \sim \dfrac{u^3}{\varepsilon} \sim \dfrac{K^{3/2}}{\varepsilon}$

Smallest scale (viscous scale) $\eta = \eta\{\nu, \varepsilon\}$ $\quad \eta \sim \left(\dfrac{\nu^3}{\varepsilon}\right)^{1/4}$

$$\frac{\text{(Largest scale)}}{\text{(Smallest scale)}} \qquad \frac{\ell}{\eta} \sim \left(\frac{u\ell}{\nu}\right)^{3/4} = O\left(Re^{3/4}\right)$$

Required grid points $\qquad N_{\mathrm{G}} = \left(\dfrac{\ell}{\eta}\right)^3 = O\left(Re^{9/4}\right)$

	Re	N_G
Walking	$O(10^4)$	$O(10^9)$
Cars	$O(10^6)$	$O(10^{13.5})$
Airplanes	$O(10^8)$	$O(10^{18})$

	Re	N_G
Earth's outer core	$O(10^8)$	$O(10^{18})$
Solar convection zone	$O(10^{10})$	$O(10^{22.5})$
Galaxies	$O(10^{11})$	$O(10^{25})$

Direct numerical simulation (DNS) is just impossible in the foreseeable future

Theoretical formulation

Nonlinearity

Renormalised perturbation expansion theory

Kraichnan, R. H. (1959) "The structure of isotropic turbulence **at very high Reynolds number**," J. Fluid Mech. **5**, 497

Velocity

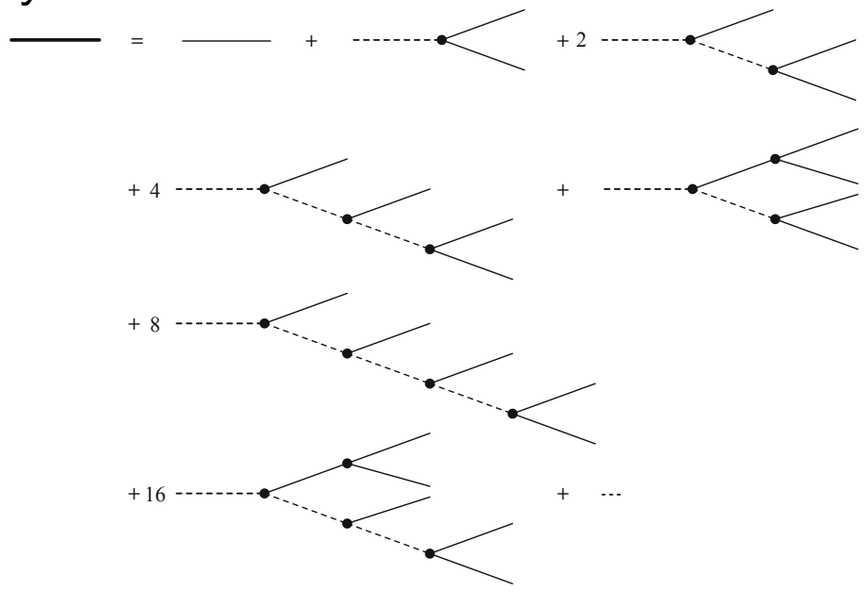
$$u_\alpha(\mathbf{k};t) = u_\alpha^{(\mathrm{L})}(\mathbf{k};t) + iM_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \times \int_{-\infty}^{t} dt_1 G_{\alpha c}^{(\mathrm{L})}(\mathbf{k};t,t_1) u_a(\mathbf{p};t_1) u_b(\mathbf{q};t_1)$$

Response function

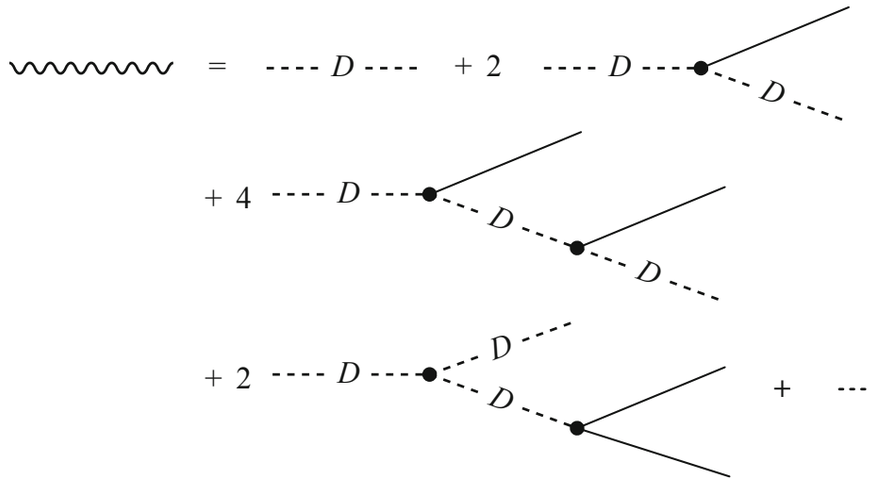
$$G'_{\alpha\beta}(\mathbf{k};t,t') = G_{\alpha\beta}^{(\mathrm{L})}(\mathbf{k};t,t') + i2M_{cab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \times \int_{t'}^{t} dt_1 \; u_a(\mathbf{p};t_1) G_{\alpha\beta}^{(\mathrm{L})}(\mathbf{q};t,t_1)$$

Perturbation expansion with respect to the non-linearity from a Gaussian random state at the infinite past

Velocity



Response function



Velocity equation

$$\frac{\partial}{\partial t}u_i(\mathbf{k};t) = -\nu k^2 u_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k})\iint u_j(\mathbf{p};t)u_\ell(\mathbf{q};t)\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

We assume that **turbulence is in a Gaussian or random state with high turbulence intensity in the infinite past**, and statistically stationary or quasi-stationary at present.

Turbulence state that is independent of the initial or past condition

Formally integrate this **with regarding the non-linear part as known**

$$u_i(\mathbf{k};t) = v_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k})\iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q} \times \int_{-\infty}^{t} dt_1\ g(k;t,t_1)u_j(\mathbf{p};t_1)u_\ell(\mathbf{q};t_1)$$

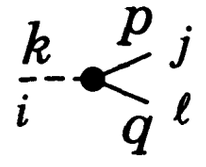
with $\mathbf{v}(\mathbf{k};t) = \mathbf{A}(\mathbf{k})\exp(-\nu k^2 t)$

$g(k;t,t') = H(t-t')\exp(-\nu k^2(t-t'))$

$u_i(k;t)$: $i,\ k,\ t$

$v_i(k;t)$: $i,\ k,\ t$

$g(k;t,t')$: $t\ \ k\ \ t'$

$iM_{ij\ell}(k)$: 

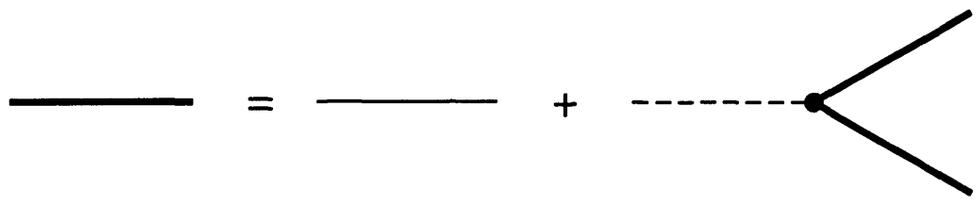
$$u_i(\mathbf{k};t) = v_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q}$$
$$\times \int_{-\infty}^{t} dt_1 \; g(k;t,t_1) u_j(\mathbf{p};t_1) u_\ell(\mathbf{q};t_1)$$

$u_i(k;t)$: $i,\ k,\ t$

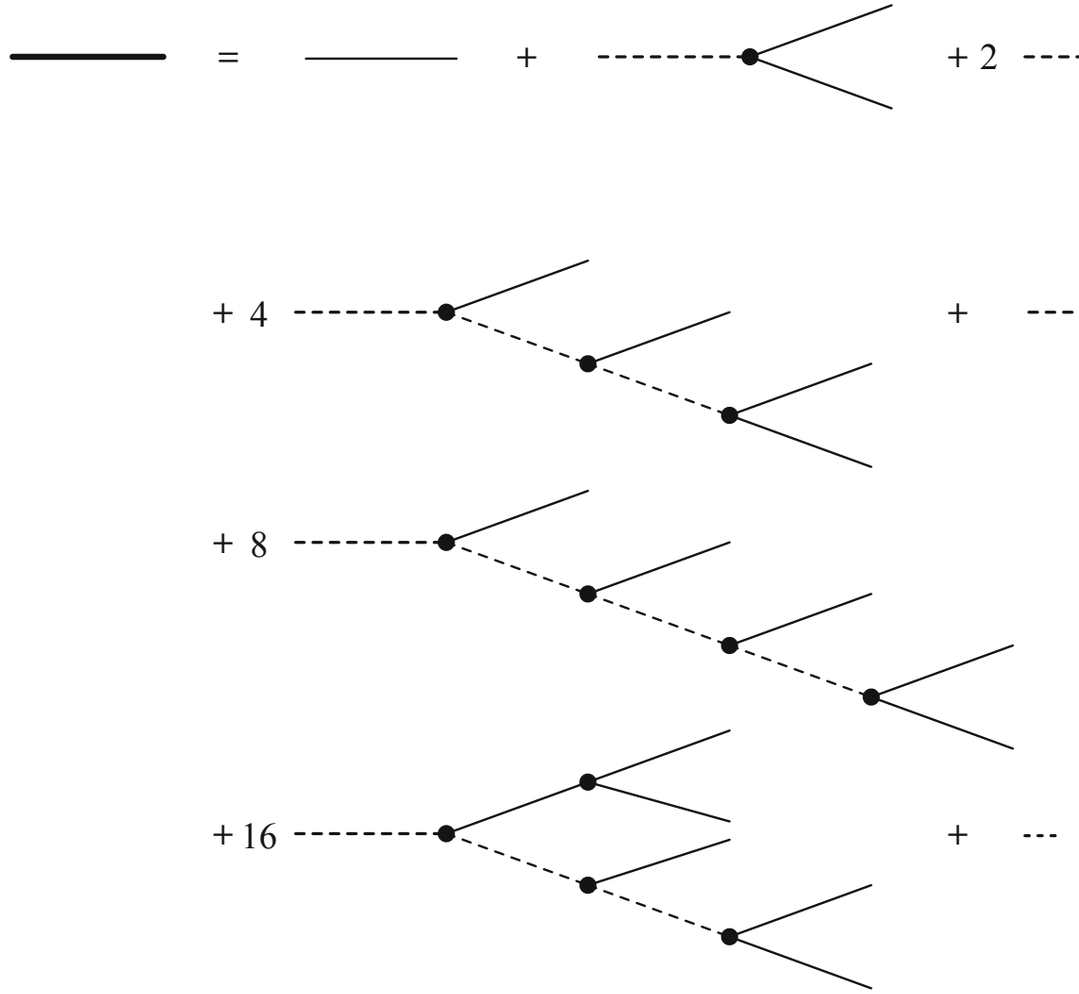
$v_i(k;t)$: $i,\ k,\ t$

$g(k;t,t')$: $t\ \ k\ \ t'$

$iM_{ij\ell}(k)$:



Solve this equation in a perturbative manner with **v**(**k**;t) as the leading term:



Response function equation

The response generated by adding an infinitesimal disturbance $\delta\mathbf{f}(\mathbf{k};t)$ to

$$\frac{\partial}{\partial t}u_i(\mathbf{k};t) = -\nu k^2 u_i(\mathbf{k};t) + iM_{ij\ell}(\mathbf{k})\iint u_j(\mathbf{p};t)u_\ell(\mathbf{q};t)\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

$$\delta u_i(\mathbf{k};t) = \int d\mathbf{k}' \int_{-\infty}^{t} G'_{ij}(\mathbf{k},\mathbf{k}';t,t_1)\delta f_j(\mathbf{k}';t_1)dt_1$$

The **response function equation** is written as

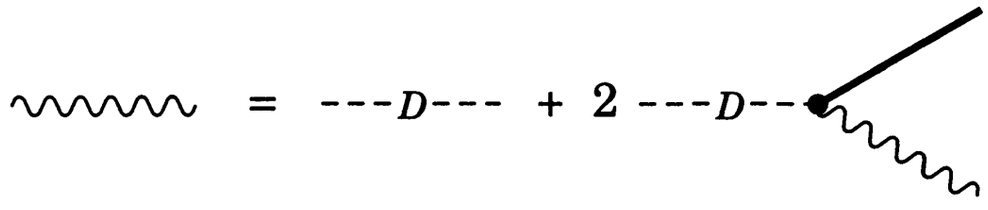
$$\frac{\partial}{\partial t}G'_{ij}(\mathbf{k},\mathbf{k}';t,t') + \nu k^2 G'_{ij}(\mathbf{k},\mathbf{k}';t,t') - D_{ij}(\mathbf{k}')\delta(\mathbf{k}-\mathbf{k}')\delta(t-t') = 2iM_{i\ell m}(\mathbf{k})\iint u_\ell(\mathbf{p};t)G'_{mj}(\mathbf{q},\mathbf{k}';t,t')\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

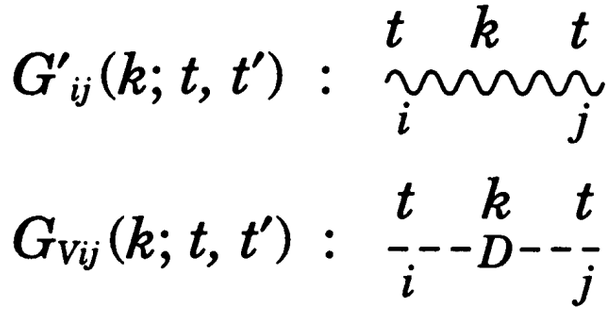
Integrate this with respect to **k**' with putting $G'_{ij}(\mathbf{k},\mathbf{k}';t,t') = G'_{ij}(\mathbf{k};t,t')\delta(\mathbf{k}-\mathbf{k}')$

$$\frac{\partial}{\partial t}G'_{ij}(\mathbf{k};t,t') + \nu k^2 G'_{ij}(\mathbf{k};t,t') - D_{ij}(\mathbf{k})\delta(t-t') = 2iM_{i\ell m}(\mathbf{k})\iint u_\ell(\mathbf{p};t)G'_{mj}(\mathbf{q};t,t')\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$

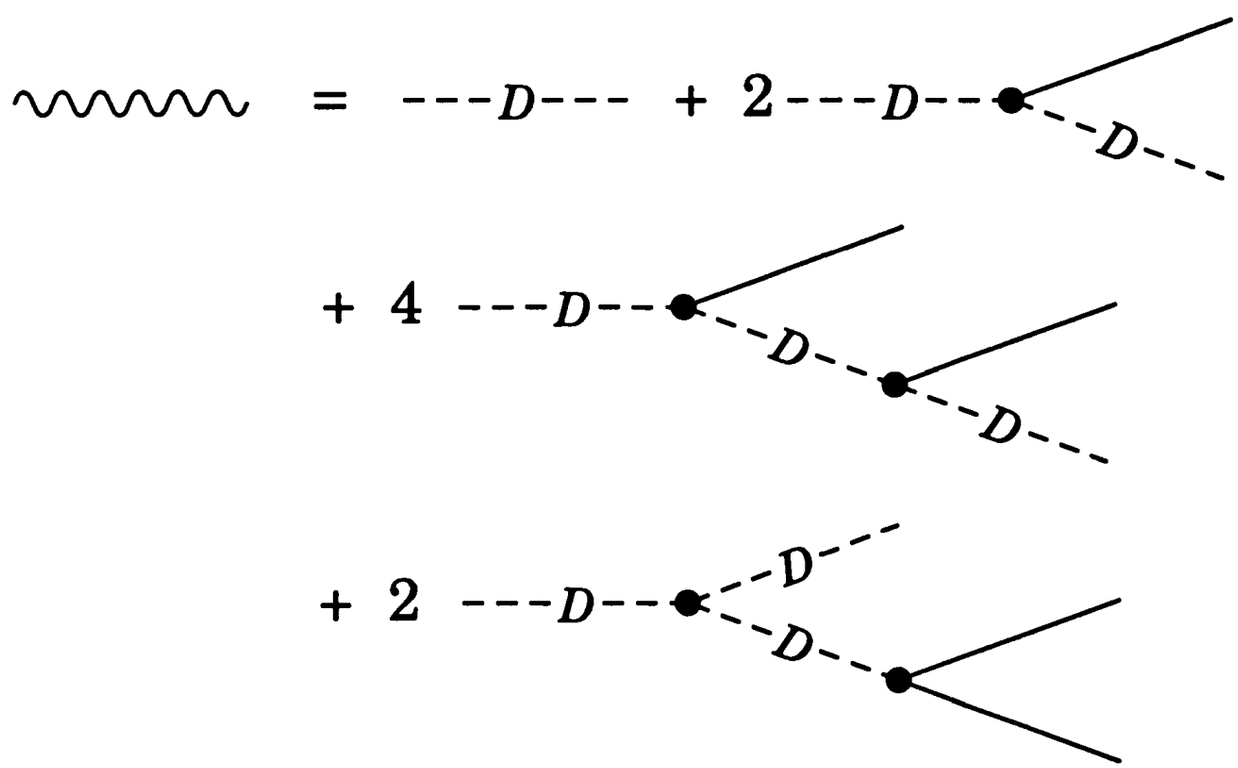
Formally integrate this **with regarding the r.h.s. as known**

$$G'_{ij}(\mathbf{k};t,t') = G_{Vij}(\mathbf{k};t,t') + 2iM_{n\ell m}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})d\mathbf{p}d\mathbf{q}$$
$$\times \int_{t'}^{t} dt_1 G_{Vin}(\mathbf{k};t,t_1)u_\ell(\mathbf{p};t_1)G'_{mj}(\mathbf{q};t_1,t')$$

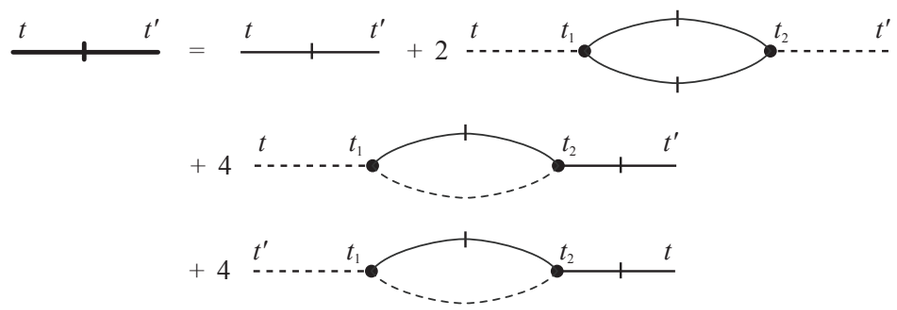




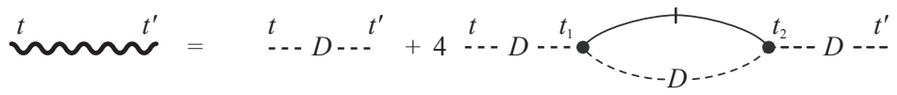
Perturbation expansion up to $O(M^2)$



Correlation function $Q_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



Response function $G_{\alpha\beta}(\mathbf{k},\mathbf{k}';t,t')$



Renormalisation

a part of the infinite series with respect to the propagators is **summed up to the infinite orders**

$$f_{\rm ex}(x) = 1 + x + x^2 + x^3 + \cdots$$

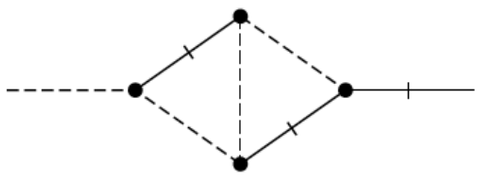
$$f_{\rm ex}(x) = 1 + x(1 + x + x^2 + \cdots)$$

$$f_{\rm ex}(x) = \frac{1}{1-x}$$

No good

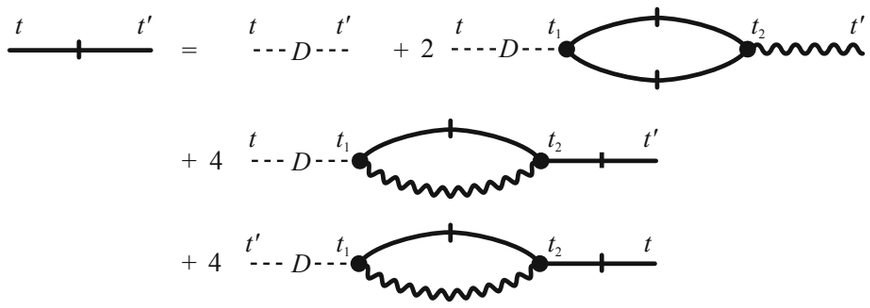
Truncation $f_{\rm ex}(x) = 1 + x(1+x)$

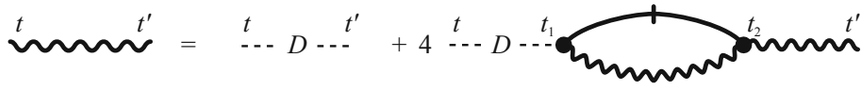
Renormalisation $f_{\rm ex}(x) = 1 + x f_{\rm ex}(x)$



Higher-order vertex not included

DIA = line (propagator) renormalisation (lowest-order in vertex)





Inhomogeneity, anisotropy, and non-equilibrium properties



RHK with Akira Yoshizawa at IIS in 1996



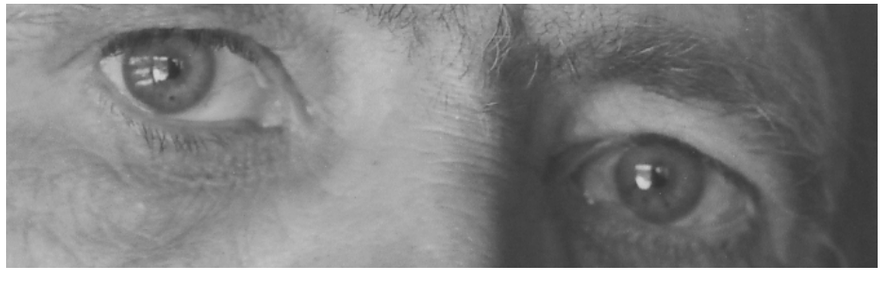
RHK with Akira Yoshizawa at IIS in 1996

The difference between the stupidity and genius is that the genius has its limit. (Albert Einstein)



RHK with Akira Yoshizawa at IIS in 1996

The difference between the stupidity and genius is that the genius has its limit. (Albert Einstein)



“Crazy”

Multiple-Scale Direct-Interaction Approximation

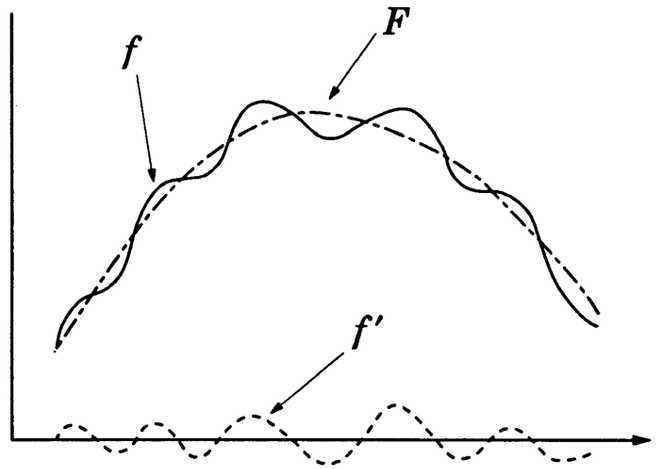
Mirror-symmetric case: Yoshizawa, Phys. Fluids **27**, 1377 (1984)
Non-mirror-symmetric case: Yokoi & Yoshizawa, Phys. Fluids A **5**, 464 (1993)

Fast and slow variables $\boldsymbol{\xi} = \mathbf{x},\ \mathbf{X} = \delta_x \mathbf{x};\ \ \tau = t,\ T = \delta_t t$

Slow variables ***X*** and *T* change only when ***x*** and *t* change much.

$$f = F(\mathbf{X}; T) + f'(\boldsymbol{\xi}, \mathbf{X}; \tau, T)$$

$$\nabla = \nabla_{\boldsymbol{\xi}} + \delta_x \nabla_{\mathbf{X}};\ \ \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} + \delta_t \frac{\partial}{\partial T}$$



Velocity-fluctuation equation $(\delta_x = \delta_t)$

$$\frac{\partial u'_\alpha}{\partial \tau} + U_a \frac{\partial u'_\alpha}{\partial \xi_a} + \frac{\partial}{\partial \xi_a} u'_a u'_\alpha + \frac{\partial p'}{\partial \xi_\alpha} - \nu \nabla_\xi^2 u'_\alpha$$

$$= \delta \left(-u'_a \frac{\partial U_\alpha}{\partial X_a} - \frac{D u'_\alpha}{DT} - \frac{\partial p'}{\partial X_\alpha} - \frac{\partial}{\partial X_a} \left(u'_a u'_\alpha - R_{a\alpha} + 2\nu \frac{\partial^2 u'_\alpha}{\partial X_a \partial \xi_a} \right) \right)$$
$$+ \delta^2 \left(\nu \nabla_X^2 u'_\alpha \right)$$

$$\frac{\partial u'_a}{\partial \xi_a} + \delta \frac{\partial u'_a}{\partial X_a} = 0$$

$$\frac{D}{DT} = \frac{\partial}{\partial T} + \mathbf{U} \cdot \nabla_X$$

Inhomogeneities, anisotropies, non-equilibrium properties

Multiple-Scale DIA calculations

Scale parameter expansion $$f' = f'_0 + \delta f'_1 + \delta^2 f'_2 + \cdots = \sum_n \delta^n f'_n$$

Basic-field (lowest-order field) equation

$$\frac{\partial u_0^i(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u_0^i(\mathbf{k};\tau) - iM^{ijk}(\mathbf{k}) \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) u_0^j(\mathbf{p};\tau) u_0^k(\mathbf{q};\tau) = 0$$

Projection operators

$$M^{ijk}(\mathbf{k}) = \frac{1}{2}\left[k^j D^{ik}(\mathbf{k}) + k^k D^{ij}(\mathbf{k})\right], \quad D^{ij}(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{k^2}$$

Green's function

$$\frac{\partial G'_{\alpha\beta}(\mathbf{k};\tau,\tau')}{\partial \tau} + \nu k^2 G'_{\alpha\beta}(\mathbf{k};\tau,\tau') - 2iM^{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p}d\mathbf{q} u'_{0a}(\mathbf{p};\tau)\, G'_{b\beta}(\mathbf{q};\tau,\tau') = D_{\alpha\beta}(\mathbf{k})\delta(\tau-\tau')$$

1st-order field

$$\frac{\partial u'_{1\alpha}(\mathbf{k};\tau)}{\partial\tau} + \nu k^2 u'_{1\alpha}(\mathbf{k};\tau)$$
$$-2iM_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} u'_{0a}(\mathbf{p};\tau) u'_{S1b}(\mathbf{q};\tau)$$
$$= -D_{\alpha b}(\mathbf{k}) u'_{0a}(\mathbf{k};\tau) \frac{\partial U_b}{\partial X_a} - D_{\alpha a}(\mathbf{k}) \frac{D u'_{0a}(\mathbf{k};\tau)}{DT_{\mathrm{I}}}$$
$$+2M_{\alpha ab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau) \frac{\partial u'_{0c}(\mathbf{q};\tau)}{\partial X_{\mathrm{I}c}}$$
$$-D_{\alpha d}(\mathbf{k}) M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau) u'_{0b}(\mathbf{q};\tau)\right)$$

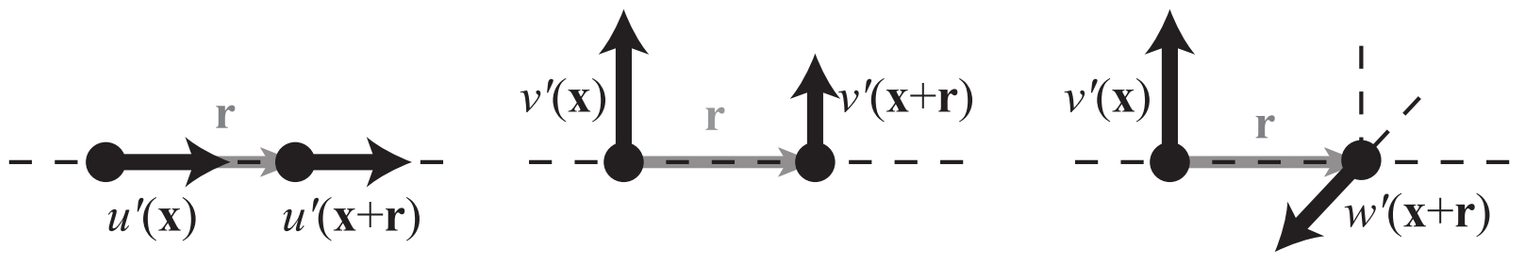
Inhomogeneities, anisotropies, and non-equilibrium properties

Formal solution in terms of Green's function

$$u'_{1\alpha}(\mathbf{k};\tau) = -\frac{\partial U_b}{\partial X_a} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha b}(\mathbf{k};\tau,\tau_1) u'_{0a}(\mathbf{k};\tau_1)$$
$$- \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha a}(\mathbf{k};\tau,\tau_1) \frac{D u'_{0a}(\mathbf{k};\tau_1)}{DT_1}$$
$$+ 2M_{dab}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \frac{q_b}{q^2} u'_{0a}(\mathbf{p};\tau_1) \frac{\partial u'_{0c}(\mathbf{q};\tau_1)}{\partial X_{\mathrm{I}c}}$$
$$- M_{abcd}(\mathbf{k}) \iint \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) d\mathbf{p} d\mathbf{q} \int_{-\infty}^{\tau} d\tau_1 G'_{\alpha d}(\mathbf{k};\tau,\tau_1) \frac{\partial}{\partial X_{\mathrm{I}c}} \left(u'_{0a}(\mathbf{p};\tau_1) u'_{0b}(\mathbf{q};\tau_1)\right)$$

Statistical assumptions on the lowest-order (basic) fields

Basic fields: homogeneous isotropic



Longitudinal Transverse Cross

$$\frac{\langle \vartheta'_{\mathrm{B}}{}^{\alpha}(\mathbf{k};\tau)\chi'_{\mathrm{B}}{}^{\beta}(\mathbf{k}';\tau')\rangle}{\delta(\mathbf{k}+\mathbf{k}')} = \underbrace{\Pi^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{C}}(k;\tau,\tau')}_{\text{Longitudinal}} + \underbrace{D^{\alpha\beta}(\mathbf{k})Q_{\vartheta\chi\mathrm{S}}(k;\tau,\tau')}_{\text{Transverse}} + \underbrace{\frac{i}{2}\frac{k^c}{k^2}\epsilon^{\alpha\beta c}H_{\vartheta\chi}(k;\tau,\tau')}_{\text{Cross}}$$

with solenoidal and dilatational projection operators

$$D^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}, \quad \Pi^{\alpha\beta}(\mathbf{k}) = \frac{k^\alpha k^\beta}{k^2}$$

Calculation of turbulent correlation by DIA

$$\langle f'(\mathbf{x};t)g'(\mathbf{x};\mathbf{t})\rangle = \int d\mathbf{k}\,\langle f'(\mathbf{k};\tau)g'(\mathbf{k};\tau)\rangle\,/\delta(\mathbf{0})$$

$$= \int d\mathbf{k}\,(\langle f'_0 g'_0\rangle + \langle f'_0 g'_1\rangle + \langle f'_1 g'_0\rangle + \cdots)\,/\delta(\mathbf{0})$$

Mean-field equations in compressible MHD

Yokoi, N., J. Plasma Phys. **84**, 735840501 & 775840603 (2018a,b)

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Turb. mass flux

Momentum

$$\frac{\partial}{\partial t}\overline{\rho}U^{\alpha} + \frac{\partial}{\partial x^{a}}\overline{\rho}U^{a}U^{\alpha}$$

$$= -\left(\gamma_0 - 1\right)\frac{\partial}{\partial x^{\alpha}}\overline{\rho}Q + \frac{\partial}{\partial x^{\alpha}}\mu S^{a\alpha} + (\mathbf{J}\times\mathbf{B})^{\alpha}$$

$$- \frac{\partial}{\partial x^{\alpha}}\left(\overline{\rho}\langle u'^{a}u'^{\alpha}\rangle - \frac{1}{\mu_0}\langle b'^{a}b'^{\alpha}\rangle + U^{a}\langle \rho' u'^{\alpha}\rangle + U^{\alpha}\langle \rho' u'^{a}\rangle\right) + R_U^{\alpha}$$

Reynolds stress

Turb. Maxwell stress

Internal energy

$$\frac{\partial}{\partial t}\overline{\rho}Q + \nabla\cdot(\overline{\rho}\mathbf{U}Q) = \nabla\cdot\left(\frac{\kappa}{C_V}\nabla Q\right) - \nabla\cdot(\overline{\rho}\langle q'\mathbf{u}'\rangle + Q\langle\rho'\mathbf{u}'\rangle + \mathbf{U}\langle\rho' q'\rangle)$$

Turb. energy flux

Turb. mass-energy correl.

$$- (\gamma_0 - 1)\left(\overline{\rho}Q\nabla\cdot\mathbf{U} + \overline{\rho}\langle q'\nabla\cdot\mathbf{u}'\rangle + Q\langle\rho'\nabla\cdot\mathbf{u}'\rangle\right) + R_Q$$

Turb. energy dilatation

Turb. mass dilatation

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B} + \langle\mathbf{u}'\times\mathbf{b}'\rangle) + \eta\nabla^2\mathbf{B}$$

Turb. electromotive force

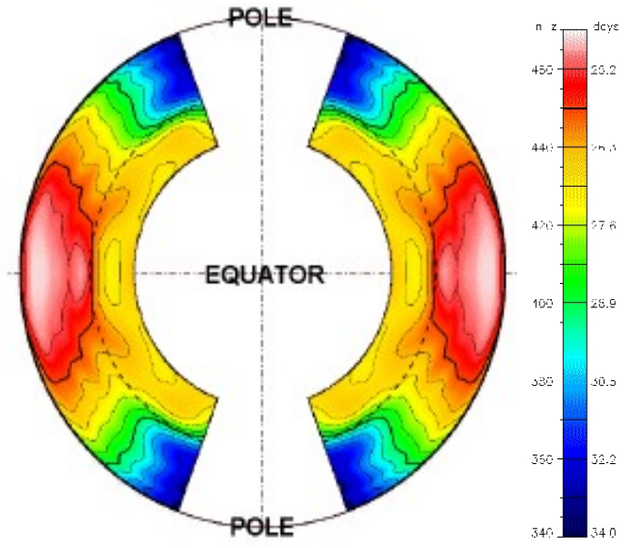
Illustrative examples

Differential Rotation

Helioseismology shows the internal structure of the Sun.

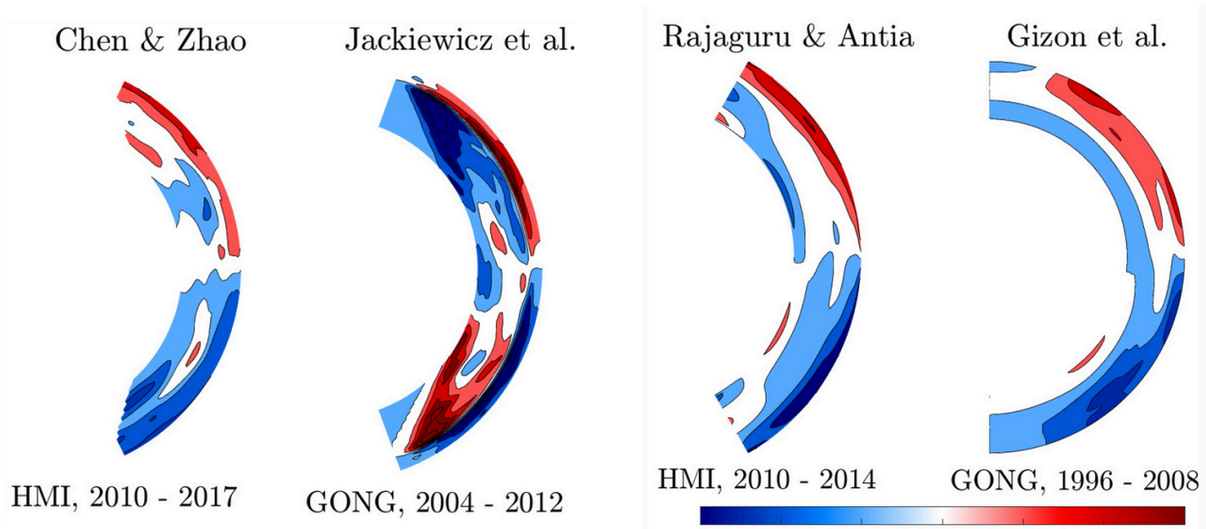
Azimuthal velocity

- Surface **differential rotation** is maintained throughout the convection zone
- Solid body rotation in the radiative interior
- Thin matching zone of shear known as the **tachocline** at the base of the solar convection zone (just in the stable region)



Meridional circulation

- Meridional flow **poleward at surface**
- Interior structure not settled
- **As stars spin faster** they tend to have **slower meridional flows** (from models)

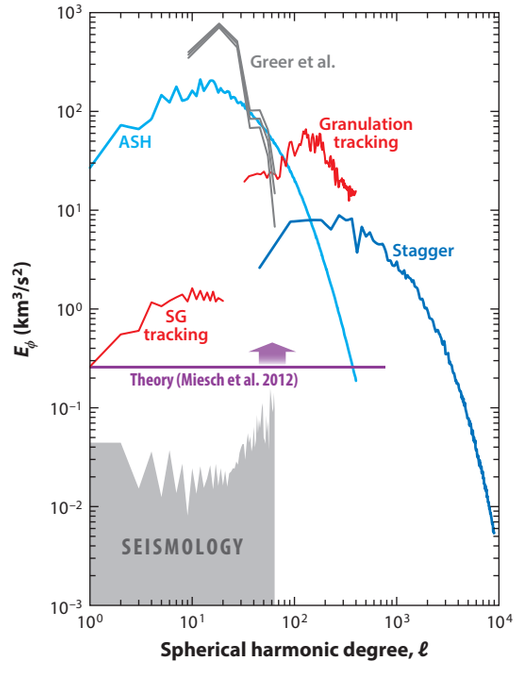


(Hanasoge 2022)

Convection conundrum

The current numerical simulations do not capture some basic characteristics of the solar convection

(Schumacher & Sreenivasan 2020)



(Hanasoge, Gizon & Sreenivasan 2016)

- ***The convective velocity amplitude at large horizontal scales observed by helioseismic investigations is much smaller*** than the one predicted by global convection simulations

- The differential rotation profile obtained by the global numerical simulations shows ***the anti-prograde profile*****, if the solar values of luminosity (energy transfer rate) and rotation rate are adopted** in the simulation.

- Also, if the large-scale convection motions are actually small, ***how such weak flows can transfer the solar luminosity and mean differential rotation rate*** observed in the helioseismology?

Approaches to the conundrum

Rotation effect

Amplitude and length-scale of convection could be smaller than we think, leading to a smaller Ro = U/ 2ΩL (Vasil et al 2022, but Käpylä 2023)

-> Inhomogeneous helicity effect coupled with rotation (Yokoi & Yoshizawa 1993, Yokoi & Brandenburg 2016)

Coherent motion effect

Non-local heat transport via entropy rain (e.g., Brandenburg 2016, Anders et al 2019)

-> Non-equilibrium effect associated with plumes and thermals (Yokoi, Masada & Takiwaki 2022)

Magnetic-field effect

Maxwell Stresses from small-scale dynamo acts to counteract Reynolds Stresses that lead to anti-solar rotation profile (e.g., Hotta et al 2022)

-> Cross-helicity effect in angular momentum transport (Yokoi 2023)

Angular-momentum transport by inhomogeneous kinetic helicity

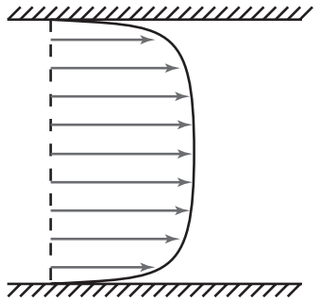
Yokoi, N. & Yoshizawa, A. “Statistical analysis of the effects of helicity in inhomogeneous turbulence,” **Phys. Fluids A 5, 464-477 (1993)**
https://doi.org/10.1063/1.858869

Yokoi, N. & Brandenburg, A. “Global flow generation by inhomogeneous helicity,” **Phys. Rev. E. 93, 033125-1-14 (2016)**
https://doi.org/10.1103/PhysRevE.93.033125

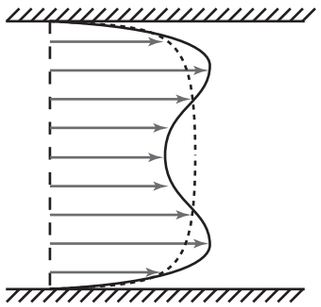
Pouquet, A. & Yokoi, N. “Helical fluid and (Hall)-MHD turbulence: a brief review,” **Phil. Trans. Roy. Soc. A 380, 20210081-1-18 (2022)**
https://doi.org/10.1098/rsta.2021.0087

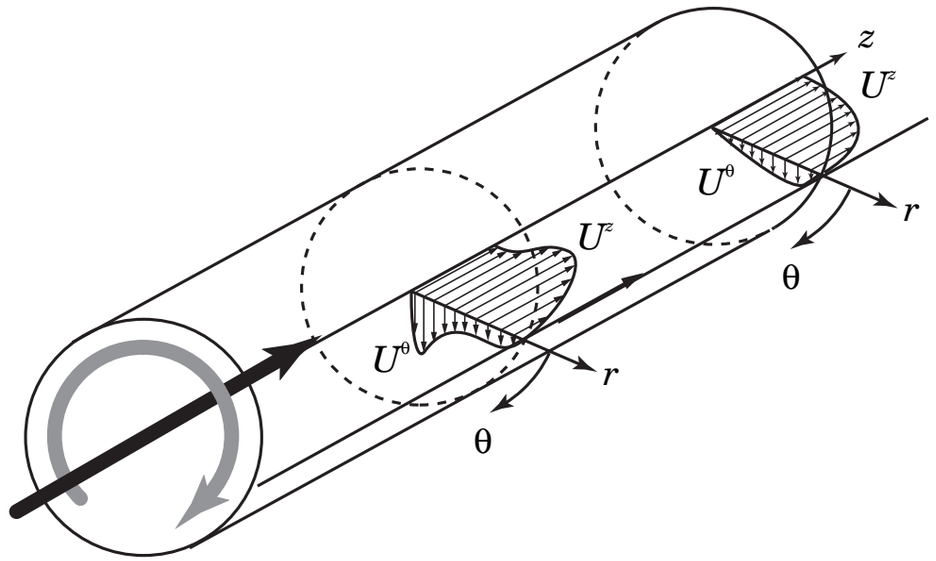
Yokoi, N. “Transports in helical fluid turbulence,” pp.25-50,
in Kuzanyan, Yokoi, Georgoulis & Stepanov (eds.) ***AGU Book: Helicities in Geophysics, Astrophysics and Beyond*** **(Wiley, 2023)**
https://doi.org/10.1002/9781119841715

Swirling flow in a circular pipe

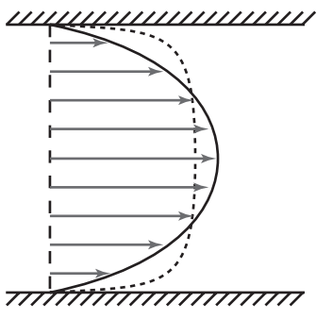


Turbulent swirling pipe flow





Axially rotating turbulent pipe flow



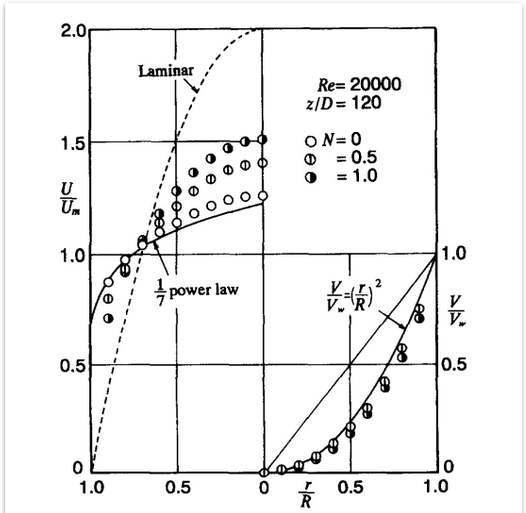


Figure 4 Time-mean velocity profiles (uncertainties in r/R are ±1%, and in U/U_m and V/V_w are ±2%)

These flow properties cannot be reproduced
by the standard eddy-viscosity representation at all.
Too much dissipative.

(Imao *et al.* 1996)

(Yokoi & Yoshizawa, PoF A**5**, 464, 1993)

Calculation of the Reynolds stress

$$R_{ij} = \langle u_i'(\boldsymbol{\xi}, \mathbf{X}; \tau, T) u_j'(\boldsymbol{\xi}, \mathbf{X}; \tau, T) \rangle = \int R_{ij}(\mathbf{k}, \mathbf{X}; \tau, T) d\mathbf{k}$$

$$\begin{aligned} \left\langle u'^{\alpha} u'^{\beta} \right\rangle = & \left\langle {u'_{\mathrm{B}}}^{\alpha} {u'_{\mathrm{B}}}^{\beta} \right\rangle + \left\langle {u'_{\mathrm{B}}}^{\alpha} {u'_{01}}^{\beta} \right\rangle + \left\langle {u'_{01}}^{\alpha} {u'_{\mathrm{B}}}^{\beta} \right\rangle + \cdots \\ & + \left\langle {u'_{\mathrm{B}}}^{\alpha} {u'_{10}}^{\beta} \right\rangle + \left\langle {u'_{10}}^{\alpha} {u'_{\mathrm{B}}}^{\beta} \right\rangle + \cdots \end{aligned}$$

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

where $\mathcal{S}^{\alpha\beta} = \dfrac{\partial U^{\alpha}}{\partial x^{\beta}} + \dfrac{\partial U^{\beta}}{\partial x^{\alpha}} - \dfrac{2}{3} \nabla \cdot \mathbf{U} \delta^{\alpha\beta}$

Eddy viscosity $\nu_{\mathrm{T}} = \dfrac{7}{15} \displaystyle\int \mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k; \tau, \tau_1) Q(k; \tau, \tau_1)$

mixing length $\nu_{\mathrm{T}} \sim \tau u^2 \sim u\ell$

Helicity-related coefficient $\boldsymbol{\Gamma} = \dfrac{1}{30} \displaystyle\int k^{-2} \mathrm{d}\mathbf{k} \int_{-\infty}^{t} d\tau_1 \; G(k; \tau, \tau_1) \nabla H(k; \tau, \tau_1)$

helicity inhomogeneity is essential

Eddy viscosity + Helicity model

Reynolds stress (Yokoi & Yoshizawa, PoF A**5**, 464, 1993)

$$\mathcal{R}_{\alpha\beta} \equiv \left\langle u'_\alpha u'_\beta \right\rangle$$
$$= \frac{2}{3} K \delta_{\alpha\beta} - \nu_{\mathrm{T}} \left(\frac{\partial U_\alpha}{\partial x_\beta} + \frac{\partial U_\beta}{\partial x_\alpha} \right) + \eta \left[\Omega_\alpha \frac{\partial H}{\partial x_\beta} + \Omega_\beta \frac{\partial H}{\partial x_\alpha} - \frac{2}{3} \delta_{\alpha\beta} \left(\boldsymbol{\Omega} \cdot \nabla \right) H \right]$$

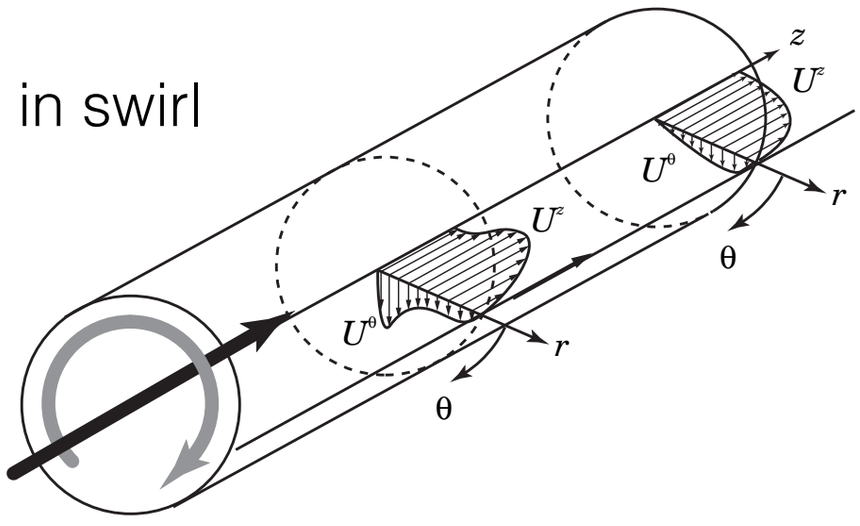
$$\nu_{\mathrm{T}} = C_\nu \tau K, \quad \tau = K/\epsilon, \quad \eta = C_H \tau (K^3/\epsilon^2)$$

Turbulence quantities

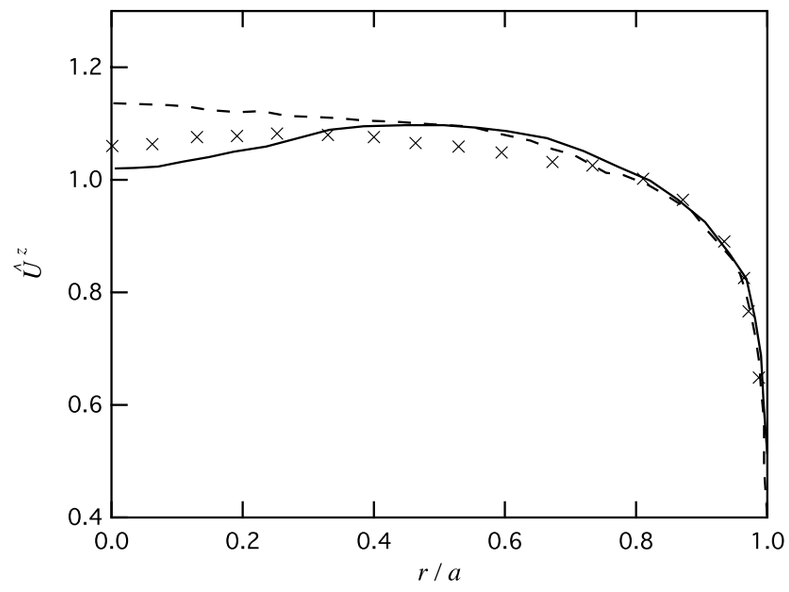
$$K \equiv \frac{1}{2} \langle \mathbf{u}'^2 \rangle, \quad \epsilon \equiv \nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial u'_b}{\partial x_a} \right\rangle,$$

$$H \equiv \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle, \quad \epsilon_H \equiv 2\nu \left\langle \frac{\partial u'_b}{\partial x_a} \frac{\partial \omega'_b}{\partial x_a} \right\rangle$$

Velocity profiles in swirl

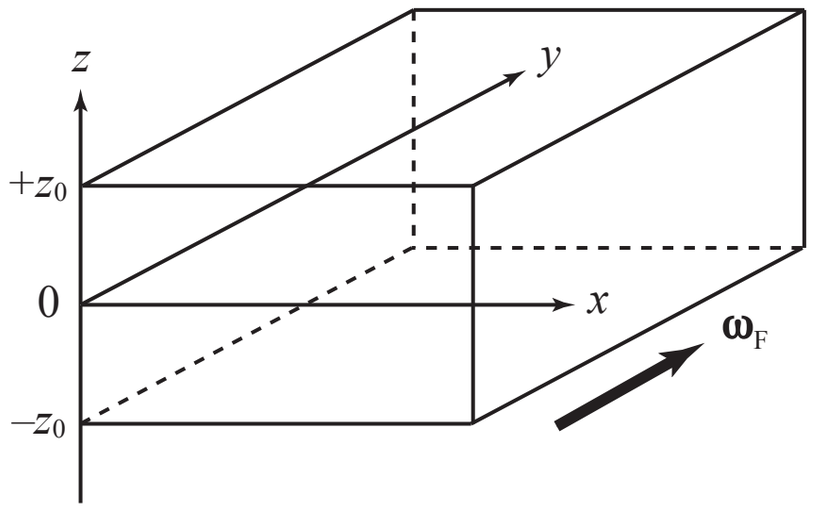


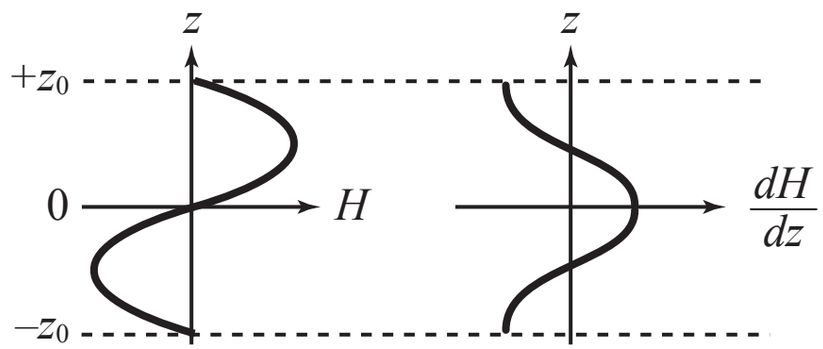
Helicity turbulence model



DNS set-up

(Yokoi & Brandenburg, Phys. Rev. E, 2016)





Set-up of the turbulence and rotation $\boldsymbol{\omega}_{\rm F}$ (left), the schematic spatial profile of the turbulent helicity H ($= \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$) (center) and its derivative dH/dz (right).

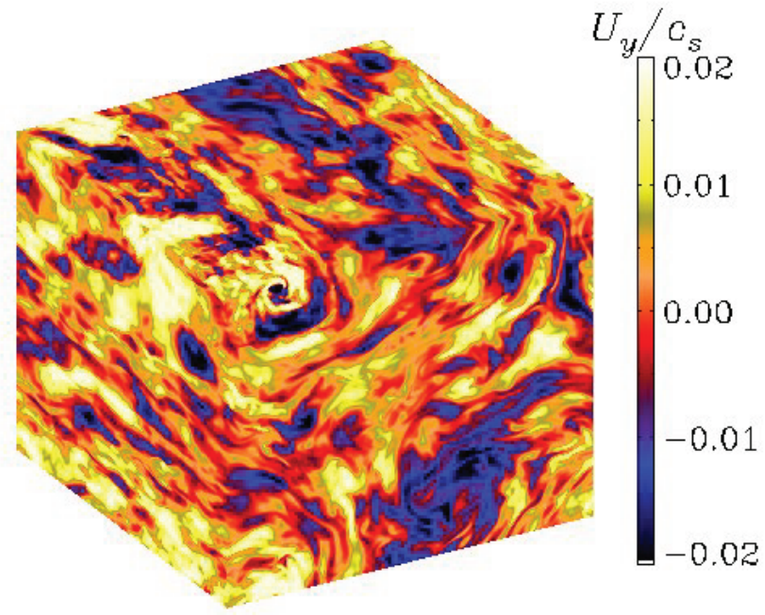
Rotation $\quad \boldsymbol{\omega}_{\rm F} = (\omega_{\rm F}^x, \omega_{\rm F}^y, \omega_{\rm F}^z) = (0, \omega_{\rm F}, 0)$

Inhomogeneous turbulent helicity $\quad H(z) = H_0 \sin(\pi z / z_0)$

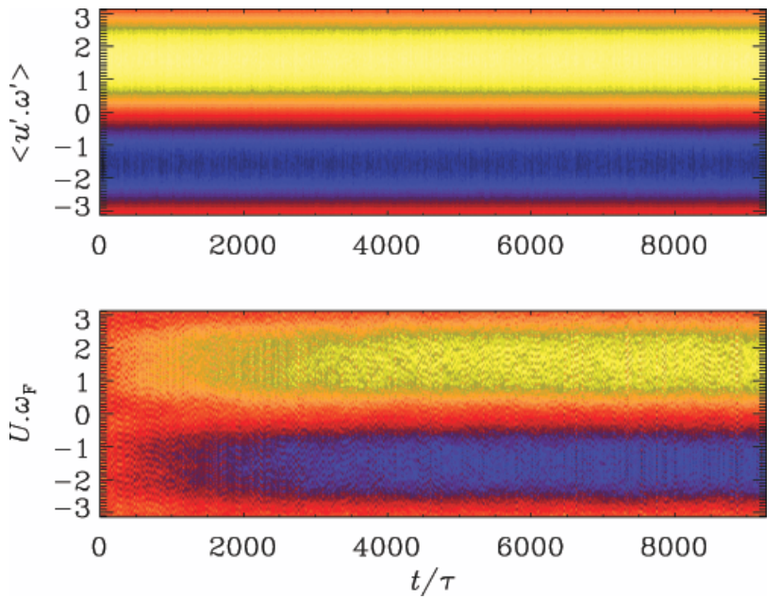
Run	$k_{\rm f}/k_1$	Re	Co	$\eta/(\nu_{\rm T}\tau^2)$
A	15	60	0.74	0.22
B1	5	150	2.6	0.27
B2	5	460	1.7	0.27
B3	5	980	1.6	0.51
C1	30	18	0.63	0.50
C2	30	80	0.55	0.03
C3	30	100	0.46	0.08

Summary of DNS results

Global flow generation



Axial flow component *Uy* on the periphery of the domain



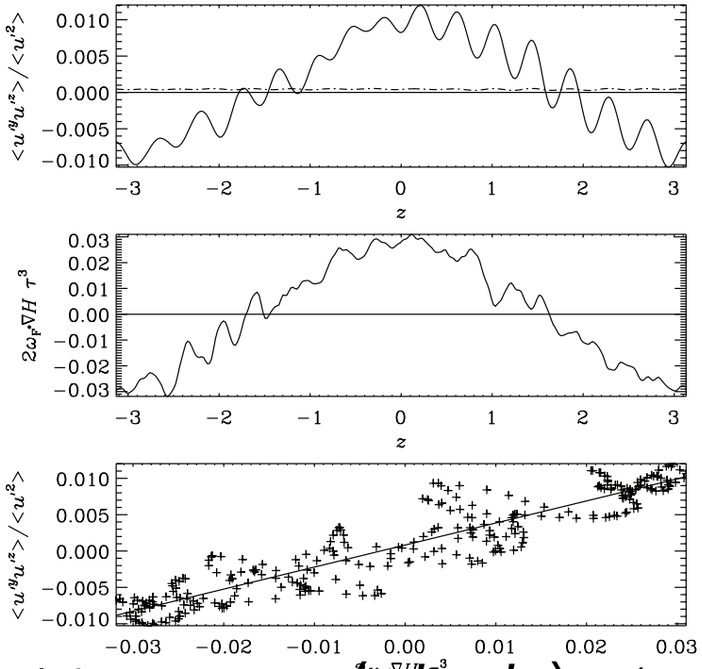
Turbulent helicity $\langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$ (top) and mean-flow helicity $\mathbf{U} \cdot 2\boldsymbol{\omega}_F$ (bottom)

Reynolds stress

$$\left\langle u'^{\alpha} u'^{\beta} \right\rangle_{\mathrm{D}} = -\nu_{\mathrm{T}} \mathcal{S}^{\alpha\beta} + \left[\Gamma^{\alpha} \left(\Omega^{\beta} + 2\omega_{\mathrm{F}}^{\beta} \right) + \Gamma^{\beta} \left(\Omega^{\alpha} + 2\omega_{\mathrm{F}}^{\alpha} \right) \right]_{\mathrm{D}}$$

Early stage

$$\langle u'^{y} u'^{z} \rangle = \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

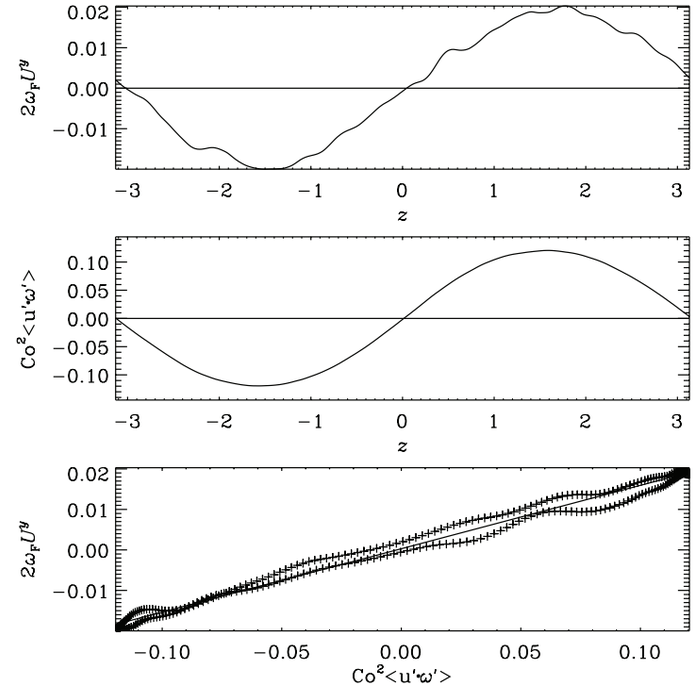


Reynolds stress $\langle u'^{y} u'^{z} \rangle$ (top), helicity-effect term $(\nabla H)^{z}\, 2\omega_{\mathrm{F}}{}^{y}$ (middle), and their correlation (bottom).

Developed stage

$$\langle u'^{y} u'^{z} \rangle = -\nu_{\mathrm{T}} \frac{\partial U^{y}}{\partial z} + \eta 2\omega_{\mathrm{F}}^{y} \frac{\partial H}{\partial z}$$

$$U^{y} = (\eta/\nu_{\mathrm{T}})\, 2\omega_{\mathrm{F}}^{y} H$$



Mean axial velocity U^{y} (top), turbulent helicity multiplied by rotation $2\omega_{\mathrm{F}} H$ (middle), and their correlation (bottom).

Physical origin

Reynolds stress $\mathcal{R}^{ij} \equiv \langle u'^i u'^j \rangle$

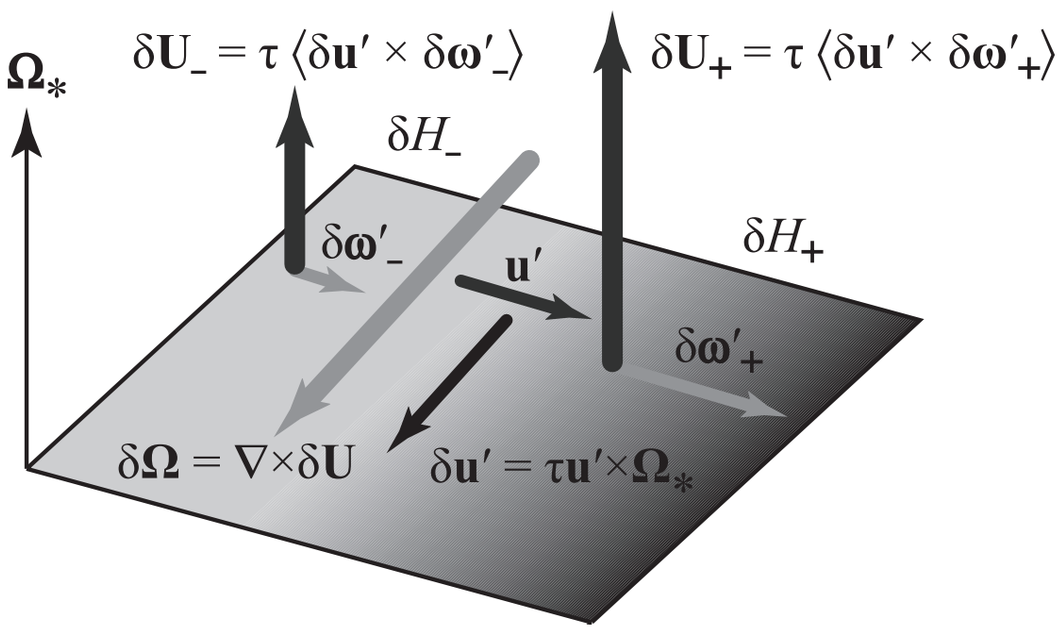
Vortexmotive force $\mathbf{V}_\mathrm{M} \equiv \langle \mathbf{u}' \times \boldsymbol{\omega}' \rangle$

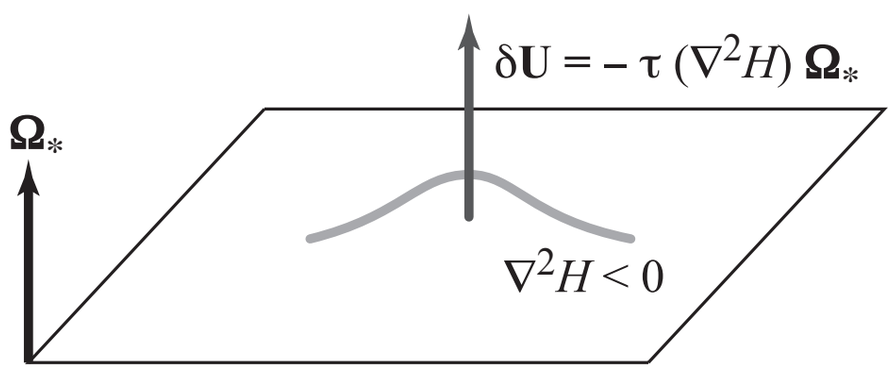
$$V^i_\mathrm{M} = -\frac{\partial \mathcal{R}^{ij}}{\partial x^j} + \frac{\partial K}{\partial x^i}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{U} \times \boldsymbol{\Omega}) + \nabla \times \mathbf{V}_\mathrm{M} + \nu \nabla^2 \boldsymbol{\Omega}$$

$$\mathbf{V}_\mathrm{M} = -D_\Gamma 2\boldsymbol{\omega}_\mathrm{F} - \nu_\mathrm{T} \nabla \times \boldsymbol{\Omega} \qquad D_\Gamma = \nabla \cdot \boldsymbol{\Gamma} \propto \nabla^2 H$$

$$\rightarrow \quad \delta\mathbf{U} \sim -(\nabla^2 H)\boldsymbol{\Omega}_* \qquad \nabla^2 H \simeq -\frac{\delta H}{\ell^2} = -\frac{\langle \mathbf{u}' \cdot \delta\boldsymbol{\omega}' \rangle}{\ell^2}$$





Mean flow induction consuming eminently localised turbulent helicity

Angular-momentum transport in the solar convection zone

Angular momentum around the rotation axis

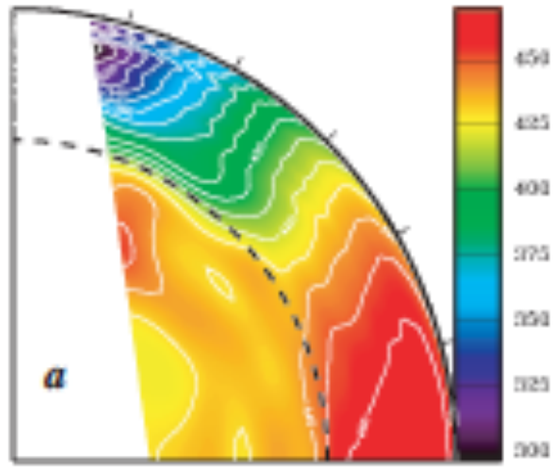
$$L = \Gamma r^2 \omega_{\mathrm{F}} + \Gamma r U^{\phi} \qquad \Gamma = \sin\theta$$

$$\frac{\partial}{\partial t}\rho L + \nabla \cdot (\rho \mathbf{F}_L) = 0$$

Vector flux of angular momentum $\mathbf{F}_L$

$$F_L^r = L U^r + r\Gamma \mathcal{R}^{r\phi}$$

$$F_L^\theta = L U^\theta + r\Gamma \mathcal{R}^{\theta\phi}$$

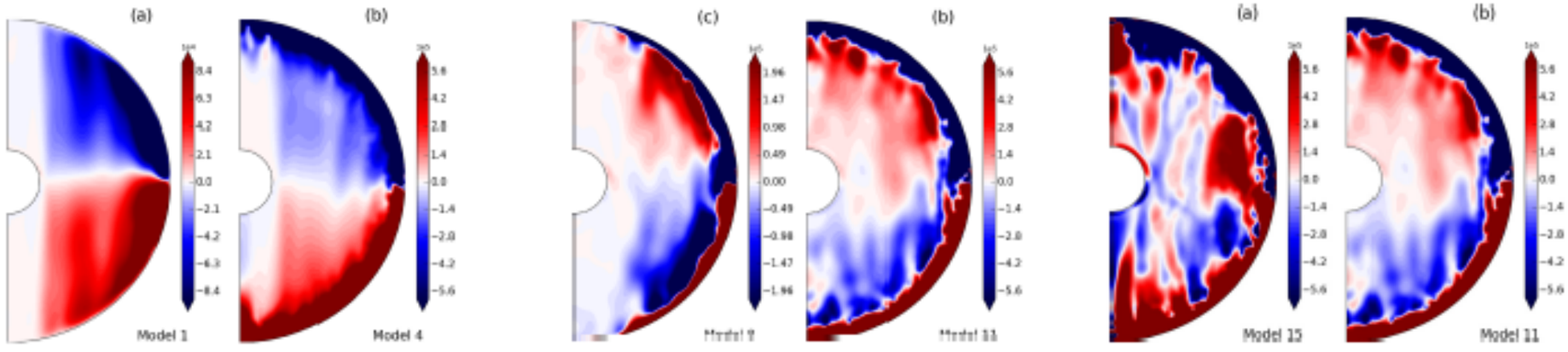


Miesch (2005) Liv. Rev. Sol. Phys. 2005-1

Helicity effect $\mathcal{R}_H^{r\phi} = +\dfrac{\partial H}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial}{\partial r} r U^\theta - \dfrac{1}{r}\dfrac{\partial U^r}{\partial \theta}\right)$

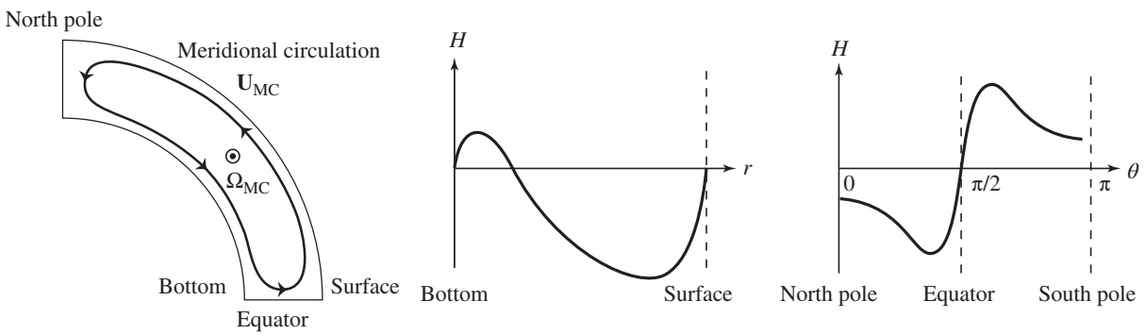
$$\mathcal{R}_H^{\theta\phi} = +\frac{1}{r}\frac{\partial H}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r} r U^\theta - \frac{1}{r}\frac{\partial U^r}{\partial \theta}\right)$$

Helicity effect in the stellar convection zone

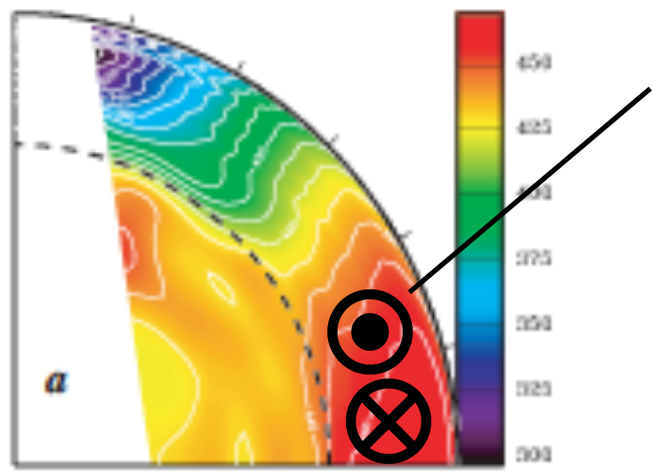


Duarte, et al, (2016) MNRAS **456**, 1708

Schematic helicity distribution



$$\nabla^2 H > 0$$



Mean vorticity due to meridional circulation

$\boldsymbol{\Omega}_{\mathrm{MC}}$

$$\mathbf{U} \sim -\tau_U (\nabla^2 H) \boldsymbol{\Omega}_{\mathrm{MC}}$$

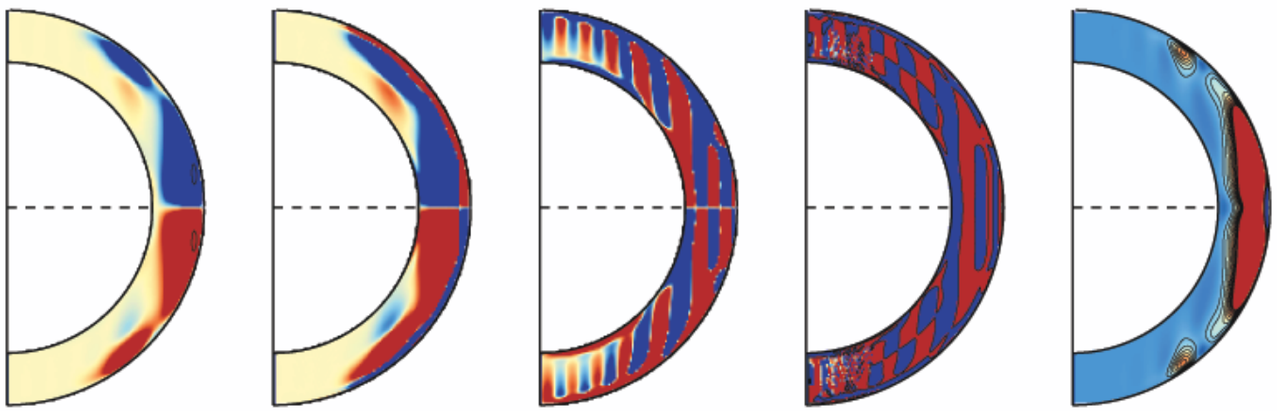
Prograde mean velocity induction

Helicity effect in the Reynolds stress

Helicity | Helicity Gradient | Azimuthal Vorticity | Helicity effect | Reynolds stress

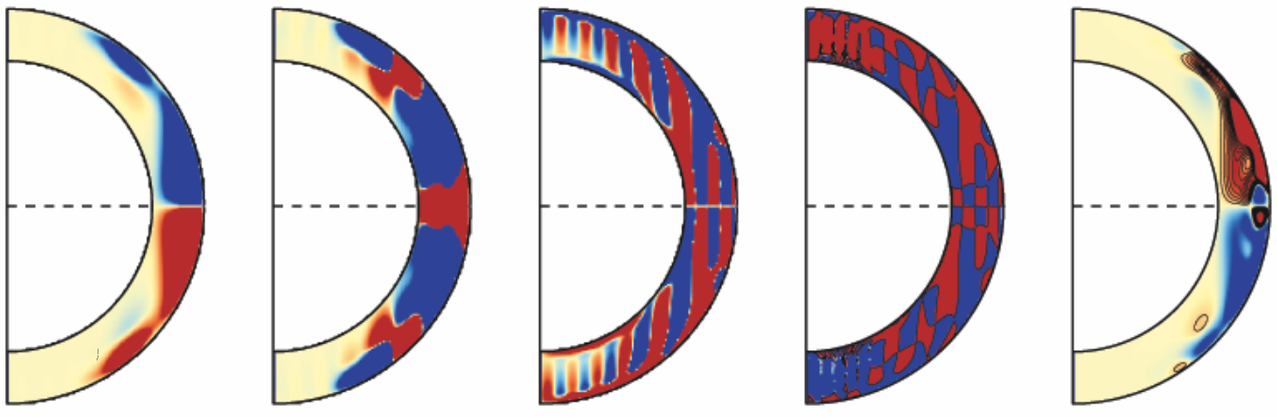
$C_\eta \tau \ell^2 |(\nabla^2 H)\Omega_*|$ Yokoi & Miesch (2025)

$\mathbf{u}\cdot\boldsymbol{\omega}$ $\quad \dfrac{\partial H}{\partial r}$ $\quad \overline{\Omega}^\phi$ $\quad \dfrac{\partial H}{\partial r}\overline{\Omega}^\phi$ $\quad \overline{u'^r u'^\phi}$



$\overline{\mathbf{u}}\cdot\overline{\boldsymbol{\omega}} - \overline{\mathbf{u}}\cdot\overline{\boldsymbol{\omega}}$ $\quad \dfrac{1}{r}\dfrac{\partial H}{\partial \theta}$ $\quad \overline{\Omega}^\phi$ $\quad \dfrac{1}{r}\dfrac{\partial H}{\partial \theta}\overline{\Omega}^\phi$ $\quad \overline{u'^\theta u'^\phi}$

$(\equiv H)$



(Yokoi & Miesch, GAFD, 2025)

Solar parameters

$$v \sim 200\ \mathrm{m\ s^{-1}} = 2\times 10^4 \mathrm{cm\ s^{-1}}$$
$$\ell \sim 200\ \mathrm{Mm} = 2\times 10^{10}\mathrm{cm}$$
$$\tau \sim \ell/v \sim 10^6\ \mathrm{s}$$

$r\phi$ component

$$\left|\overline{u'^r u'^\phi}\right| \sim 1.2\times 10^9$$
$$\left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 9.4\times 10^{-15}$$
$$\tau\ell^2\left|\frac{\partial H}{\partial r}\overline{\Omega}^\phi\right| \sim 10^{12} \longrightarrow 10^9$$

with $C_\eta = O(10^{-3})$

$\theta\phi$ component

$$\left|\overline{u'^\theta u'^\phi}\right| \sim 5.6\times 10^8$$
$$\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 2.6\times 10^{-15}$$
$$\tau\ell^2\left|\frac{1}{r}\frac{\partial H}{\partial \theta}\overline{\Omega}^\phi\right| \sim 10^{11} \longrightarrow 10^8$$

Magnitude same as the Reynolds stress

Validation of the helicity SGS model by DNSs

Yokoi, Mininni, Pouquet, Rosenberg & Marino, Phys. Fluids (to be submitted)

Problem of Constant Adjustment

Smagorinsky model $\nu_{\mathrm{S}} = (C_{\mathrm{S}}\Delta)^2 S$

Smagorinsky constant needs to be adjusted such as

Isotropic flow	$C_{\mathrm{S}} \simeq 0.18$
Mixing-layer flow	$C_{\mathrm{S}} \simeq 0.15$
Channel flow	$C_{\mathrm{S}} \simeq 0.1$

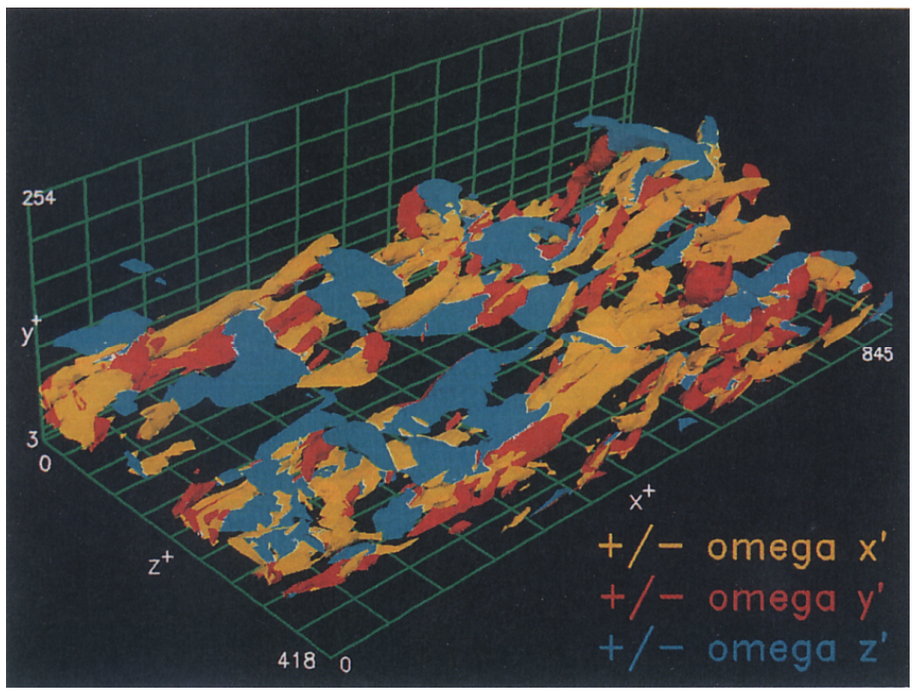
To alleviate

- Dynamic procedure to determine the coefficient C_S
- Alternatives to the generic form

$$\mathcal{T}_{\alpha\beta} = C f_{\alpha\beta}\left(\nabla\overline{\mathbf{u}}; \Delta\right) \longrightarrow \mathcal{T}_{\alpha\beta} = C f_{\alpha\beta}\left(\nabla\overline{\mathbf{u}}; \Delta, \cdots\right)$$

- Evolution equations of the SGS quantities

Implication to SGS modelling



Fluctuating vorticity
(Robinson, Kline & Spalart 1988)

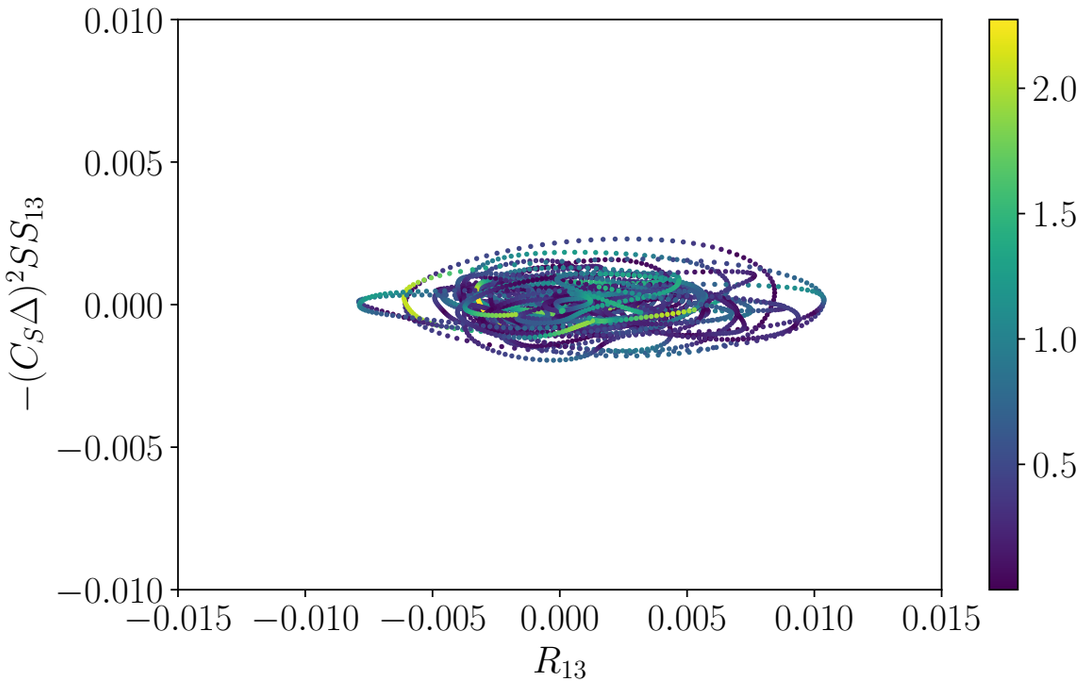
	Smagorinsky constant C_S	Dissipative nature	Vortical structures
Isotropic flow	0.18	more	weak
Mixing-layer flow	0.15	↕	intermediate
Channel flow	0.1	less	strong

Coherent vortical structures (streetwise vorticity)
may be related to the less dissipative nature

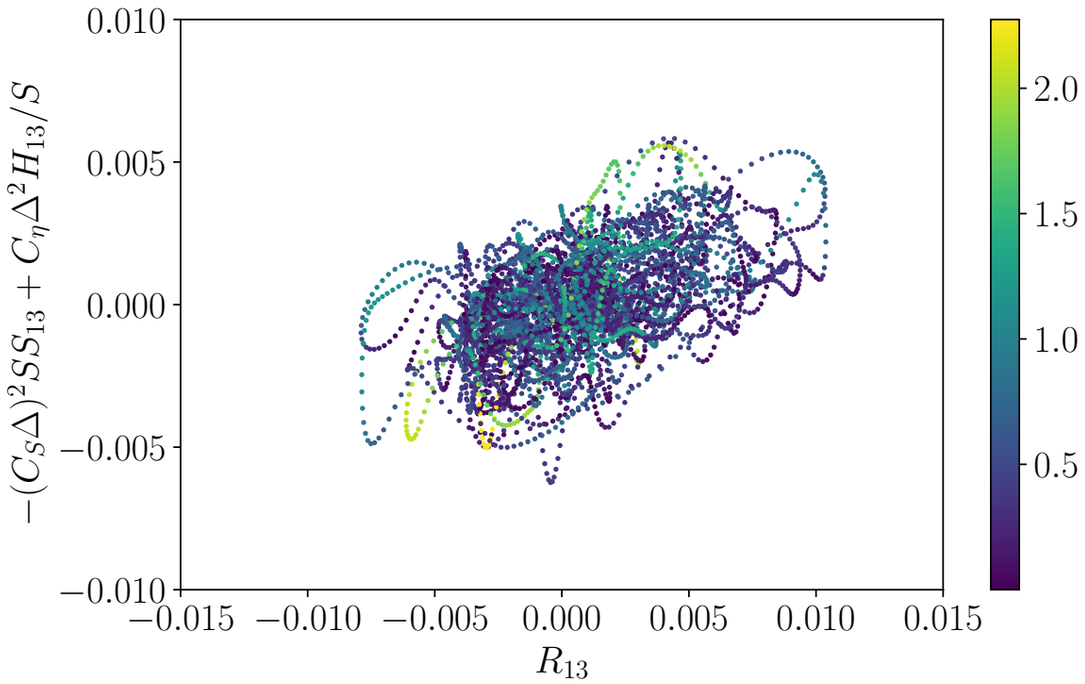
SGS stress

HIT, with a Gaussian filter at kc=14

Smagorinsky model



With helicity correction



$$R_{13} = -(C_\mathrm{S}\Delta)^2 S S_{13}$$

$$R_{13} = -(C_\mathrm{S}\Delta)^2 S S_{13} + C_\eta \Delta^2 H_{13}/S$$

SGS helicity correction improves the SGS stress evaluation

Cross-response effects in dynamos

Yokoi, N. “Unappreciated cross-helicity effects in plasma physics: Anti-diffusion effects in dynamo and momentum transport,” **Rev. Mod. Plasma Phys. 7, 33-1-98 (2023)** https://doi.org/10.1007/s41614-023-00133-4

Mizerski, K., Yokoi, N. & Brandenburg, A. “Cross-helicity effect on α-type dynamo in non-equilibrium turbulence,” **J. Plasma Phys. 89, 905890412 (2023)** https://doi.org/10.1017/S0022377823000545

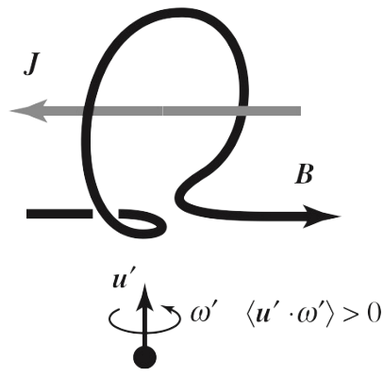
Modelling in dynamos

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \frac{\partial B^a}{\partial x^b} + \cdots$$

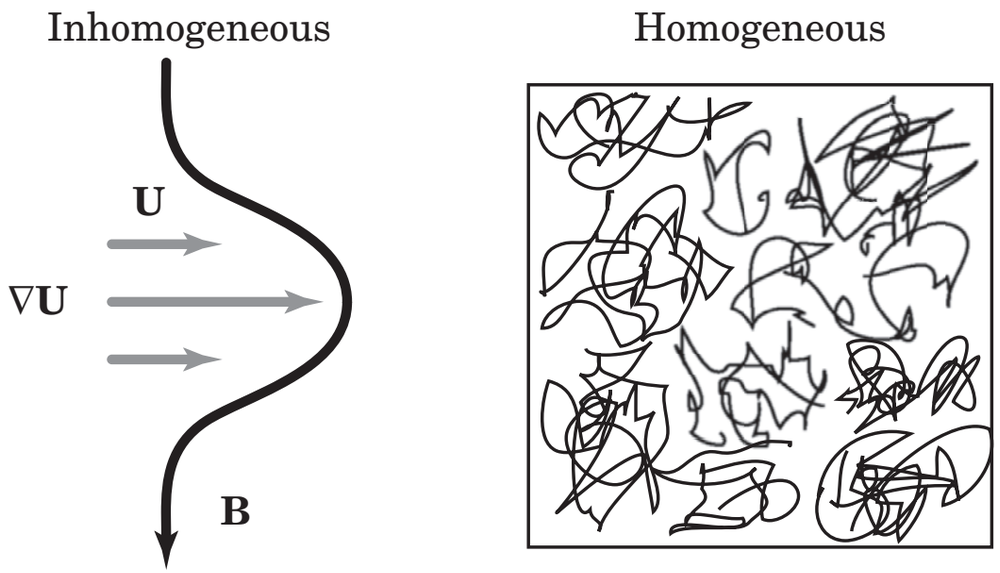
Mean field

$$\mathbf{b} = \mathbf{B} + \mathbf{b}', \quad \mathbf{B} = \langle \mathbf{b} \rangle$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{U} \times \mathbf{B}) + \nabla \times \langle \mathbf{u}' \times \mathbf{b}' \rangle + \eta \nabla^2 \mathbf{B}$$



$(\mathbf{B} \cdot \nabla)\mathbf{U}$ → differential rotation, "Ω effect"



Turbulence

$$\mathbf{U} = \mathbf{U}_0 (\text{constant}) \text{ or } \mathbf{0}$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u}' = (\mathbf{B} \cdot \nabla)\mathbf{b}' + (\mathbf{b}' \cdot \nabla)\mathbf{B} - (\mathbf{u}' \cdot \nabla)\mathbf{U} + \cdots$$

$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{b}' = (\mathbf{B} \cdot \nabla)\mathbf{u}' - (\mathbf{u}' \cdot \nabla)\mathbf{B} + (\mathbf{b}' \cdot \nabla)\mathbf{U} + \cdots$$

→ $\langle \mathbf{u}' \times \mathbf{b}' \rangle^\alpha = \alpha^{\alpha a} B^a + \beta^{\alpha a b} \frac{\partial B^a}{\partial x^b} + \cdots$ "Ansatz"

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{u}' = (\mathbf{B}\cdot\nabla)\mathbf{b}' + (\mathbf{b}'\cdot\nabla)\mathbf{B} - (\mathbf{u}'\cdot\nabla)\mathbf{U} + \cdots$$

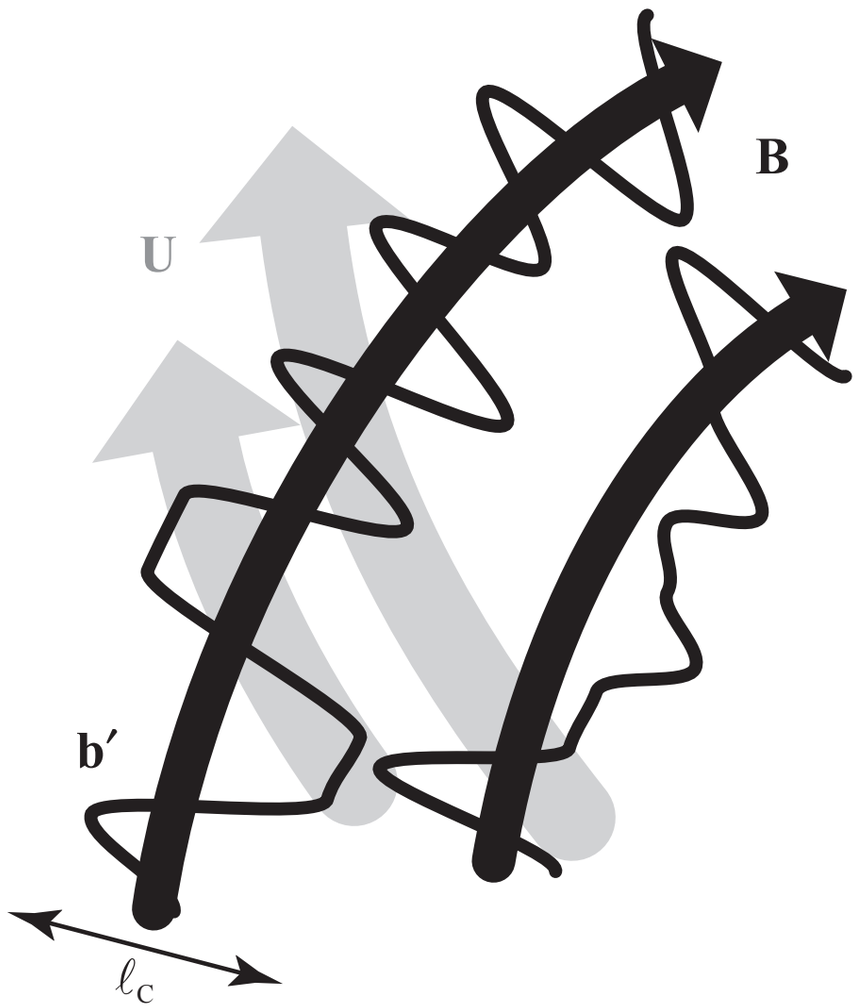
$$\frac{\partial \mathbf{b}'}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{b}' = (\mathbf{B}\cdot\nabla)\mathbf{u}' - (\mathbf{u}'\cdot\nabla)\mathbf{B} + (\mathbf{b}'\cdot\nabla)\mathbf{U} + \cdots$$

$$\left\langle \frac{\partial \mathbf{u}'}{\partial t} \times \mathbf{b}' \right\rangle + \left\langle \mathbf{u}' \times \frac{\partial \mathbf{b}'}{\partial t} \right\rangle = \cdots$$

$$\tau \langle \mathbf{u}' \times [(\mathbf{b}'\cdot\nabla)\mathbf{U}] + [(\mathbf{u}'\cdot\nabla)\mathbf{U}] \times \mathbf{b}' \rangle^{\alpha}$$

$$= \epsilon^{\alpha a b} \tau \langle u'^a b'^c \rangle \frac{\partial U^b}{\partial x^c} - \epsilon^{\alpha b a} \tau \langle b'^a u'^c \rangle \frac{\partial U^b}{\partial x^c}$$

$$= \tau \left(\langle u'^a b'^c \rangle + \langle u'^c b'^a \rangle \right) \epsilon^{\alpha a b} \frac{\partial U^b}{\partial x^c}$$



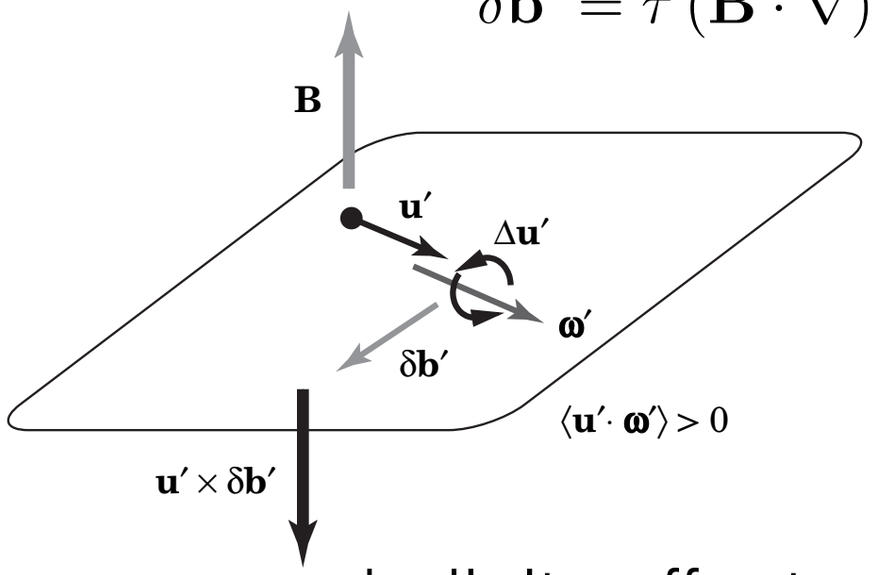
$$\rightarrow \quad \langle \mathbf{u}' \times \mathbf{b}' \rangle = \cdots + \tau \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \times \mathbf{U} + \cdots$$

cross helicity

α and cross-helicity effects

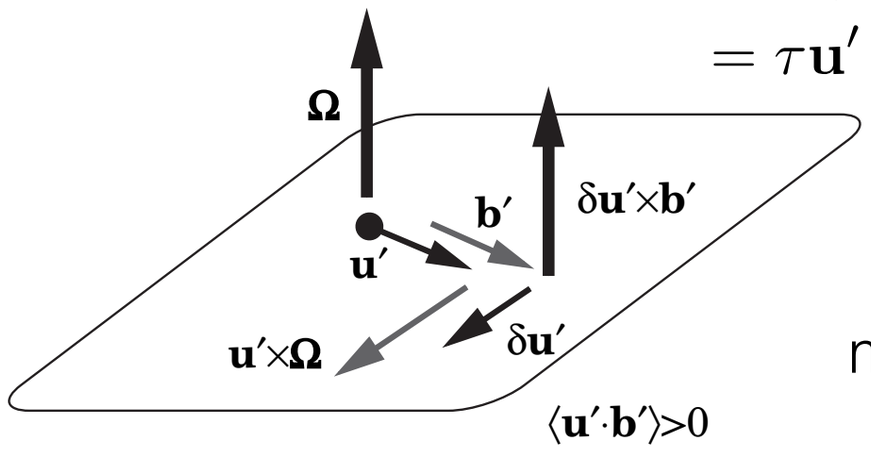
(Yokoi, GAFD **107**, 114, 2013)

$$\delta \mathbf{b}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{u}' + \cdots$$



helicity effect

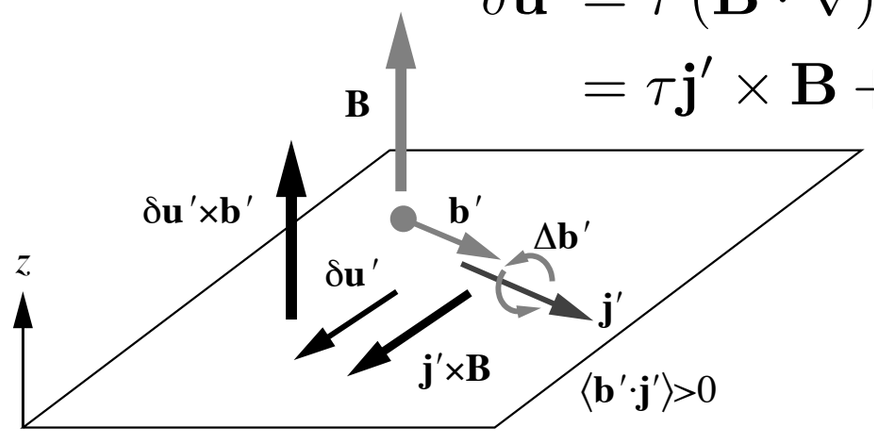
$$\delta \mathbf{u}' = \tau \left(\mathbf{u}' \cdot \nabla \right) \mathbf{U} + \cdots$$
$$= \tau \mathbf{u}' \times \mathbf{\Omega} + \cdots$$



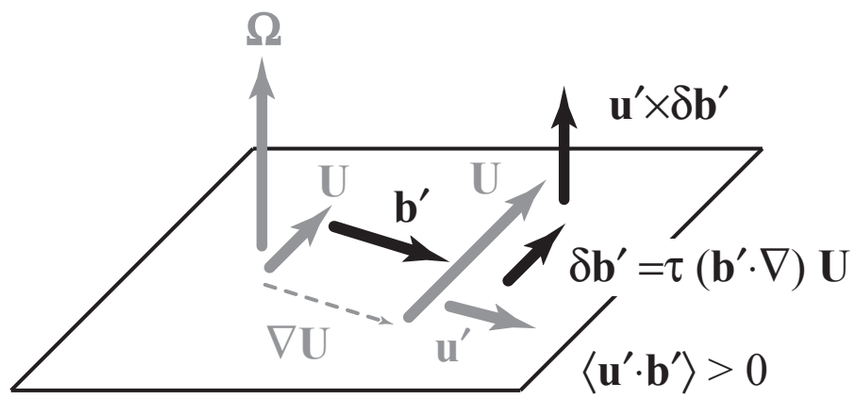
mean vorticity
$\mathbf{\Omega} = \nabla \times \mathbf{U}$

cross-helicity effect

$$\delta \mathbf{u}' = \tau \left(\mathbf{B} \cdot \nabla \right) \mathbf{b}' + \cdots$$
$$= \tau \mathbf{j}' \times \mathbf{B} + \cdots$$



$$\delta \mathbf{b}' = \tau \left(\mathbf{b}' \cdot \nabla \right) \mathbf{U} + \cdots$$



α dynamo

$$\langle \mathbf{u}' \times \mathbf{b}' \rangle = \alpha \mathbf{B} - \beta \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U}$$

cross-helicity dynamo

Global flow generation by cross helicity

Reynolds and turbulent Maxwell stress

Theoretical Result

eddy viscosity inhomogeneous helicity

$$\langle \mathbf{u}'\mathbf{u}' - \mathbf{b}'\mathbf{b}' \rangle_{\mathrm{D}} = -\nu_{\mathrm{K}} \boldsymbol{\mathcal{S}} + \nu_{\mathrm{M}} \boldsymbol{\mathcal{M}} + \eta_H \boldsymbol{\Omega}_* \nabla H + \cdots$$

cross helicity

D: deviatoric part

$\boldsymbol{\mathcal{S}}$: mean velocity strain $\quad \boldsymbol{\mathcal{S}} = \nabla\mathbf{U} + (\nabla\mathbf{U})^\dagger$

$\boldsymbol{\mathcal{M}}$: mean magnetic-field strain $\quad \boldsymbol{\mathcal{M}} = \nabla\mathbf{B} + (\nabla\mathbf{B})^\dagger$

$\boldsymbol{\Omega}_*$: absolute mean vorticity (mean vorticity + rotation)

cf. $\langle \mathbf{u}' \times \mathbf{b}' \rangle = -\eta_{\mathrm{T}} \nabla \times \mathbf{B} + \gamma \nabla \times \mathbf{U} + \alpha \mathbf{B} + \cdots$

Turbulent cross helicity coupled with **mean magnetic-field strain** may contribute to **transport suppression** and/or **global flow generation** against the eddy-viscosity effect

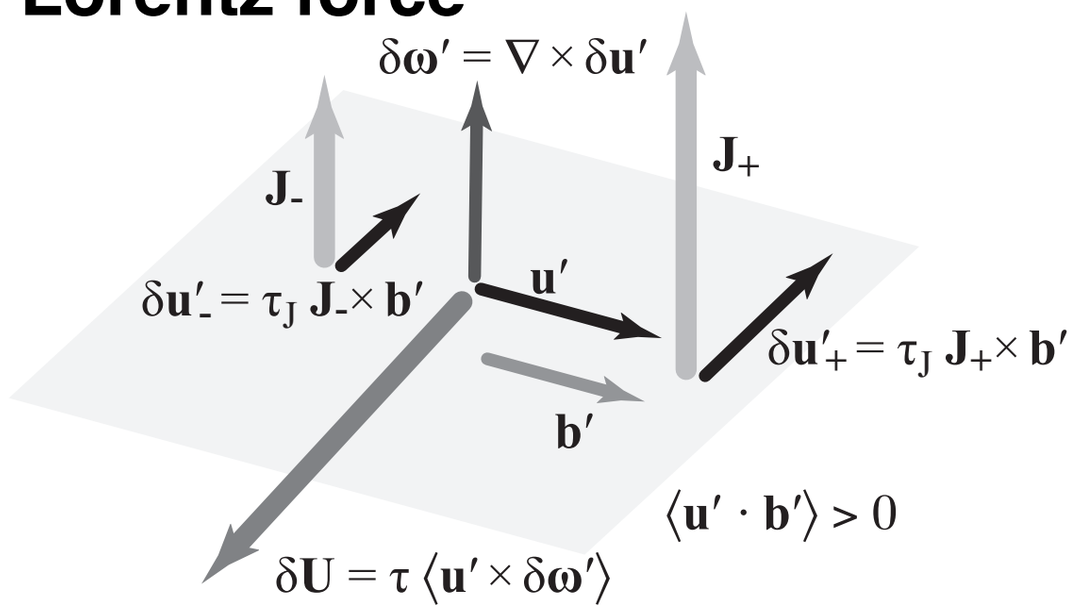
Physical interpretation of large-scale flow generation by cross helicity

Velocity fluctuation
induced by **fluctuating Lorentz force**

$$\delta\mathbf{u}' = \tau_J \mathbf{J} \times \mathbf{b}'$$

Associated vorticity

$$\begin{aligned}\delta\boldsymbol{\omega}' &= \nabla \times \delta\mathbf{u}' \\ &= \tau_J \nabla \times (\mathbf{J} \times \mathbf{b}') \\ &\simeq \tau_J (\mathbf{b}' \cdot \nabla)\mathbf{J}\end{aligned}$$



Non-trivial mean electric-current distribution (Inhomogeneous **J**) is required

$$\delta\mathbf{U} = \tau \langle \mathbf{u}' \times \delta\boldsymbol{\omega}' \rangle \propto \langle \mathbf{u}' \cdot \mathbf{b}' \rangle \nabla \times \mathbf{J}$$

in the direction of $\nabla \times \mathbf{J}$

Theoretical formulation (e.g., incompressible MHD)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = (\mathbf{b}\cdot\nabla)\mathbf{b} - \nabla p_{\mathrm{M}} - 2\boldsymbol{\omega}_{\mathrm{F}}\times\mathbf{u} + \nu\nabla^2\mathbf{u}$$

$$\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} = (\mathbf{b}\cdot\nabla)\mathbf{u} + \eta\nabla^2\mathbf{b}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0$$

Multiple-scale analysis $\quad \boldsymbol{\xi}(=\mathbf{x}),\ \mathbf{X}(=\delta\mathbf{x}),\ \tau(=t),\ T(=\delta t)$

$$\nabla_{\mathbf{x}} = \nabla_{\boldsymbol{\xi}} + \delta\nabla_{\mathbf{X}};\quad \frac{\partial}{\partial t} = \frac{\partial}{\partial\tau} + \delta\frac{\partial}{\partial T}$$

$$f(\mathbf{x};t) = F(\mathbf{X};T) + f'(\boldsymbol{\xi},\mathbf{X};\tau,T)$$

Two-scale equations in configuration space

$$\frac{\partial u'^i}{\partial\tau} + U^j\frac{\partial u'^i}{\partial\xi^j} + \frac{\partial}{\partial\xi^j}(u'^j u'^i - b'^j b'^i) + \frac{\partial p'_{\mathrm{M}}}{\partial\xi^i} - \nu\frac{\partial^2 u'^i}{\partial\xi^j\partial\xi^j} - B^j\frac{\partial b'^i}{\partial\xi^j}$$

$$= \delta\left[b'^j\frac{\partial B^i}{\partial X^j} - u'^j\left(\frac{\partial U^i}{\partial X^j} + \epsilon^{jik}\Omega_0^k\right) + B^j\frac{\partial b'^i}{\partial X^j} - \frac{\overline{D}u'^i}{DT} - \frac{\partial}{\partial X^j}\left(u'^j u'^i - b'^j b'^i - \langle u'^j u'^i - b'^j b'^i\rangle\right) - \frac{\partial p'_{\mathrm{M}}}{\partial X^i}\right]$$

$$\frac{\partial u'^i}{\partial\xi^i} + \delta\frac{\partial u'^i}{\partial X^i} = 0,$$

$$\frac{\partial b'^i}{\partial\tau} + U^j\frac{\partial b'^i}{\partial\xi^j} + \frac{\partial}{\partial\xi^j}(u'^j b'^i - b'^j u'^i) - \eta\frac{\partial^2 b'^i}{\partial\xi^j\partial\xi^j} - B^j\frac{\partial u'^i}{\partial\xi^j}$$

$$= \delta\left[u'^j\frac{\partial B^i}{\partial X^j} - b'^j\left(\frac{\partial U^i}{\partial X^j} + \epsilon^{jik}\Omega_0^k\right) + B^j\frac{\partial u'^i}{\partial X^j} - \frac{\overline{D}b'^i}{DT} - \frac{\partial}{\partial X^j}\left(u'^j b'^i - b'^j u'^i - \langle u'^j b'^i - b'^j u'^i\rangle\right)\right]$$

$$\frac{\partial b'^i}{\partial\xi^i} + \delta\frac{\partial b'^i}{\partial X^i} = 0$$

inhomogeneities,
anisotropies,
non-equilibrium properties

Scale parameter expansion

$$f^i(\mathbf{k};\tau) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \delta^n f^i_{nm}(\mathbf{k};\tau) - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \delta^{n+1} i \frac{k^i}{k^2} \frac{\partial}{\partial X^j_{\mathrm{I}}} f^j_{nm}(\mathbf{k};\tau)$$

Basic-field (lowest-order field) equations

$$\frac{\partial u^i_{00}(\mathbf{k};\tau)}{\partial \tau} + \nu k^2 u^i_{00}(\mathbf{k};\tau)$$
$$- iM^{ijk}(\mathbf{k}) \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \left[u^j_{00}(\mathbf{p};\tau) u^k_{00}(\mathbf{q};\tau) - b^j_{00}(\mathbf{p};\tau) b^k_{00}(\mathbf{q};\tau) \right] = 0$$

$$\frac{\partial b^i_{00}(\mathbf{k};\tau)}{\partial \tau} + \eta k^2 b^i_{00}(\mathbf{k};\tau)$$
$$- iN^{ijk}(\mathbf{k}) \iint d\mathbf{p}d\mathbf{q}\ \delta(\mathbf{k}-\mathbf{p}-\mathbf{q}) \left[u^j_{00}(\mathbf{p};\tau) b^k_{00}(\mathbf{q};\tau) - b^j_{00}(\mathbf{p};\tau) u^k_{00}(\mathbf{q};\tau) \right] = 0$$

Projection operators

$$M^{ijk}(\mathbf{k}) = \frac{1}{2}\left[k^j D^{ik}(\mathbf{k}) + k^k D^{ij}(\mathbf{k}) \right], \quad D^{ij}(\mathbf{k}) = \delta^{ij} - \frac{k^i k^j}{k^2}$$

$$N^{ijk}(\mathbf{k}) = k^j \delta^{ik} - k^k \delta^{ij}$$

$f_{01}(\mathbf{k};\tau)$ equation

$$
\begin{pmatrix} \dfrac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \dfrac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix}
\begin{pmatrix} u^i_{01}(\mathbf{k};\tau) \\ \\ b^i_{01}(\mathbf{k};\tau) \end{pmatrix}
$$

$$
+i \begin{pmatrix} -2M^{ikm}(\mathbf{k})\displaystyle\int_\Delta u^k_{00}(\mathbf{p};\tau) & 2M^{ikm}(\mathbf{k})\displaystyle\int_\Delta b^k_{00}(\mathbf{p};\tau) \\ \\ N^{ikm}(\mathbf{k})\displaystyle\int_\Delta b^k_{00}(\mathbf{p};\tau) & -N^{ikm}(\mathbf{k})\displaystyle\int_\Delta u^k_{00}(\mathbf{p};\tau) \end{pmatrix}
\begin{pmatrix} u^m_{01}(\mathbf{q};\tau) \\ \\ b^m_{01}(\mathbf{q};\tau) \end{pmatrix}
$$

$$
= -ik^j B^j \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} u^i_{00}(\mathbf{k}) \\ \\ b^i_{00}(\mathbf{k}) \end{pmatrix}
$$

with $\displaystyle\int_\Delta = \iint d\mathbf{p}d\mathbf{q}\;\delta(\mathbf{k}-\mathbf{p}-\mathbf{q})$

$f_{10}(\mathbf{k};\tau)$ equation

$$
\begin{pmatrix} \dfrac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \dfrac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix}
\begin{pmatrix} u^i_{10}(\mathbf{k};\tau) \\ \\ b^i_{10}(\mathbf{k};\tau) \end{pmatrix}
$$

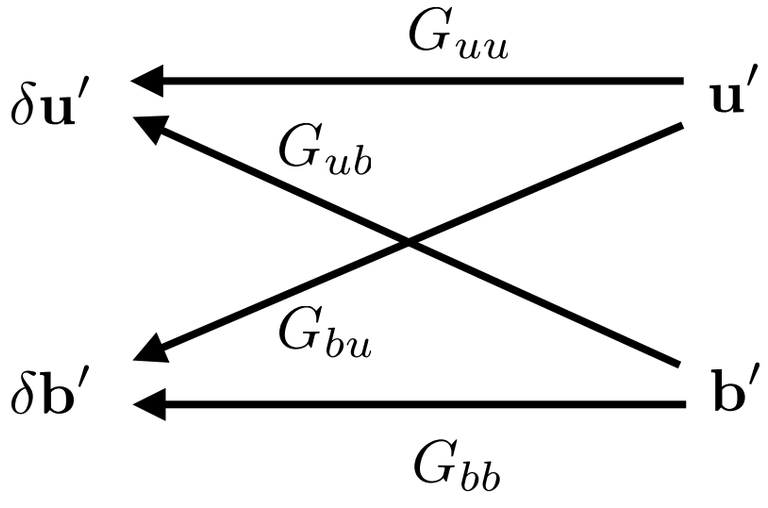
$$
+i \begin{pmatrix} -2M^{ikm}(\mathbf{k})\displaystyle\int_\Delta u^k_{00}(\mathbf{p};\tau) & 2M^{ikm}(\mathbf{k})\displaystyle\int_\Delta b^k_{00}(\mathbf{p};\tau) \\ \\ N^{ikm}(\mathbf{k})\displaystyle\int_\Delta b^k_{00}(\mathbf{p};\tau) & -N^{ikm}(\mathbf{k})\displaystyle\int_\Delta u^k_{00}(\mathbf{p};\tau) \end{pmatrix}
\begin{pmatrix} u^m_{10}(\mathbf{q};\tau) \\ \\ b^m_{10}(\mathbf{q};\tau) \end{pmatrix}
$$

$$
= B^k \frac{\partial}{\partial X^k_{\mathrm{I}}} \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} u^i_{00}(\mathbf{k}) \\ \\ b^i_{00}(\mathbf{k}) \end{pmatrix}
- D^{jk}(\mathbf{k}) \frac{\overline{D}}{DT_{\mathrm{I}}} \begin{pmatrix} 1 & 0 \\ \\ 0 & 1 \end{pmatrix}
\begin{pmatrix} u^k_{00}(\mathbf{k}) \\ \\ b^k_{00}(\mathbf{k}) \end{pmatrix}
$$

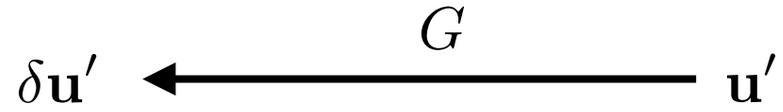
$$
+ \begin{pmatrix} -D^{jk}(\mathbf{k})\left(\dfrac{\partial U^k}{\partial X^m}+\epsilon^{mkn}\Omega^n_0\right) & D^{jk}(\mathbf{k})\dfrac{\partial B^k}{\partial X^m} \\ -D^{jk}(\mathbf{k})\dfrac{\partial B^k}{\partial X^m} & D^{jk}(\mathbf{k})\left(\dfrac{\partial U^k}{\partial X^m}+\epsilon^{mkn}\Omega^n_0\right) \end{pmatrix}
\begin{pmatrix} u^m_{00}(\mathbf{k}) \\ \\ b^m_{00}(\mathbf{k}) \end{pmatrix}
$$

Response functions in MHD

Cross-interaction responses in incompressible MHD



cf. Hydrodynamic case



Green's function equations

$$\begin{pmatrix} \frac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \frac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix} \begin{pmatrix} G^{ij}_{uu} & G^{ij}_{bu} \\ G^{ij}_{ub} & G^{ij}_{bb} \end{pmatrix}$$
$$+i \begin{pmatrix} -2M^{ikm}\int_\Delta u^k_{00} & 2M^{ikm}\int_\Delta b^k_{00} \\ N^{ikm}\int_\Delta b^k_{00} & -N^{ikm}\int_\Delta u^k_{00} \end{pmatrix} \begin{pmatrix} G^{mj}_{uu} & G^{mj}_{bu} \\ G^{mj}_{ub} & G^{mj}_{bb} \end{pmatrix} = \delta^{ij}\delta(\tau-\tau') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$f_{10}(\mathbf{k};\tau)$ solution

$$u^i_{10}(\mathbf{k};\tau) = \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{uu}(\mathbf{k};\tau,\tau_1) \left[-D^{jk}(\mathbf{k}) \frac{\overline{D} u^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} - D^{jk}(\mathbf{k}) \left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn} \Omega^n_0 \right) u^m_{00}(\mathbf{k};\tau_1) \right.$$
$$\left. + \; B^k \frac{\partial b^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} + D^{jk}(\mathbf{k}) \frac{\partial B^k}{\partial X^m} b^m_{00}(\mathbf{k};\tau_1) \right]$$
$$+ \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{ub}(\mathbf{k};\tau,\tau_1) \left[B^k \frac{\partial u^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} - D^{jk}(\mathbf{k}) \frac{\partial B^k}{\partial X^m} u^m_{00}(\mathbf{k};\tau_1) \right.$$
$$\left. - D^{jk}(\mathbf{k}) \frac{\overline{D} b^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} + D^{jk}(\mathbf{k}) \left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn} \Omega_0 \right) b^m_{00}(\mathbf{k};\tau_1) \right]$$

$$b^i_{10}(\mathbf{k};\tau) = \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{bu}(\mathbf{k};\tau,\tau_1) \left[-D^{jk}(\mathbf{k}) \frac{\overline{D} u^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} - D^{jk}(\mathbf{k}) \left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn} \Omega^n_0 \right) u^m_{00}(\mathbf{k};\tau_1) \right.$$
$$\left. + \; B^k \frac{\partial b^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} + D^{jk}(\mathbf{k}) \frac{\partial B^k}{\partial X^m} b^m_{00}(\mathbf{k};\tau_1) \right]$$
$$+ \int_{-\infty}^{\tau} d\tau_1 \; G^{ij}_{bb}(\mathbf{k};\tau,\tau_1) \left[B^k \frac{\partial u^i_{00}(\mathbf{k};\tau_1)}{\partial X^k_{\mathrm{I}}} - D^{jk}(\mathbf{k}) \frac{\partial B^k}{\partial X^m} u^m_{00}(\mathbf{k};\tau_1) \right.$$
$$\left. - D^{jk}(\mathbf{k}) \frac{\overline{D} b^k_{00}(\mathbf{k};\tau_1)}{DT_{\mathrm{I}}} + D^{jk}(\mathbf{k}) \left(\frac{\partial U^k}{\partial X^m} + \epsilon^{mkn} \Omega_0 \right) b^m_{00}(\mathbf{k};\tau_1) \right]$$

Turbulent electromotive force (EMF)

$$E^i_{\rm M} \equiv \epsilon^{ijk}\langle u'^j b'^k\rangle = \epsilon^{ijk}\int d\mathbf{k}\ \langle u^j(\mathbf{k};\tau)b^k(\mathbf{k}';\tau)\rangle/\delta(\mathbf{k}+\mathbf{k}').$$

$$\langle u^j b^k\rangle = \langle u^j_{00}b^k_{00}\rangle + \langle u^j_{01}b^k_{00}\rangle + \langle u^j_{00}b^k_{01}\rangle + \delta\langle u^j_{10}b^k_{00}\rangle + \delta\langle u^j_{00}b^k_{10}\rangle + \cdots .$$

$\boldsymbol{\alpha}$ effect — Turb. Mag. Diffusivity — Turb. Pumping — Cross-helicity effect

$$\langle \mathbf{u}'\times\mathbf{b}'\rangle = \alpha\mathbf{B} - (\beta+\zeta)\nabla\times\mathbf{B} - (\nabla\zeta)\times\mathbf{B} + \gamma\nabla\times\mathbf{U}$$

$$I\{A,B\} = \int d\mathbf{k}\int_{-\infty}^{\tau} d\tau_1 A(k;\tau,\tau_1)B(k;\tau,\tau_1)$$

Cross interaction mediated by G_{ub} and G_{bu}

$$\alpha = \frac{1}{3}\left[-I\{G_{bb},H_{uu}\} + I\{G_{uu},H_{bb}\} - I\{G_{bu},H_{ub}\} + I\{G_{ub},H_{bu}\}\right]$$

$$\beta = \frac{1}{3}\left[I\{G_{bb},Q_{uu}\} + I\{G_{uu},Q_{bb}\} - I\{G_{bu},Q_{ub}\} - I\{G_{ub},Q_{bu}\}\right]$$

$$\zeta = \frac{1}{3}\left[I\{G_{bb},Q_{uu}\} - I\{G_{uu},Q_{bb}\} + I\{G_{bu},Q_{ub}\} - I\{G_{ub},Q_{bu}\}\right]$$

$$\gamma = \frac{1}{3}\left[I\{G_{bb},Q_{ub}\} + I\{G_{uu},Q_{bu}\} - I\{G_{bu},Q_{uu}\} - I\{G_{ub},Q_{bb}\}\right]$$

Dynamo due to the cross-interaction responses

Torsional cross correlations

$$\alpha_X = \frac{1}{3}\left[-I\{G_{bu}, H_{ub}\} + I\{G_{ub}, H_{bu}\}\right] \longrightarrow \alpha_X = -\tau_{bu}\langle \mathbf{u}' \cdot \mathbf{j}' \rangle + \tau_{ub}\langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle$$

$$\beta_X = \frac{1}{3}\left[-I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\}\right] \longrightarrow \beta_X = -(\tau_{bu} + \tau_{ub})\langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\zeta_X = \frac{1}{3}\left[+I\{G_{bu}, Q_{ub}\} - I\{G_{ub}, Q_{bu}\}\right] \longrightarrow \zeta_X = (\tau_{bu} - \tau_{ub})\langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\gamma_X = \frac{1}{3}\left[-I\{G_{bu}, Q_{uu}\} - I\{G_{ub}, Q_{bb}\}\right] \longrightarrow \gamma_X = -\tau_{bu}\langle \mathbf{u}'^2 \rangle - \tau_{ub}\langle \mathbf{b}'^2 \rangle$$

- Vanish if $G_{ub} = G_{bu} = 0$ No cross-interaction response effects

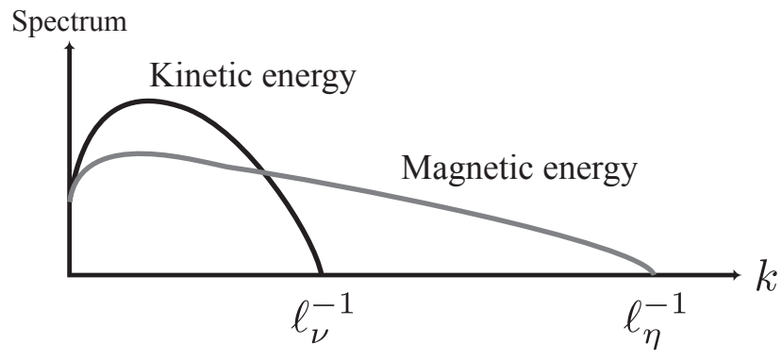
- Finite cross-interaction responses with same timescales

$$\alpha_X = \tau_{bu}(-\langle \mathbf{u}' \cdot \mathbf{j}' \rangle + \langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle) \qquad \tau_{bu} = \tau_{ub}$$
$$= \tau_{bu}\nabla \cdot \langle \mathbf{u}' \times \mathbf{b}' \rangle \quad \text{EMF flux across the boundary}$$

$$G_{ub} = G_{bu} \neq 0$$

- Timescale difference in responses $G_{ub} \neq G_{bu}$

$$\tau_{bu} \neq \tau_{ub} \qquad Pm \gg 1 \ \text{ or } \ Pm \ll 1$$



- Non-equilibrium properties $H_{ub}(\tau, \tau_1) \neq H_{ub}(\tau_1, \tau)$

Present talk

Cross-interaction α_X

$$\alpha_{\rm X} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1\; G_{bu}(k,\mathbf{X};\tau,\tau_1,T)H_{ub}(k,\mathbf{X};\tau,\tau_1,T)$$
$$+\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1\; G_{bu}(k,\mathbf{X};\tau,\tau_1,T)H_{bu}(k,\mathbf{X};\tau,\tau_1,T)$$

Helical functions satisfy $\quad H_{bu}(\tau,\tau_1) = H_{ub}(\tau_1,\tau)$

$$\alpha_{\rm X} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1\; G_{bu}(k,\mathbf{X};\tau,\tau_1,T)H_{ub}(k,\mathbf{X};\tau,\tau_1,T)$$
$$+\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1\; G_{bu}(k,\mathbf{X};\tau,\tau_1,T)H_{ub}(k,\mathbf{X};\tau_1,\tau,T)$$

Symmetric and anti-symmetric parts of H_{ub}
with respect to the exchange of time variables

$$H^{(\rm S)}_{ub}(\tau,\tau_1) = \frac{1}{2}\left(H_{ub}(\tau,\tau_1) + H_{ub}(\tau_1,\tau)\right)$$

$$H^{(\rm A)}_{ub}(\tau,\tau_1) = \frac{1}{2}\left(H_{ub}(\tau,\tau_1) - H_{ub}(\tau_1,\tau)\right)$$ Non-equilibrium effect

$$H_{ub}(\tau,\tau_1) \neq H_{ub}(\tau_1,\tau)$$

$$\alpha_{\rm X} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) + G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\rm A)}(k,\mathbf{X};\tau,\tau_1,T)$$
$$+\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) - G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\rm S)}(k,\mathbf{X};\tau_1,\tau,T)$$

Non-equilibrium alpha

$$\alpha_{\rm neq} = -\frac{1}{3}\int d^3k \int_{-\infty}^{\tau} d\tau_1 \; [G_{ub}(k,\mathbf{X};\tau,\tau_1,T) + G_{bu}(k,\mathbf{X};\tau,\tau_1,T)] H_{ub}^{(\rm A)}(k,\mathbf{X};\tau,\tau_1,T)$$

Simple model

$\mathcal{G}(\tau,\tau_1) = G_{ub}(\tau,\tau_1) + G_{bu}(\tau,\tau_1)$ Independent of *k*

$$\alpha_{\rm neq} = -\frac{1}{3}\int_{-\infty}^{\tau} d\tau_1 \; \mathcal{G}(\tau,\tau_1)\langle \mathbf{u}'_{00}\cdot\mathbf{j}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1)$$

where $\langle \mathbf{u}'_{00}\cdot\mathbf{j}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1) = \frac{1}{2}\left[\langle \mathbf{u}'_{00}(\mathbf{x};\tau)\cdot\mathbf{j}'_{00}(\mathbf{x};\tau_1)\rangle - \langle \mathbf{u}'_{00}(\mathbf{x};\tau_1)\cdot\mathbf{j}'_{00}(\mathbf{x};\tau)\rangle\right]$

Memory effect is crucial since $\langle \mathbf{u}'_{00}\cdot\mathbf{j}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau) = 0$

$$\begin{pmatrix} \dfrac{\partial}{\partial\tau}+\nu k^2 & 0 \\ 0 & \dfrac{\partial}{\partial\tau}+\eta k^2 \end{pmatrix} \begin{pmatrix} G^{ij}_{uu} & G^{ij}_{bu} \\ G^{ij}_{ub} & G^{ij}_{bb} \end{pmatrix}$$

$$+i \begin{pmatrix} -2M^{ikm}\int_\Delta u^k_{00} & 2M^{ikm}\int_\Delta b^k_{00} \\ N^{ikm}\int_\Delta b^k_{00} & -N^{ikm}\int_\Delta u^k_{00} \end{pmatrix} \begin{pmatrix} G^{mj}_{uu} & G^{mj}_{bu} \\ G^{mj}_{ub} & G^{mj}_{bb} \end{pmatrix} = \delta^{ij}\delta(\tau-\tau') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The response functions $\boldsymbol{G}_{ub}$ must be an odd function of $\mathbf{b}_{00}$

$\langle \mathbf{u}'_{00}\cdot\mathbf{j}'_{00}\rangle$ pure scalar → G_{ub} skew

Dynamic quantity that is odd in $\mathbf{b}_{00}$ and skew → Cross helicity $\langle \mathbf{u}'_{00}\cdot\mathbf{b}'_{00}\rangle$

$$\alpha_{\rm neq} \sim -\frac{2}{3}\int_{-\infty}^{\tau} d\tau_1 \Upsilon^{(\rm S)}(\mathbf{x};\tau,\tau_1)\langle \mathbf{u}'_{00}\cdot\mathbf{j}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1)$$

where $\Upsilon(\mathbf{x};\tau,\tau_1) \equiv \dfrac{\langle \mathbf{u}'_{00}(\mathbf{x};\tau)\cdot\mathbf{b}'_{00}(\mathbf{x};\tau_1)\rangle}{\sqrt{\langle {u'_{00}}^2\rangle(\mathbf{x};\tau)}\sqrt{\langle {b'_{00}}^2\rangle(\mathbf{x};\tau_1)}}$ Normalised cross helicity

$$\Upsilon^{(\rm S)}(\mathbf{x};\tau,\tau_1) = \frac{1}{2}[\Upsilon(\mathbf{x};\tau,\tau_1)+\Upsilon(\mathbf{x};\tau_1,\tau)]$$

→

$$\alpha_{\rm neq} \sim -\frac{2}{3}\int_{-\infty}^{\tau} d\tau_1 \left(\Upsilon^{(\rm S)}(\mathbf{x};\tau,\tau_1)\right)^2 \langle \mathbf{u}'_{00}\cdot\boldsymbol{\omega}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1)$$

Non-equilibrium alpha

$$\alpha_{\rm neq} \sim -\frac{2}{3}\int_{-\infty}^{\tau} d\tau_1 \left(\Upsilon^{(\rm S)}(\mathbf{x};\tau,\tau_1)\right)^2 \langle \mathbf{u}'_{00}\cdot\boldsymbol{\omega}'_{00}\rangle^{(\rm A)}(\mathbf{x};\tau,\tau_1)$$

Cross helicity Non-equilibrium helicity

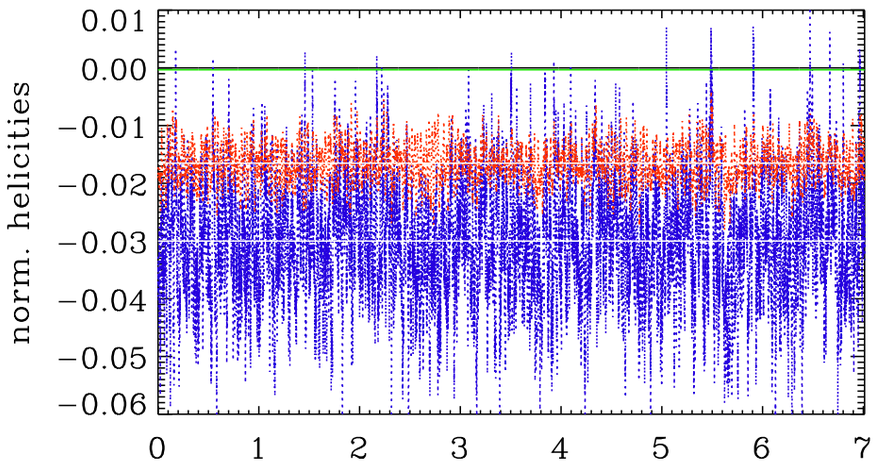
Validation of $\alpha_{\rm neq}$ by DNSs

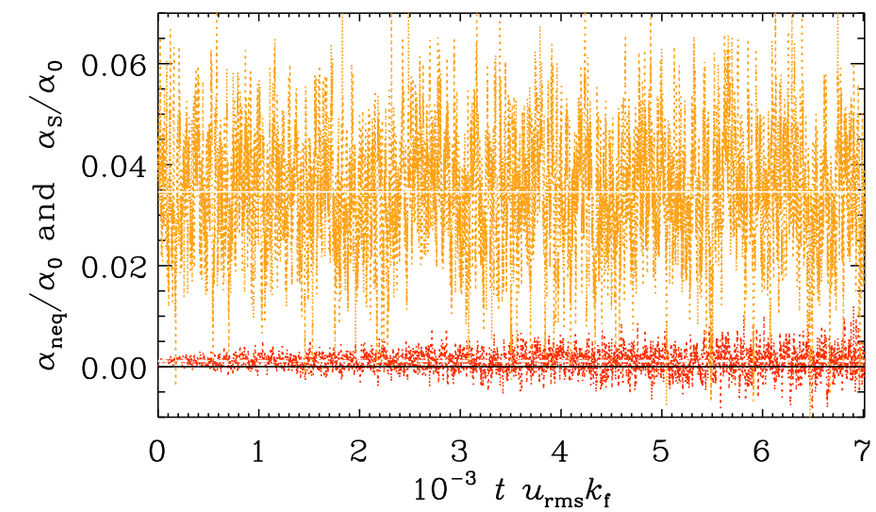
$$\mathbf{g} \parallel \nabla\rho \parallel \mathbf{B}_0 \parallel \omega_{\rm F}$$

$$\boldsymbol{\Omega}_{\rm F} \parallel \nabla\rho \longrightarrow \text{Kinetic helicity}$$

$$\mathbf{B} \parallel \nabla\rho \longrightarrow \text{Cross helicity}$$

Co-existence of kinetic and cross helicities





	$\frac{g}{c_{\rm s}^2 k_1}$	$\frac{v_{A0}}{c_{\rm s}}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}{\sqrt{\langle u'^2\rangle\langle b'^2\rangle}}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{w}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\langle \mathbf{b}'\cdot\mathbf{j}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\alpha_{\rm neq}}{\alpha_0}$	$\frac{\alpha_{\rm S}}{\alpha_0}$	$\frac{u_{\rm rms}}{c_{\rm s}}$	$\frac{v_A^{\rm rms}}{c_{\rm s}}$
C	0.5	0.01	-9.8×10^{-3}	-1.6×10^{-2}	-2.0×10^{-4}	7.8×10^{-4}	1.8×10^{-2}	0.10	0.03
A	1.0	0.01	-1.7×10^{-2}	-3.0×10^{-2}	-3.3×10^{-4}	1.1×10^{-3}	3.5×10^{-2}	0.11	0.04
D	2.0	0.01	-2.0×10^{-2}	-3.6×10^{-2}	-2.8×10^{-4}	6.1×10^{-4}	4.1×10^{-2}	0.16	0.04
E	0.5	0.10	-5.5×10^{-2}	-1.9×10^{-2}	-6.2×10^{-4}	-5.6×10^{-3}	1.5×10^{-2}	0.08	0.07
B	1.0	0.10	-5.3×10^{-2}	-3.2×10^{-2}	-1.2×10^{-2}	2.3×10^{-3}	1.8×10^{-2}	0.09	0.12

TABLE 1. Summary of the simulation results for Runs A–E.

	$k_{\rm f}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{b}'\rangle}{\sqrt{\langle u'^2\rangle\langle b'^2\rangle}}$	$\frac{\langle \mathbf{u}'\cdot\mathbf{w}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\langle \mathbf{b}'\cdot\mathbf{j}'\rangle}{\sqrt{\langle u'^2\rangle\langle w'^2\rangle}}$	$\frac{\alpha_{\rm neq}}{\alpha_0}$	$\frac{\alpha_{\rm S}}{\alpha_0}$	$\frac{u_{\rm rms}}{c_{\rm s}}$	$\frac{v_A^{\rm rms}}{c_{\rm s}}$
A	30	-1.7×10^{-2}	-3.0×10^{-2}	-3.3×10^{-4}	1.1×10^{-3}	3.5×10^{-2}	0.11	0.04
A2	10	-1.3×10^{-1}	-1.2×10^{-1}	1.3×10^{-3}	-1.7×10^{-2}	6.9×10^{-2}	0.12	0.12
A3	3	-6.4×10^{-2}	-2.1×10^{-1}	-3.0×10^{-2}	-6.0×10^{-3}	5.5×10^{-2}	0.19	0.09

TABLE 2. Summary of the simulation results for Runs A, A2, and A3.

Summary of non-equilibrium effects on dynamos

Cross-interaction response functions

$$G_{ub}, \quad G_{bu}$$

Torsional cross correlations

$$\langle \mathbf{u}' \cdot \mathbf{j}' \rangle, \quad \langle \boldsymbol{\omega}' \cdot \mathbf{b}' \rangle$$

Non-equilibrium properties of turbulence

$$\langle \mathbf{u}'_{00}(\mathbf{x};\tau) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau_1) \rangle \neq \langle \mathbf{u}'_{00}(\mathbf{x};\tau_1) \cdot \mathbf{j}'_{00}(\mathbf{x};\tau) \rangle$$

Co-existence of kinetic and cross helicities

$$\langle \mathbf{u}' \cdot \mathbf{b}' \rangle, \quad \langle \mathbf{u}' \cdot \boldsymbol{\omega}' \rangle$$

Beyond heuristic modelling

Mean-field equations in compressible MHD

Yokoi, N., J. Plasma Phys. **84**, 735840501 & 775840603 (2018a,b)

Density

$$\frac{\partial \overline{\rho}}{\partial t} + \nabla \cdot (\overline{\rho}\mathbf{U}) = -\nabla \cdot \langle \rho' \mathbf{u}' \rangle$$

Turb. mass flux

Momentum

$$\frac{\partial}{\partial t}\overline{\rho}U^{\alpha} + \frac{\partial}{\partial x^{a}}\overline{\rho}U^{a}U^{\alpha}$$

$$= -\left(\gamma_0 - 1\right)\frac{\partial}{\partial x^{\alpha}}\overline{\rho}Q + \frac{\partial}{\partial x^{\alpha}}\mu S^{a\alpha} + (\mathbf{J}\times\mathbf{B})^{\alpha}$$

$$- \frac{\partial}{\partial x^{\alpha}}\left(\overline{\rho}\langle u'^{a}u'^{\alpha}\rangle - \frac{1}{\mu_0}\langle b'^{a}b'^{\alpha}\rangle + U^{a}\langle \rho' u'^{\alpha}\rangle + U^{\alpha}\langle \rho' u'^{a}\rangle\right) + R_U^{\alpha}$$

Reynolds stress

Turb. Maxwell stress

Internal energy

$$\frac{\partial}{\partial t}\overline{\rho}Q + \nabla\cdot(\overline{\rho}\mathbf{U}Q) = \nabla\cdot\left(\frac{\kappa}{C_V}\nabla Q\right) - \nabla\cdot(\overline{\rho}\langle q'\mathbf{u}'\rangle + Q\langle\rho'\mathbf{u}'\rangle + \mathbf{U}\langle\rho' q'\rangle)$$

Turb. energy flux

Turb. mass-energy correl.

$$- (\gamma_0 - 1)\left(\overline{\rho}Q\nabla\cdot\mathbf{U} + \overline{\rho}\langle q'\nabla\cdot\mathbf{u}'\rangle + Q\langle\rho'\nabla\cdot\mathbf{u}'\rangle\right) + R_Q$$

Turb. energy dilatation

Turb. mass dilatation

Magnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B} + \langle\mathbf{u}'\times\mathbf{b}'\rangle) + \eta\nabla^2\mathbf{B}$$

Turb. electromotive force

Mean fields

$\overline{\rho}$

$\mathbf{U}$

$\mathbf{B}$

Q

represented by

One-point
turbulent statistical quantities

$$\left[K_\rho \equiv \langle \rho'^2 \rangle\right]$$

$$K \equiv \frac{1}{2}\langle \mathbf{u}'^2 + \mathbf{b}'^2 \rangle$$

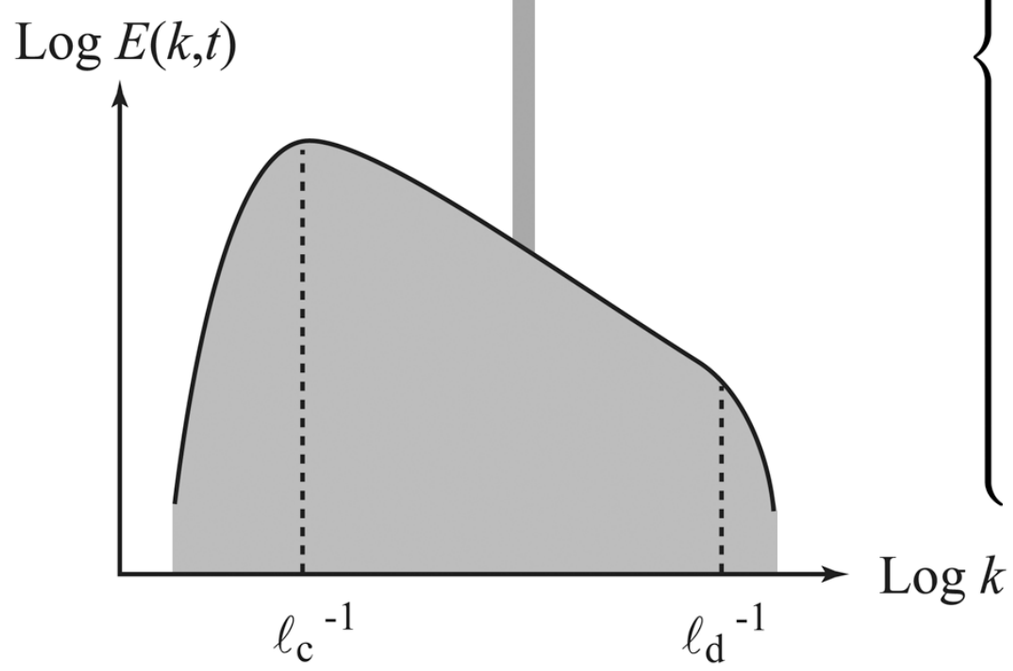
$$\varepsilon \equiv \nu \left\langle \left(\frac{\partial u'_a}{\partial x_b}\right)^2 \right\rangle + \eta \left\langle \left(\frac{\partial b'_a}{\partial x_b}\right)^2 \right\rangle$$

$$W \equiv \langle \mathbf{u}' \cdot \mathbf{b}' \rangle$$

$$\left[K_{\mathrm{R}} \equiv \frac{1}{2}\langle \mathbf{u}'^2 - \mathbf{b}'^2 \rangle\right]$$

$$\left[\varepsilon_W \equiv (\nu+\eta)\left\langle \frac{\partial u'_a}{\partial x_b}\frac{\partial b'_a}{\partial x_b} \right\rangle\right]$$

$$\left[H \equiv \langle -\mathbf{u}' \cdot \boldsymbol{\omega}' + \mathbf{b}' \cdot \mathbf{j}' \rangle\right]$$



Beyond the heuristic modelling

- Model structures
- Transport coefficients

MEAN
LARGE SCALES

Transport suppression

- Helicity effect
- Cross-helicity effect

Transport enhancement

- Eddy viscosity
- Turbulent magnetic diffusivity

Inhomogeneities

- Velocity strain
- Electric current
- Magnetic strain
- Vorticity

Transport coefficients

Production rates

Intensity of turbulence: energies

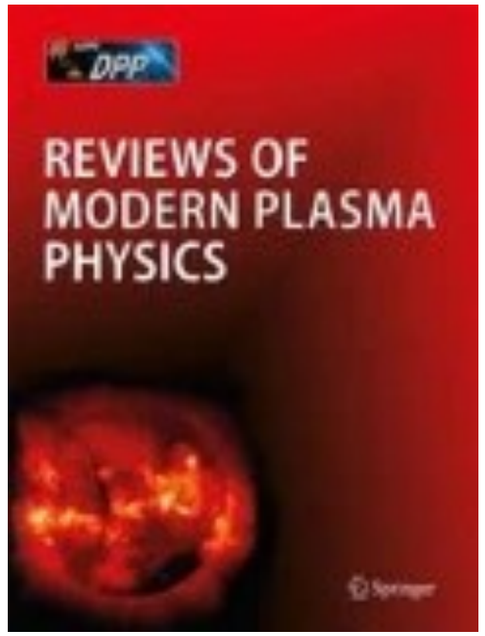
TURBULENCE
SMALL SCALES

Structure of turbulence: helicities

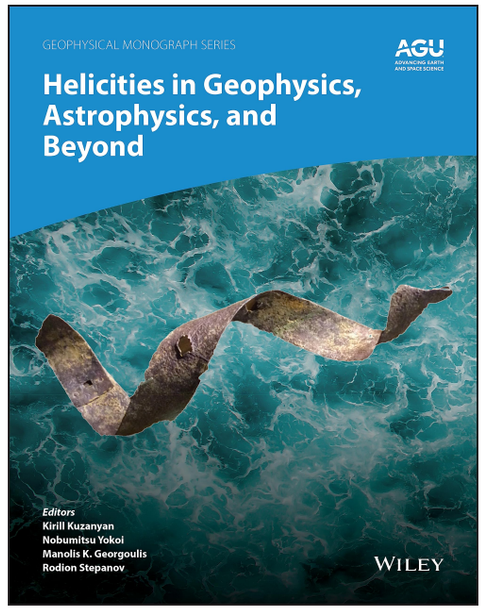
Advertisement: Helicities…

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https://doi.org/10.1080/03091929.2012.754022 (arXiv:2205.14453)

Happy Birthday!
Please keep leading our community for long!

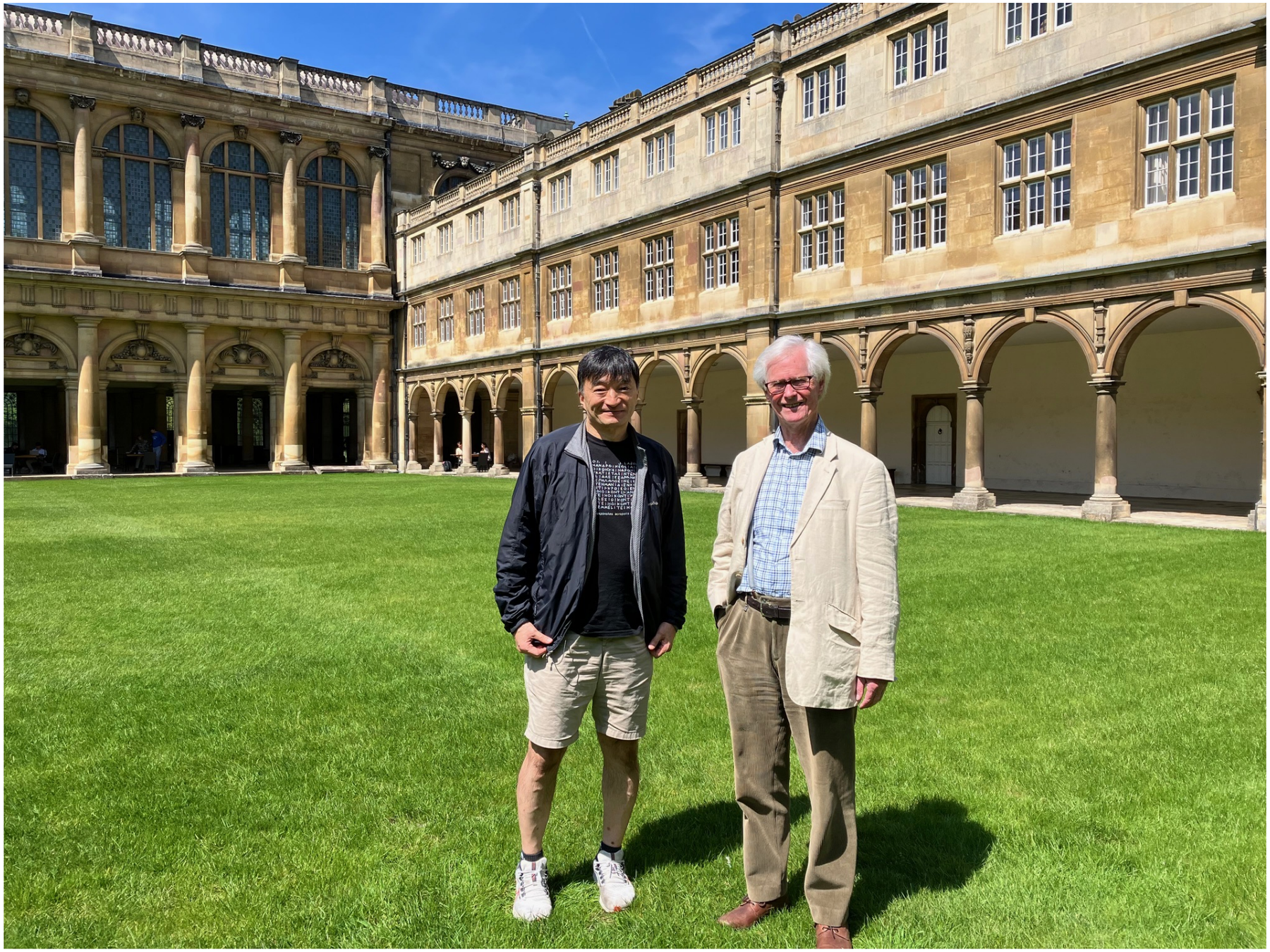


Photo taken by Yoshi Kimura in May 2024