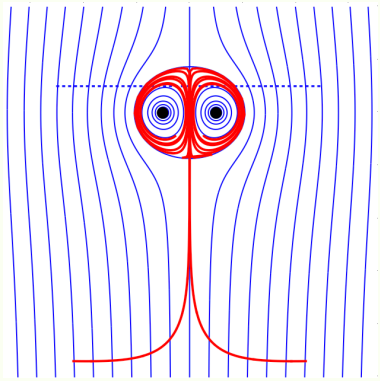


Puzzle: What is this?

Some Reflections on Fluid Mechanics and MHD as it was in the 1960s

Keith Moffatt
Trinity College, Cambridge

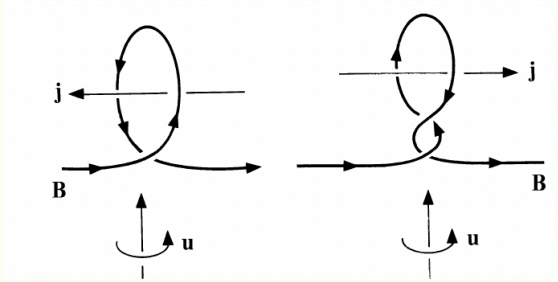
Some comments on works prior to 1960:

Cowling 1934: the first anti-dynamo theorem: "axisymmetric dynamos are impossible"; he was the pioneer of anti-dynamo theory — an over-pessimistic approach.

Batchelor 1950: The **ω - B** analogy and its limitations [$\omega = \nabla \wedge \mathbf{u}$, but **B** is not so constrained] — an over-optimistic approach

Parker 1955: 'cyclonic events' ; a great hand-waving approach!

Cowling 1957: Was grudging in his comments on Parker's approach: "*The argument is not altogether satisfactory; a more detailed analysis is really needed. Parker does not attempt such an analysis; his discussion is limited to elucidating the consequences, if his picture of what occurs is accepted. But clearly his suggestion deserves a good deal of attention.*"



So this was the situation when I started my PhD research in 1958 on MHD Turbulence

I



George Batchelor in discussion with Theodore von Karman, 1961

Kraichnan's (1959) DIA gave $k^{-3/2}$; the theory was challenged by Ian Proudman, and the criticisms were rebutted by Kraichnan

Batchelor abandoned turbulence theory! Houdini! Intermittency now ruled the waves

But all was sweetness and light at the banquet

Stresa (ICTAM 1960) and Marseille 1961: *Colloque International sur la Mécanique de la Turbulence*

Participants included: Favre, Kovasznay, G.I.Taylor, Kolmogorov, Obukhof, Yaglom, Millionchikov, Kraichnan, Ian Proudman, Corrsin, Lumley, Ed Spiegel, Bob Stewart, . . .

$k^{-5/3}$ or not $k^{-5/3}$

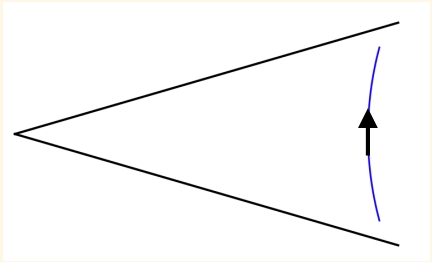
Theory [Kolmogorov 1961] and ocean measurements [Grant, Stewart, Moilliet 1962] were now in conflict.



II

Digression: Corner Eddies

1963/4 I gave a course of lectures on *Viscous Flow* in Part III of the Mathematical Tripos. There was a need to set examination questions. My co-examiners found this one too difficult for a three-hour exam, and it was not allowed



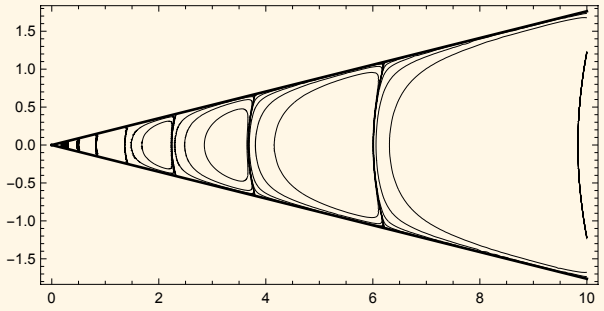
Fluid stirred far from the corner; determine the asymptotic flow as $r \to 0$

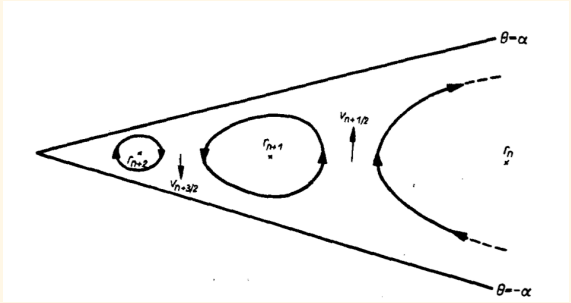
$$\nabla^4 \psi = 0 \qquad \text{Try } \psi \sim r^\lambda f(\theta)$$

No-slip b.c.s on $\theta = \pm\alpha$ Eigenvalue problem for λ

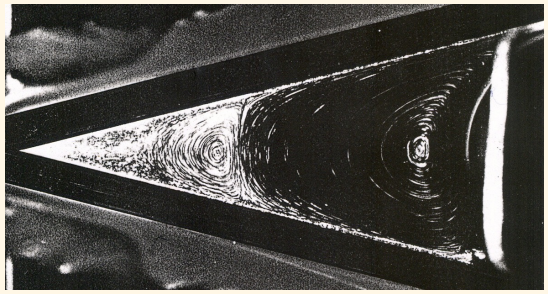
SURPRISE: for $2\alpha \lesssim 147^o$, λ is complex: $\lambda = p + iq$

$r^{p+iq} = r^p \cos[q \log r]$: infinite oscillations as $r \to 0$

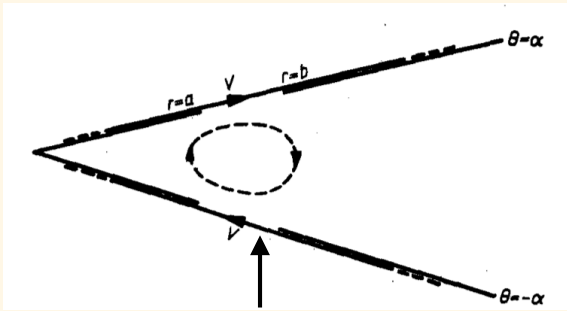




Now these corner eddies are found everywhere [even in Zakopane]



Taneda 1979, long time exposure photography



Flow can be driven by 'moving sleeves' [Zakopane 1963]

This flow minimises the rate of energy dissipation!

hkm 1964 *JFM*, **18**, 1- 18, and *Arch. Mech. Stosowanej* **2**, 365-372.

III *Helicity*

[A pseudo-scalar: non-zero only for flows that are not mirror-symmetric]

1964 I inherited a graduate course on Magnetohydrodynamics from Arthur Shercliff, who had moved to Warwick as the Founding Professor of Engineering Science.

Each year, in one of the lectures, I derived 'Woltjer's (1957) invariant': $\int \mathbf{A}\cdot\mathbf{B}\,dV = \text{const}$, where $\mathbf{B} = \nabla\wedge\mathbf{A}$,

but I struggled to find a physical interpretation of this invariant.

1967 Finally, 'the penny dropped': I realised that it represents the linkage between magnetic flux tubes, and is invariant because, in a perfectly conducting fluid, these flux tubes are 'frozen in the fluid', i.e. they are transported with the flow, so that their linkage is invariant.

1968 But then an analogous invariant must exist for vortex lines which are similarly transported in the flow of an ideal fluid, governed by Euler's equations; hence I was led to prove the invariance of

'helicity' $$\mathcal{H} = \int \mathbf{u}\cdot\omega\,dV = \text{const.}, \quad \text{where } \mathbf{n}\cdot\omega = 0 \text{ on } \partial V$$

hkm 1969 *The degree of knottedness of tangled vortex lines*, *JFM*, **35**, 117- 129

This was quite amazing, and very exciting: a 'topological invariant' of Euler's 1756 equations, in addition to the known invariants, energy, momentum and angular momentum.

1978 **But,** much later I learnt that the result had been discovered earlier by J.-J. Moreau (1961), *C.R.Acad Sci.* **252**, 2810-2812, '*Constantes d'un îlot tourbillonnaire en fluide parfait barotrope*'. I had been **scooped**! I cited this paper in 1981; I can't find any earlier citation!

IV *Mean Field Electrodynamics*

1966-68 Glyn Roberts was my PhD student and developed his space-periodic dynamo [later motivating the Karlsruhe dynamo (Müller & Stieglitz 2002)]. Glyn moved to Newcastle in 1968, and was awarded his PhD in 1969. His first paper was published in 1970.

I extended Glyn's 2-scale approach to the problem of turbulence, and the relevance of helicity then became clear: the parameter α in the mean-field relation $\langle \mathbf{u} \wedge \mathbf{b} \rangle = \alpha \mathbf{B_0}$ was, like helicity, a pseudo-scalar, and in fact linearly related to the helicity spectrum.

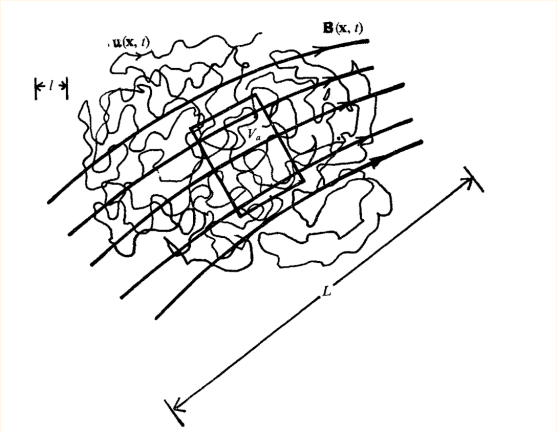
'Chiral turbulence', i.e.turbulence that lacks statistical reflexion symmetry, is generated in a rotating fluid (rotation vector $\mathbf{\Omega}$) when there is a flux of energy $\mathbf{F}$ with $\mathbf{F} \cdot \mathbf{\Omega} \neq 0$.

I submitted my paper to *JFM* in 1968 and sent a copy to Glyn; he had just been at a meeting in Madrid, and had heard a lecture by Fritz Krause, from the DDR (East Germany), presenting the work of the Potsdam group Steenbeck, Krause & Râdler (1966) [*Berechnung der mittleren Lorentz-Feldstarke für ein elektrisch leitendes Medium in turbulenter, durch Coriolis-Kräfte beeinflusster Bewegung*. *Z. Naturf.* **21**a, *369*]

I spoke no German, but a German researcher in DAMTP, translated the paper for me. This revealed that again, I had been scooped.

1970 **My revised paper was published in 1970 in *JFM*, 41, 435-452, with opening paragraph:**

"*A theory that is likely to be of the greatest significance in geomagnetism and in cosmical electrodynamics has been developed recently by Steenbeck, Krause & Rädler (1966), The theory is concerned with the effect of a turbulent velocity field on a magnetic field in an electrically conducting fluid, it being supposed that there is no external source of magnetic field, the only source being the electric current distribution within the fluid itself.*"



Mean-field theory was revolutionary: it showed that Batchelor, Biermann & Schlüter, and Saffman had all been wrong, and that dynamo action could occur even when $R_m << 1$, provided the turbulence is chiral, and the flow domain is large enough. Parker (1955) and Braginskii (1964) were retrospectively justified.

Convection and diffusion
By turb'lence with helicity
Yields order from confusion
In cosmic electricity.

Before 1960, there was no convincing theory of turbulent dynamo action
After 1970, mean-field electrodynamics and the turbulent dynamo, and its application to the generation of magnetic fields in planets, stars, galaxies and accretion discs, became firmly established

I am prejudiced, but I see this as the most significant advance in turbulence research since 1960

And now for something completely different, 50+ years later . . .

Interaction of vortex tubes and bridges/threads/sheets

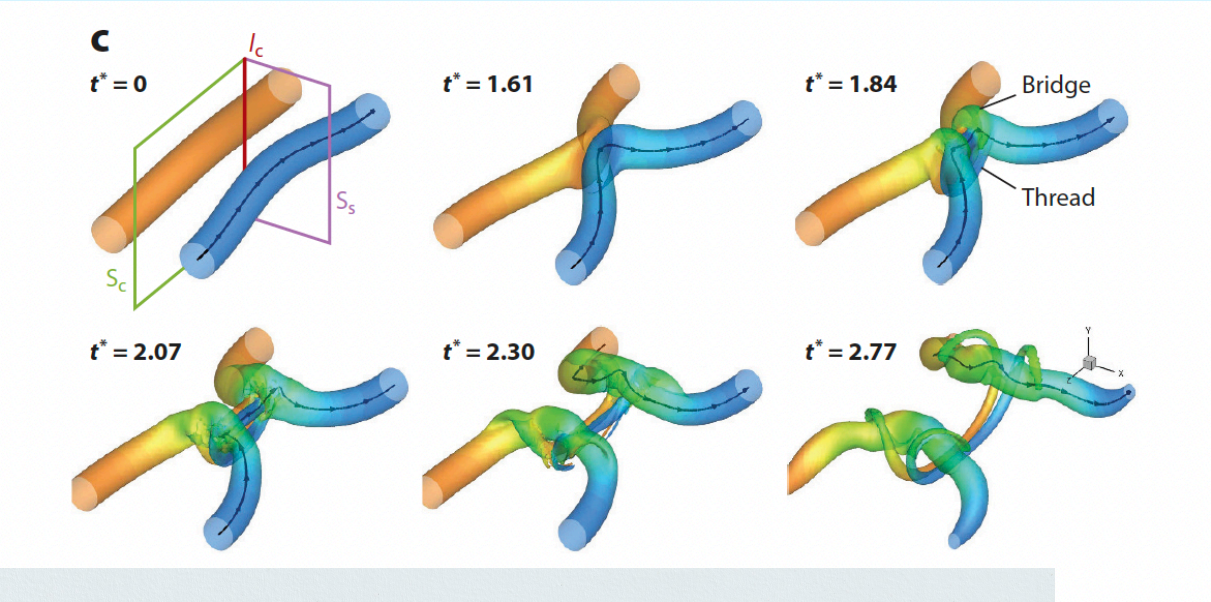
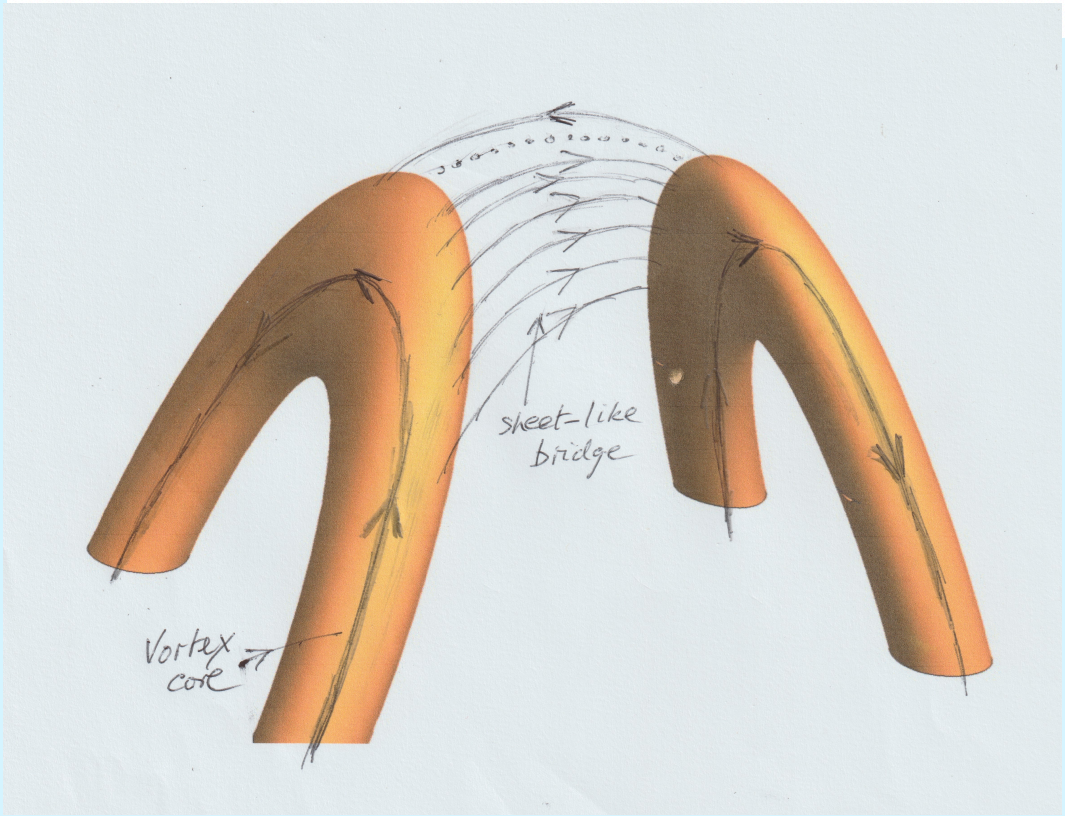


Figure 1 from Yau & Hussain 2022 Reconnection of antiparallel vortex tubes at vortex Reynolds number Re = 2,000: time evolution of vorticity magnitude iso-surfaces; this shows wind-up of 'threads' *after* reconnection

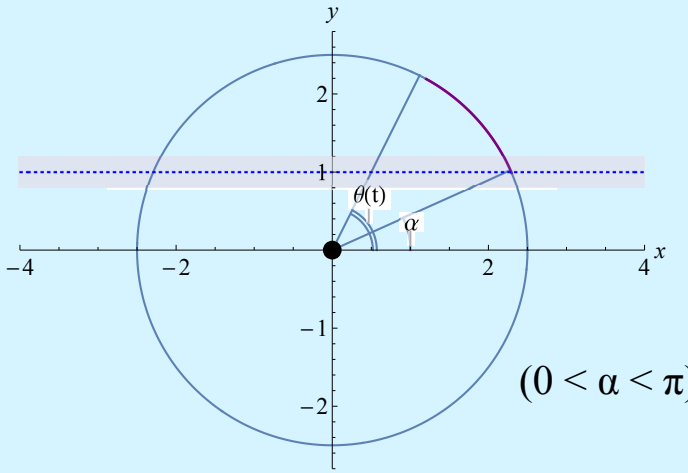


This is a sketch

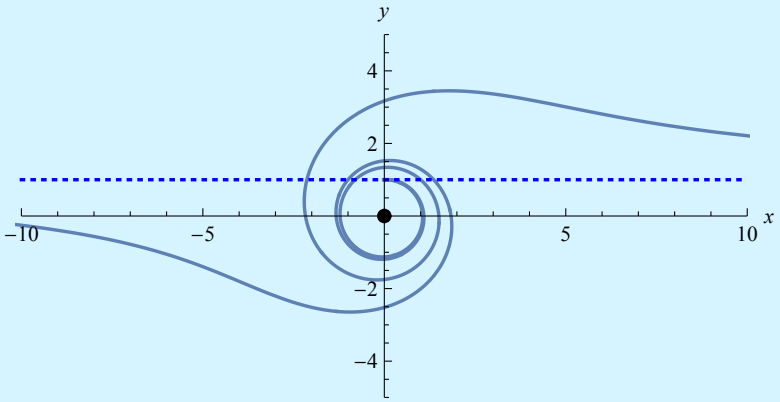
representing the *beginning* of vortex reconnection: bridging creates a vortex sheet that is potentially wound up by the approaching vortex rings

This suggests it may be useful to investigate interaction of vortex tubes and vortex sheets

Interaction of vortex tubes and sheets: a model problem



$(0 < \alpha < \pi)$

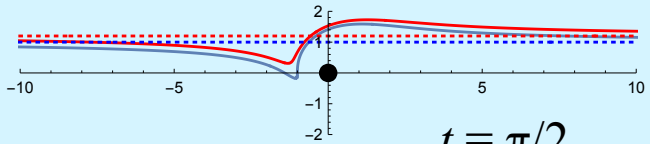


The point vortex at (0,0) winds up the vortex sheet of thickness δ_0 on y = b (=1)

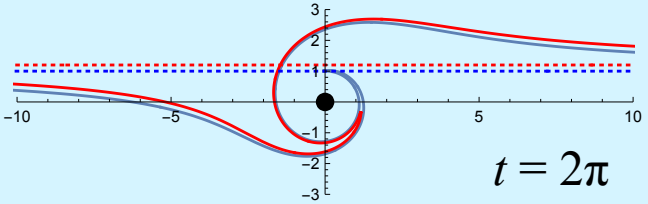
at $t = 0$, $\boldsymbol{\omega}_0 = \{\omega_0(y),\ 0,\ 0\}$ **NB: initial vorticity is in *x*-direction**

e.g. if $\omega_0(y) = \left(U/(\sqrt{\pi}\,\delta_0\right) \exp\left[-(y-b)^2/\delta_0^2\right]$

then $\mathbf{u}_0 = (U/2)\{0,\ 0,\ \mathrm{erf}[(y-b)/\delta_0] - c\}$ in *z*-direction, so doesn't affect point vortex

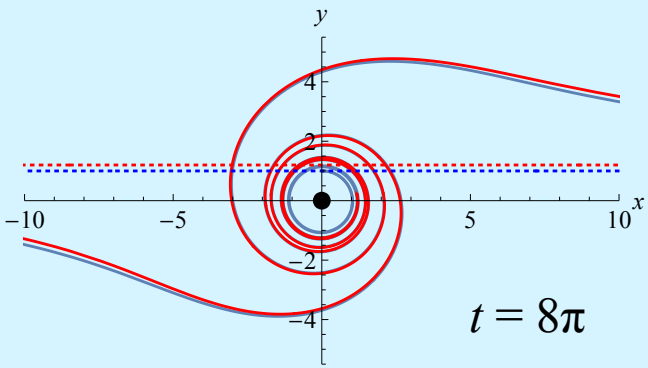


$t = \pi/2$

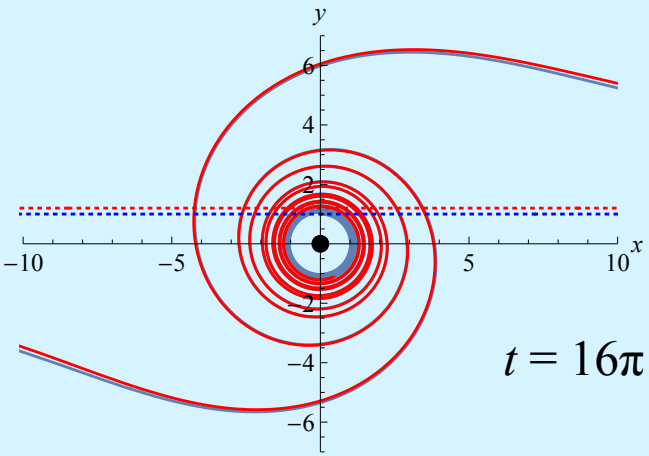


$t = 2\pi$

Two vortex lines in the sheet, red and blue; for $t > 0$ and zero viscosity, a double spiral forms, and the curves get closer in the spiral region (because of incompressibility); and blue rotates faster than red



$t = 8\pi$



$t = 16\pi$

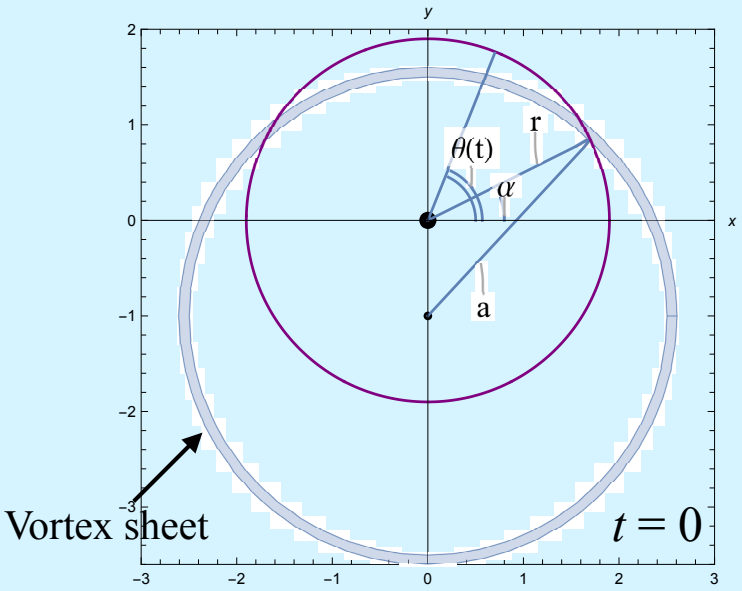
The velocity induced by the spiral vortex sheet remains unidirectional in z-direction for all t > 0

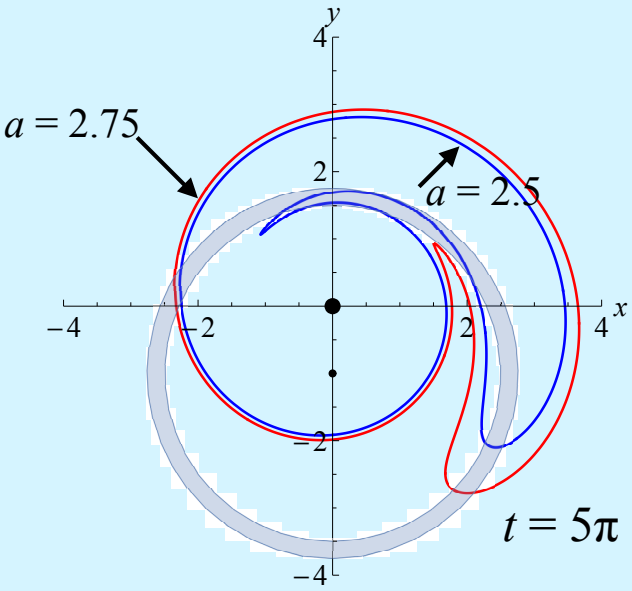
Parametric equations of spiral starting from $y = y_0$ with parameter α $(0 < \alpha < \pi)$:

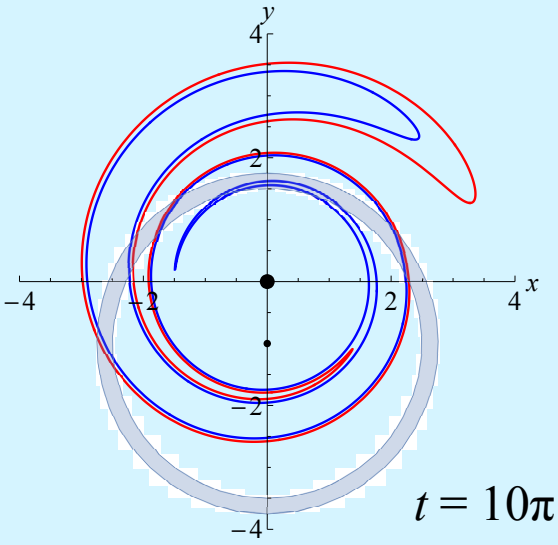
$$x(\alpha, y_0, t) = y_0 \csc\alpha \cos\left[\alpha + t\, y_0^{-2} \sin^2\alpha\right], \quad y(\alpha, y_0, t) = y_0 \csc\alpha \sin\left[\alpha + t\, y_0^{-2} \sin^2\alpha\right].$$

This is reminiscent of Gilbert (JFM 1988), who treated spiral wind-up of a 2D patch of weak vorticity by a point vortex; that was a linearised analysis, whereas the present configuration remarkably provides an exact solution with no linearisation

Spiral wind-up of cylindrical vortex sheet



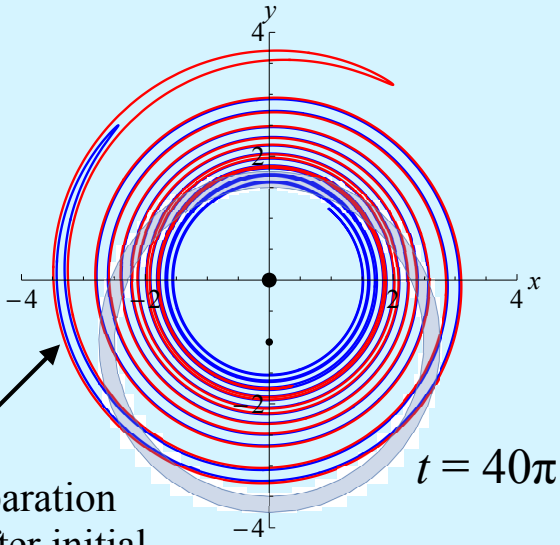


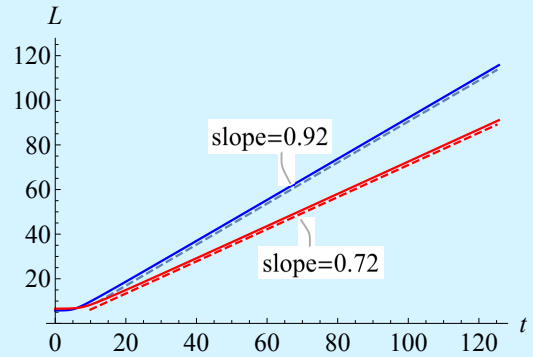


Parametric equation of spiral in this case:

$$r(\alpha, a) = -\sin\alpha + \sqrt{\sin^2\alpha + a^2 - 1}\,, \quad \theta(\alpha, a, t) = \alpha + t/r(\alpha, a)^2$$

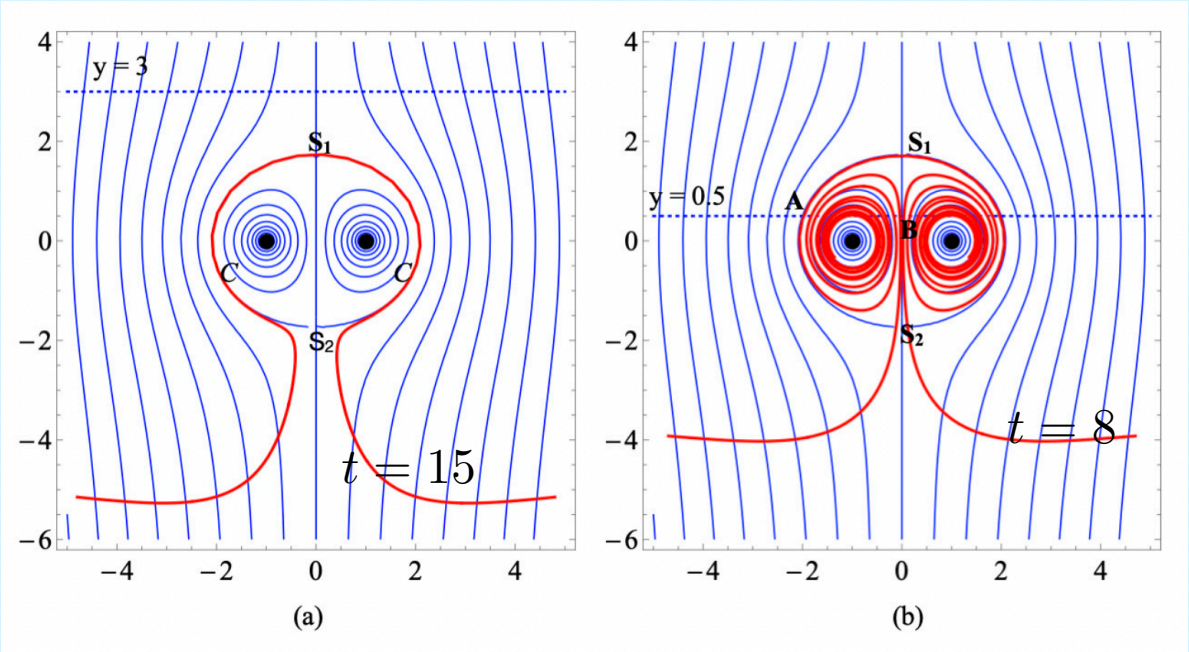
$$\mathbf{x} = \mathbf{X}(\alpha, a, t) = r(\alpha, a)\{\cos[\theta(\alpha, a, t)],\ \sin[\theta(\alpha, a, t)]\}$$

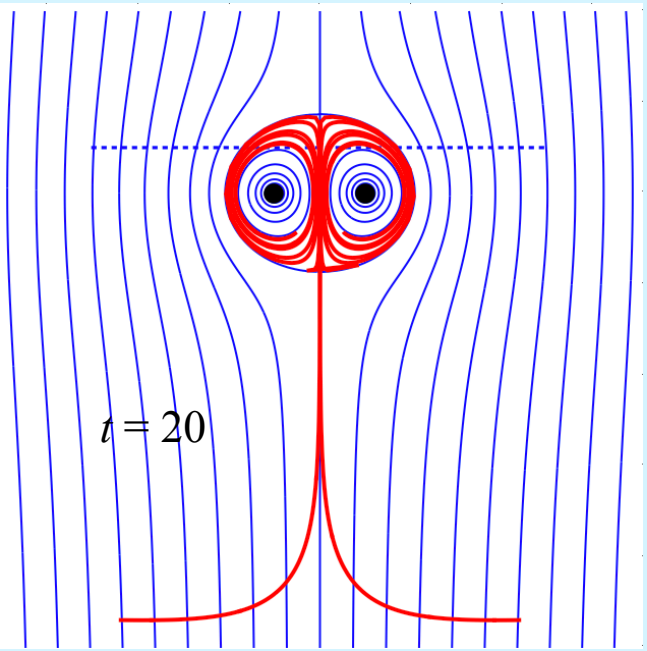




Length L of sheet grows linearly. The mean separation between blue and red curves decreases as $1/t$ after initial transient phase; this is ultimately limited by viscous diffusion.

Sheet distortion by a vortex pair ($v = 0$)





Vortex pair, marked by bullets, propagates upwards with velocity ½; vortex sheet initially at $y = y_0$; take axes moving with the vortex pair; in this frame, stream function is

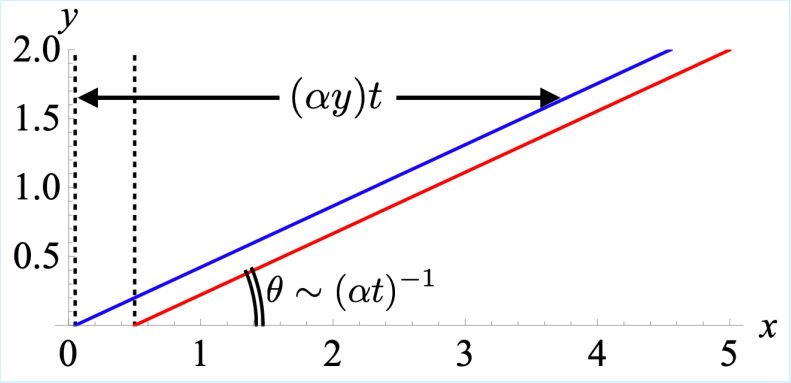
$$\psi(x,y) = \tfrac{1}{2}x - \log\left[r_1/r_2\right], \quad \text{where } r_{1,2} = (x \pm 1)^2 + y^2 ,$$

and distorting velocity field is $\mathbf{u}_1(x,y) = \{\partial\psi/\partial y,\ -\partial\psi/\partial x,\ 0\}$.

(a) $y_0 > \sqrt{3}$: The sheet moves down and is wrapped round the closed streamline C, with exponential stretching at the saddle points S_1 and S_2

(b) $y_0 < \sqrt{3}$: The sheet is wound up inside C, and still with exponential stretching at the saddle points S_1 and S_2

Finite Reynolds number effect



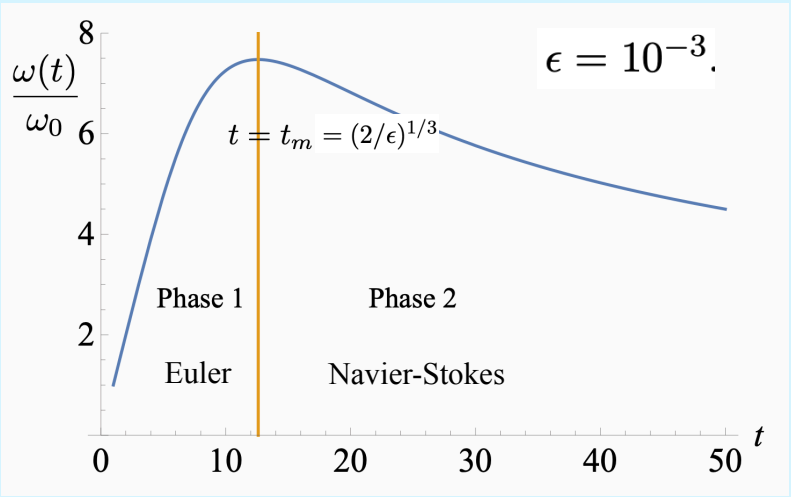
Shearing decreases lateral scale as t^{-1}

So consider a model problem:

Burger's vortex sheet with strain $\alpha = t^{-1}$

$$\boldsymbol{\omega} = \{0, \omega(x,t), 0\}\,, \quad \omega(x,1) = \omega_0 \exp\left[-x^2/\delta_0^2\right], \quad \frac{\partial \omega}{\partial t} - \alpha(t) x \frac{\partial \omega}{\partial x} = \alpha(t)\,\omega + \nu \frac{\partial^2 \omega}{\partial x^2}$$

Solution is $\omega(x,t) = \omega(t) \exp\left[-x^2/\delta(t)^2\right], \quad \text{with } \omega(t) = U/[\sqrt{\pi}\,\delta(t)]$

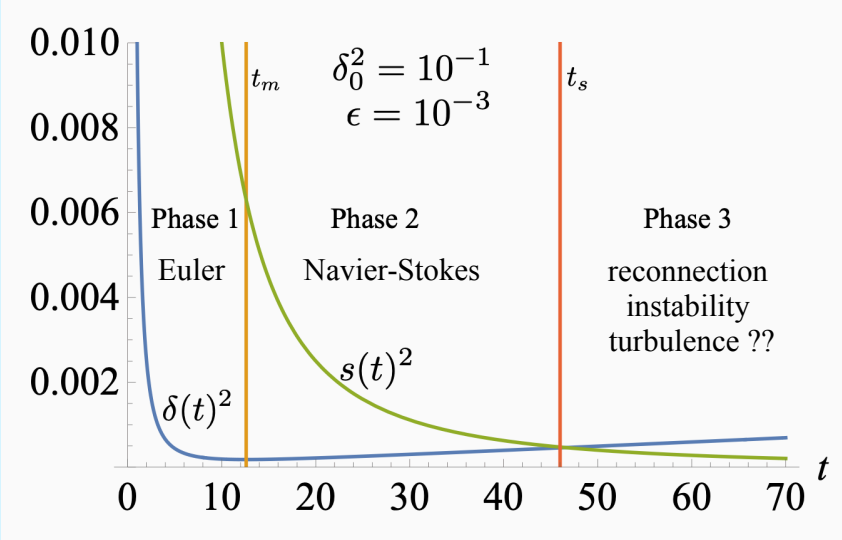


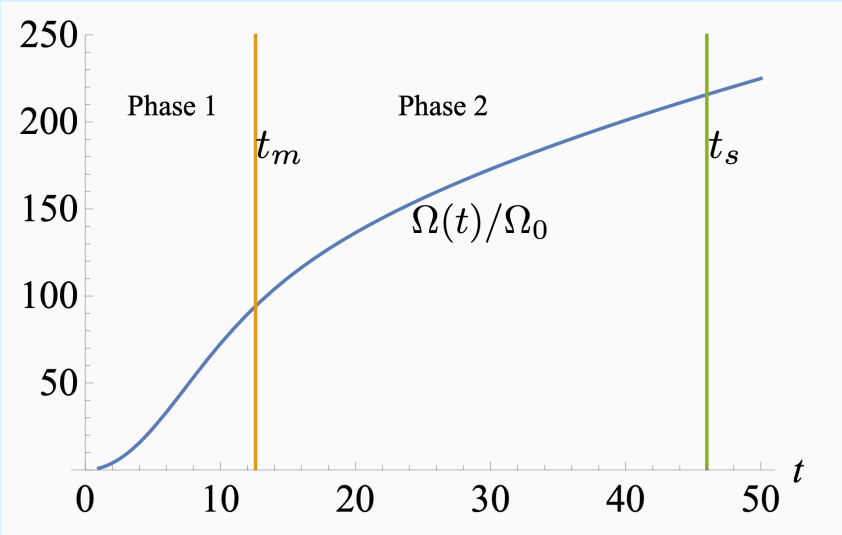
$$\delta^2/\delta_0^2 = t^{-2} + \epsilon\,(t-1) \quad \text{with } \epsilon = 4\nu/3\delta_0^2$$

$$\omega(t_m) = \omega_m \approx 0.727\,\epsilon^{-1/3} \approx 0.661\,\delta_0^{2/3}\,\mathrm{Re}^{1/3}\,.$$

So maximum vorticity increases like $\mathrm{Re}^{1/3}$

Growth of enstrophy





Separation $s(t)$ between turns of spiral also decreases like t^{-1}

When $\delta(t) \sim s(t)$ $[\text{at } t \sim t_s = (\delta_0^2 \epsilon)^{-1/3}]$ turns begin to overlap and reconnections begin (Phase 3)

$$\Omega/\Omega_0 = t[t^{-2} + \epsilon(t-1))]^{-1/2}$$

Enstrophy $\Omega(t)$ continues to grow until $t \sim t_s$, and maximum value is then

$$\Omega_{max}/\Omega_0 \sim (\delta_0 \epsilon^2)^{1/3} \sim \delta_0 \nu^{-2/3}$$

e.g. if $\delta_0 = 10^{-1}$ and $\epsilon = 10^{-3}$, then $\Omega_{max}/\Omega_0 \sim 10^{7/3} \approx 215$

but rate of dissipation $\Phi_{max} = \nu\,\Omega_{max} \sim \Omega_0\,\delta_0\,\nu^{1/3} \to 0$ as $\nu \to 0$

no dissipation anomaly

Possible generalisations

1. Any distribution of point vortices could be chosen to provide the 'distorting' velocity field; e.g. two co-rotating vortices.

2. If we choose 3 or more point vortices, then there are regions of Lagrangian chaos; the growth of the distorted vortex sheet is then likely to be exponential rather than linear, leading to much greater maximum rate of dissipation; but still no dissipation anomaly.

3. The axisymmetric counterpart of this theory could be considered: e.g. distortion of a toroidal vortex sheet by a propagating ring vortex, or by any distribution of ring vortices; here we would be considering a family of unsteady axisymmetric flows with a swirl component; nicely compact in 3D.

4. Tempting to extend to MHD; but no obvious analogy, because a distorted sheet of magnetic flux immediately exerts a restoring Lorentz force, associated with Maxwell tension in the curved lines of force; this has no counterpart in a distorted vortex sheet.

Summary

A vortex sheet that is spiral

Can wind up at a rate that's quite viral;

Distorted by vortex

*In cerebral cortex *,*

A whole lot of fun in retiral !

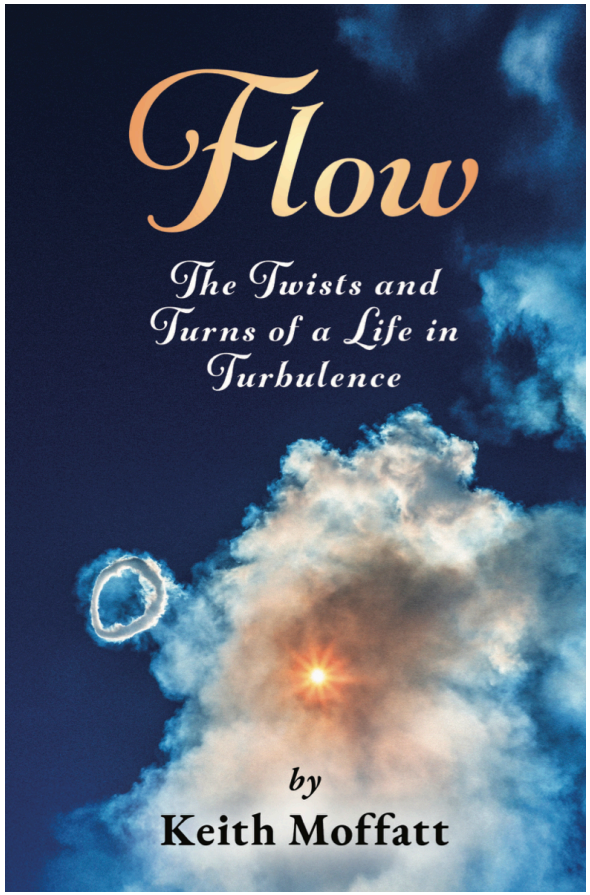


Yoshi Kimura on a recent visit to Cambridge

Moffatt & Kimura, 2024 Spiral wind-up of vortex sheets, GAFD, 1-11

The End

* Your cerebral cortex plays a key role in memory, thinking, learning, reasoning, problem-solving, emotions, consciousness and functions related to your senses. [Cleveland Clinic]



Cover photo: © Jürg Alean. A steam vortex ring emitted by eruption of Mount Etna, 24 February 2000

The 'stepping stones' of my scientific life: the quadrennial ICTAMs since 1960:

Stresa 1960:
Munich 1964:
Stanford 1968:
Moscow 1972:
Delft 1976:
Toronto 1980:
Lyngby 1984:
Grenoble 1988:
Haifa 1992:
Kyoto 1996
Chicago 2000
Warsaw 2004
Adelaide 2008
Beijing 2012
Montreal 2016
Milan 2020/21 [zoomed]
Daegu, S.Korea 2024
—-> Vienna 2028

This brief autobiography is due to be published later this month, and will be available on Amazon