When boundaries matter, the case of inertia-gravity modes

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«Mathematical Fluid Dynamics», IES Cargèse, April 2025











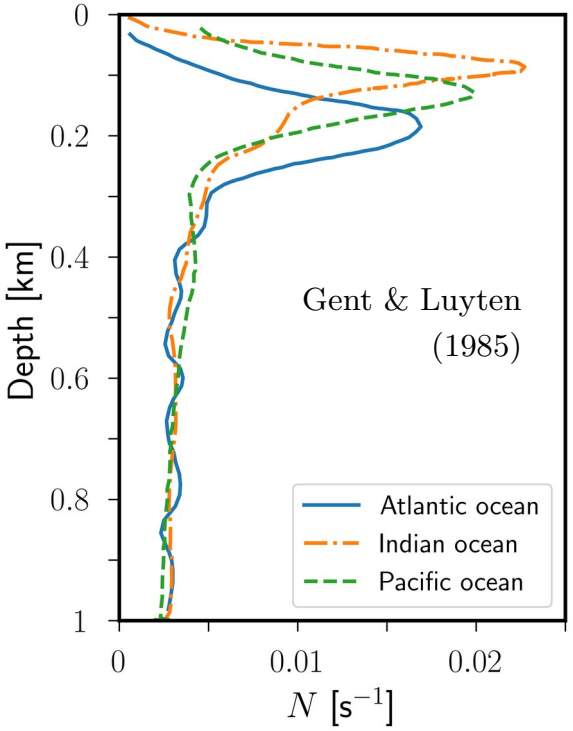

INTRODUCTION

Geo and astrophysical flows: two specific effects

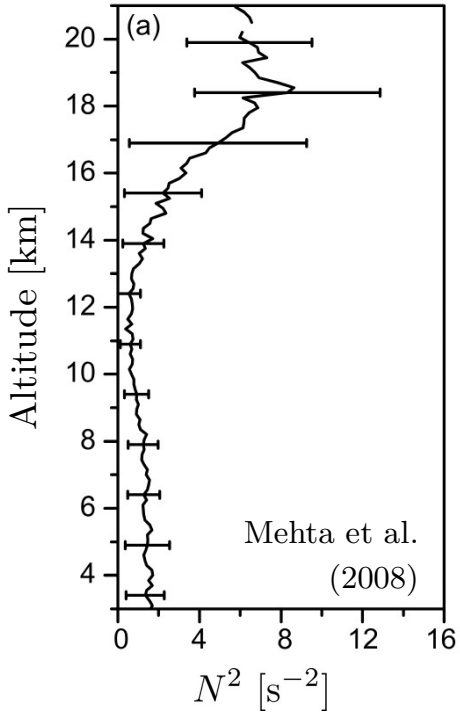
INTRODUCTION

Geo and astrophysical flows: two specific effects

Oceans



Atmosphere

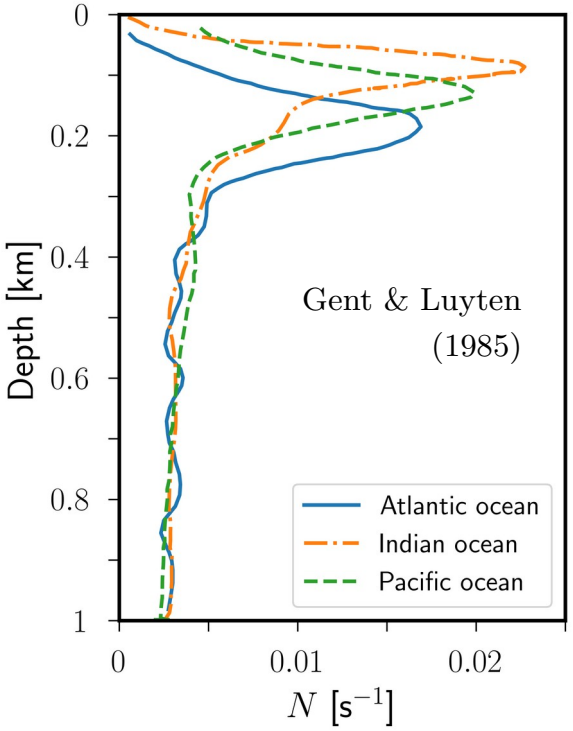


$$N^2 = \frac{\boldsymbol{g}}{\rho} \cdot \left(\nabla \rho - \frac{\rho \boldsymbol{g}}{c_s^2} \right) \geq 0$$

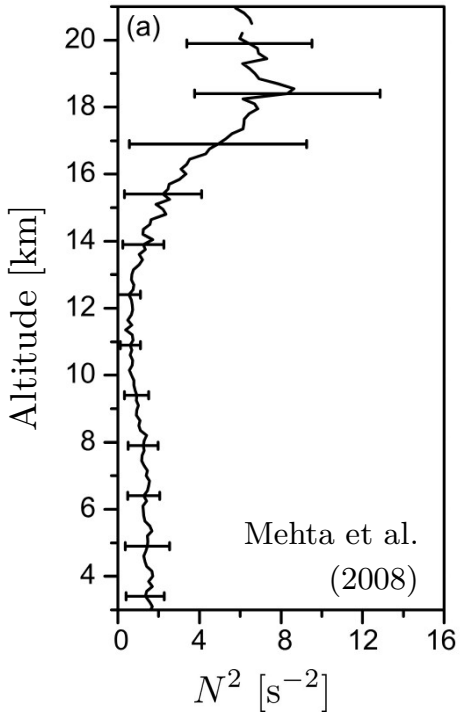
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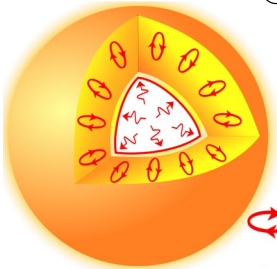


Atmosphere



Astrophysics

$0.5 - 1.5 M_\odot$

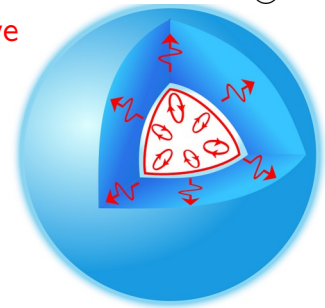


$N \sim 10^{-4} - 10^{-3}\,\mathrm{s}^{-1}$

Convective

Stratified

$> 1.5 M_\odot$



$N \sim 10^{-4} - 10^{-1}\,\mathrm{s}^{-1}$

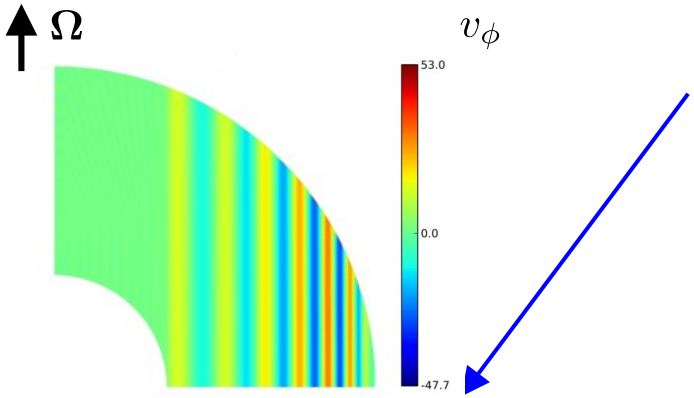
$$N^2 = \frac{\boldsymbol{g}}{\rho} \cdot \left(\nabla \rho - \frac{\rho \boldsymbol{g}}{c_s^2} \right) \geq 0$$

Inertia – Gravity

Waves (IGWs)

INTRODUCTION

Geophysical waves and/or modes



Inertia – Gravity

Waves (IGWs)

Rotation

$2\boldsymbol{\Omega} \times \boldsymbol{v}$

$0 \leq |\omega| < 2|\boldsymbol{\Omega}|$

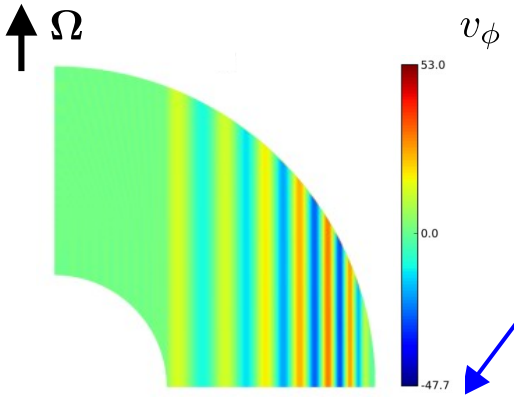
Effect

Restoring force

Existence

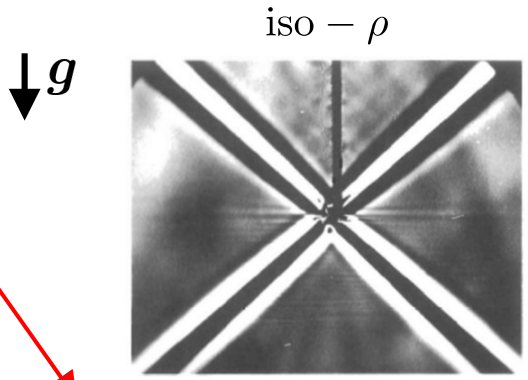
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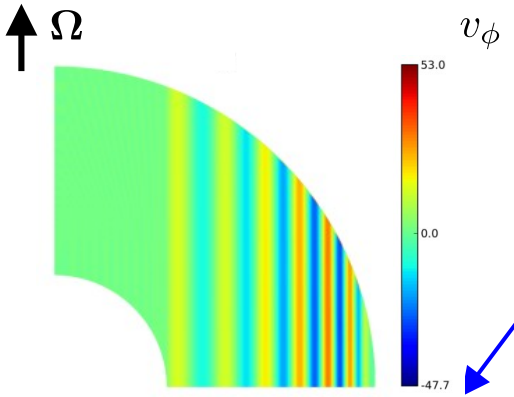
Stratification

$\rho \boldsymbol{g}$

$0 \leq |\omega| < N$

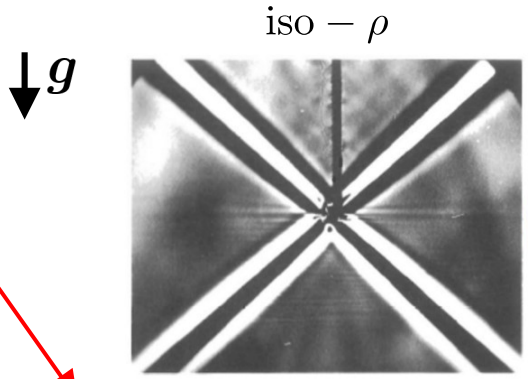
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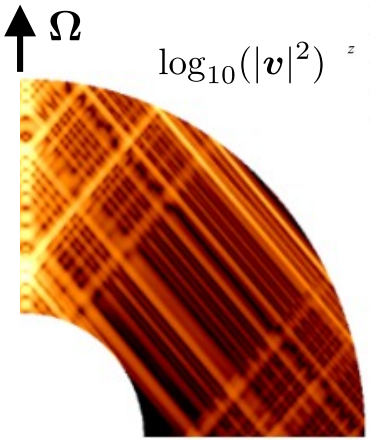
$0 \leq |\omega| < N$

Waves are **ubiquitous** in geophysical flows

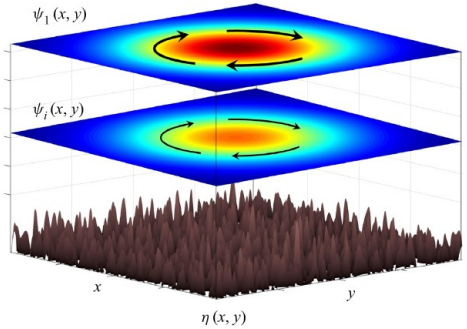
Waves / **Modes** have many **applications**

Waves / Modes have many applications

Dissipation



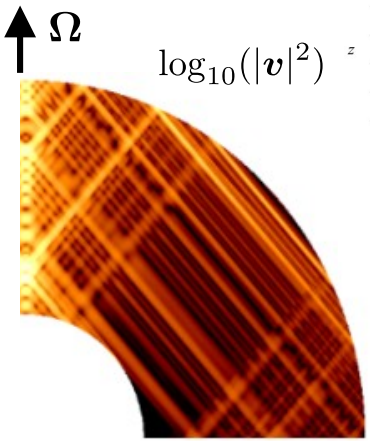
Rieutord *et al.* (1999)



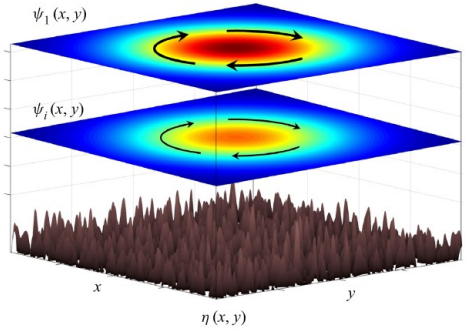
Radko (2023)

Waves / Modes have many applications

Dissipation

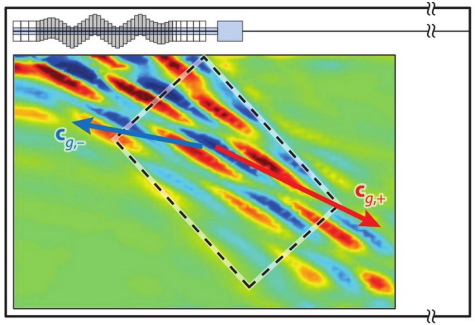


Rieutord *et al.* (1999)



Radko (2023)

$$\omega_0 = \omega_1 + \omega_2$$

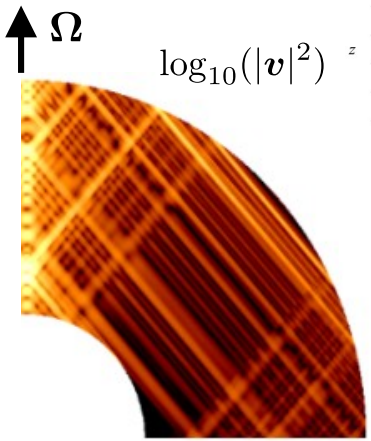


Bourget el al. (2013)

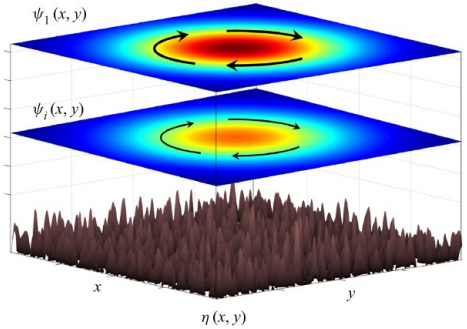
Flow instabilities

Waves / Modes have many **applications**

Dissipation



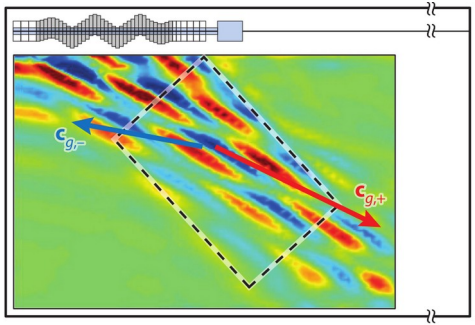
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Radko (2023)

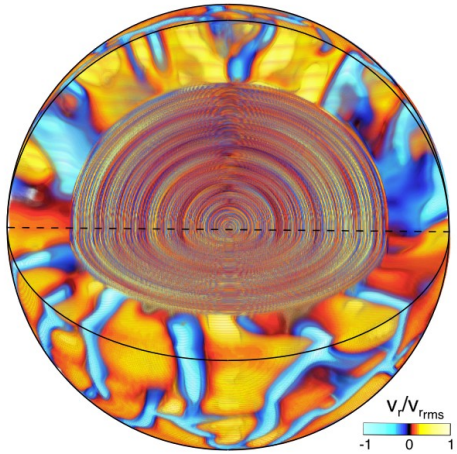
Flow instabilities

$$\omega_0 = \omega_1 + \omega_2$$



Bourget el al. (2013)

Transport/Mixing



Alvan et al. (2014)

INTRODUCTION

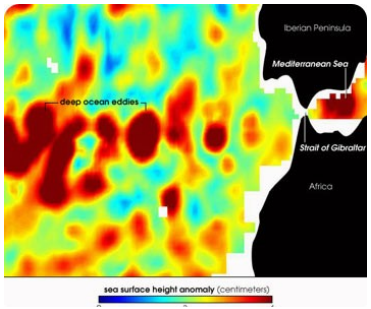
An interdisciplinary collaboration

APPLIED MATHS

PURE MATHS

Realism

Rigour





INTRODUCTION

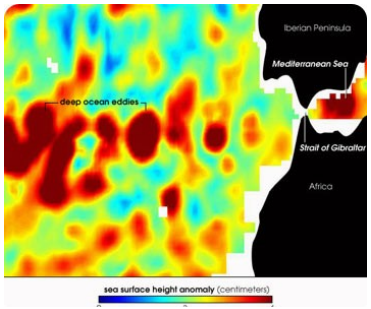
An interdisciplinary collaboration

APPLIED MATHS

PURE MATHS

Realism

Rigour





Y. Colin de Verdière



Recent research **papers**

Vidal & Colin de Verdière, 2024, *Inertia-gravity waves in geophysical vortices,* Proc. R. Soc. A.

Colin de Verdière & **Vidal**, 2025, *The spectrum of the Poincaré operator in an ellipsoid,* J. Spec. Theo.

Colin de Verdière & **Vidal**, 2025, *On gravito-inertial surface waves,* Contemp. Math.

FLUID MODEL

Linearised Boussinesq equations for IGWs

$$\partial_t \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p + \rho \boldsymbol{g} + \nu \nabla^2 \boldsymbol{v}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

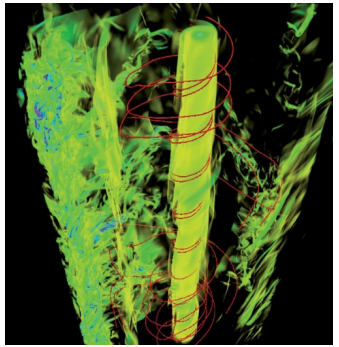
$$\partial_t \rho + \boldsymbol{v} \cdot \nabla \rho_0 = 0$$

$+\mathbf{BC}$

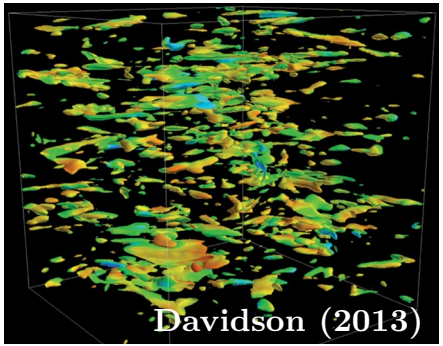
➔ **Rotation**

➔ **Stratification**

R >> S



R << S



$\mathrm{iso} - |\boldsymbol{\omega}|^2$

FLUID MODEL

Linearised Boussinesq equations for IGWs

$$\mathrm{i}\omega \boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p + \rho \boldsymbol{g}$$

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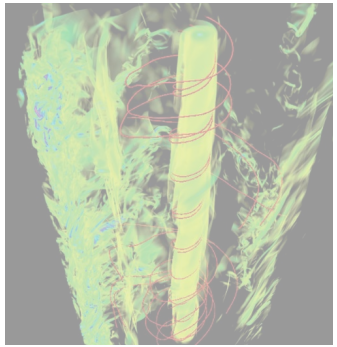
$$\mathrm{i}\omega \rho + \boldsymbol{v} \cdot \nabla \rho_0 = 0$$

$$\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial V} = 0$$

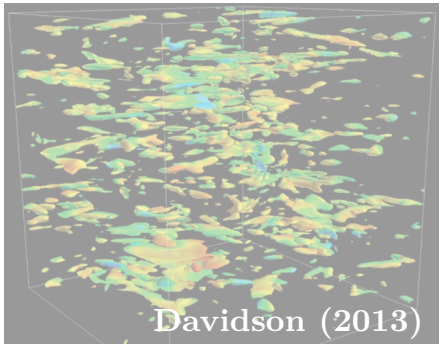
→ **Rotation**

→ **Stratification**

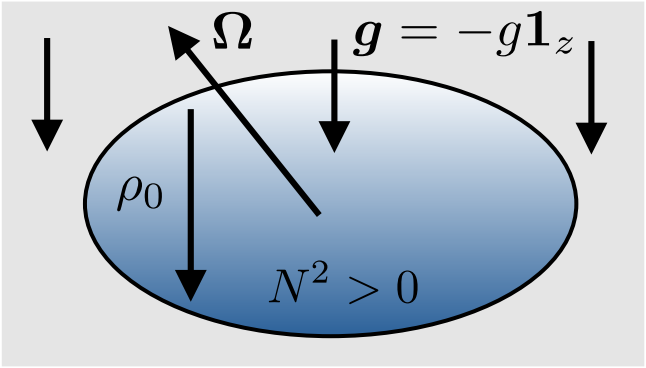
R >> S



R << S



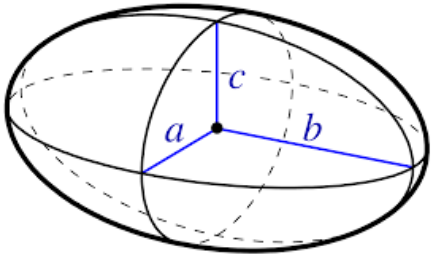
iso $-|\boldsymbol{\omega}|^2$



Mathematical approximations

- **Ellipsoid** with a **rigid** boundary (Dirichlet BC)
- Constant $N^2 \geq 0$ (i.e. constant $\boldsymbol{g}$ and linear density)
- **Inviscid** fluid
- Normal-mode perturbations

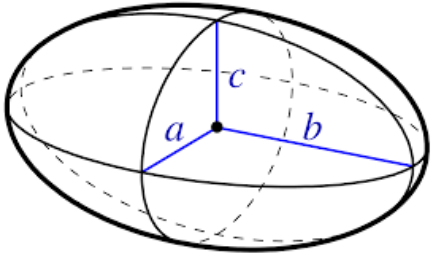
Pure $\oplus$ Applied



FLUID MODEL

Mathematical motivations

Pure ⊕ Applied



1800 - 1880



Jacobi, Dirichlet, Riemann...

1880 - 1930

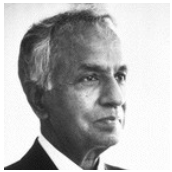


Poincaré

1960 - 1990



Cartan



Chandra



Friedlander



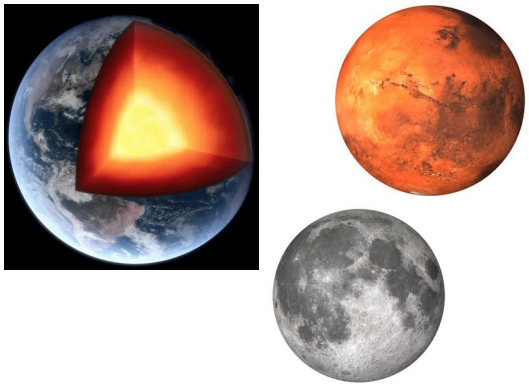
Lebovitz

Backus

Pure ⊕ Applied

Pure $\oplus$ Applied

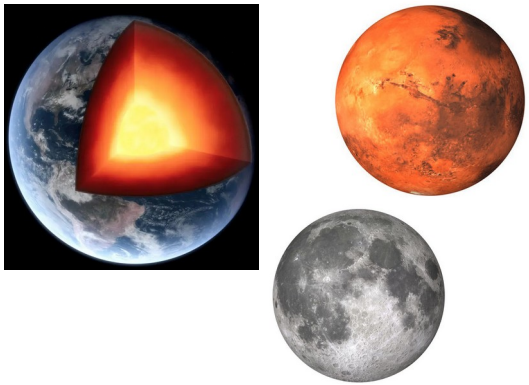
~ 0 $\longrightarrow$ $N/|\mathbf{\Omega}|$

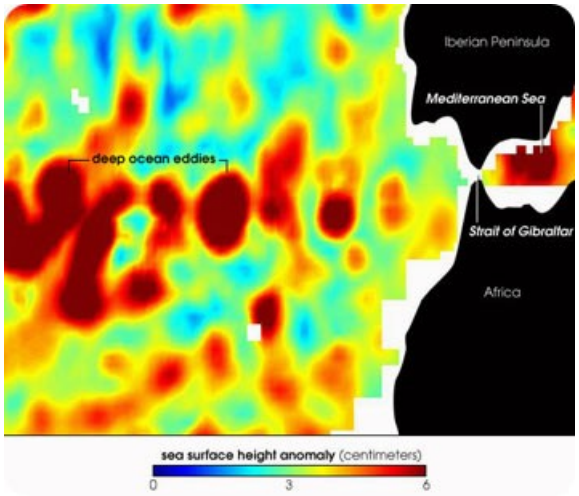


Inertial Waves

Pure $\oplus$ Applied

~ 0 ~ 10 ~ 100 $N/|\mathbf{\Omega}|$







Inertial Waves

Inertia-Gravity Waves

INERTIAL WAVES

Remainder in unbounded fluids

Velocity equation

$$\mathrm{i}\omega\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Poincaré equation

$$\omega^2 \nabla^2 p = (2\boldsymbol{\Omega} \cdot \nabla)^2 p$$

INERTIAL WAVES

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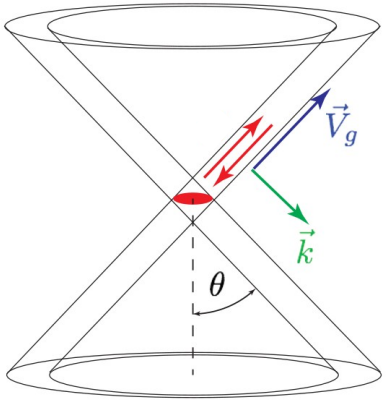
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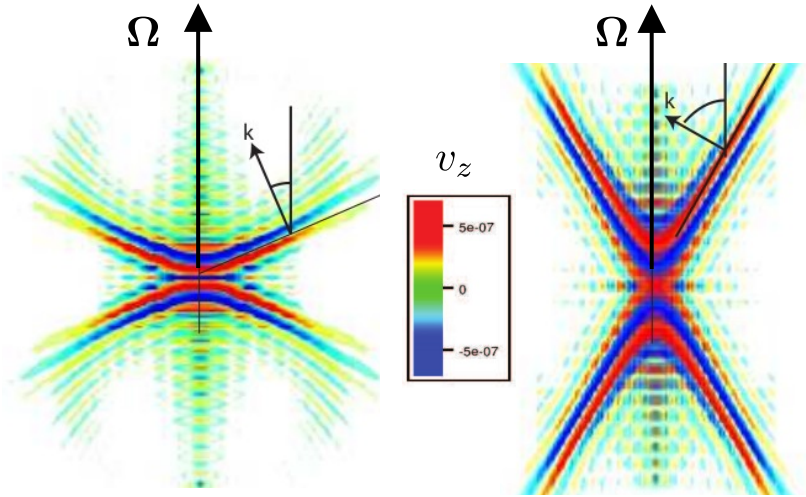
Principal symbol

(i.e. dispersion relation)

$$\mathfrak{p} := \omega^2 |\boldsymbol{k}|^2 - (2\boldsymbol{\Omega} \cdot \boldsymbol{k})^2$$

$$\begin{cases} |\omega| < 2|\boldsymbol{\Omega}| : & \text{Hyperbolic} \\ |\omega| > 2|\boldsymbol{\Omega}| : & \text{Elliptic} \end{cases}$$



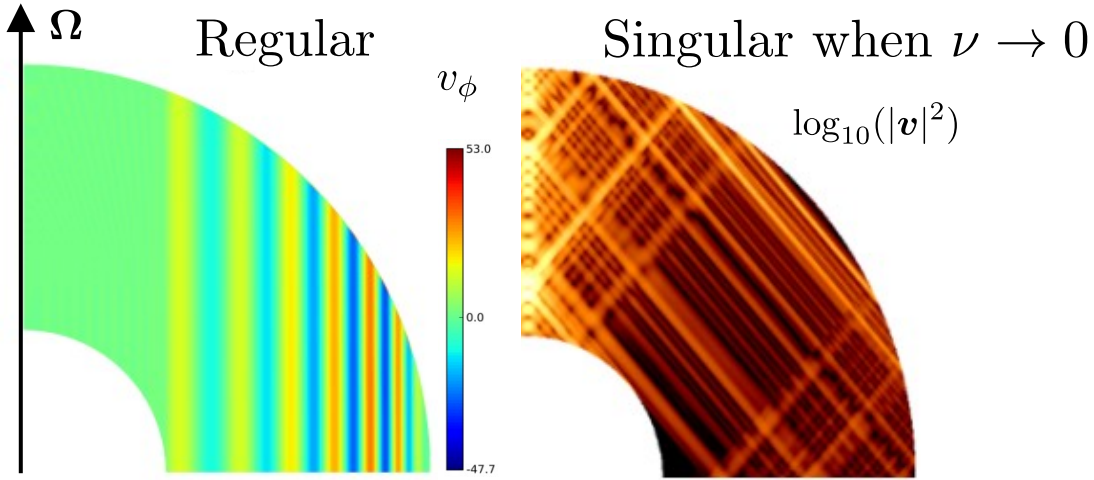


Favier (2009)

INERTIAL WAVES

General case in a bounded geometry

$$0 \leq |\omega| < 2|\mathbf{\Omega}|$$



Rieutord *et al.* (2001)

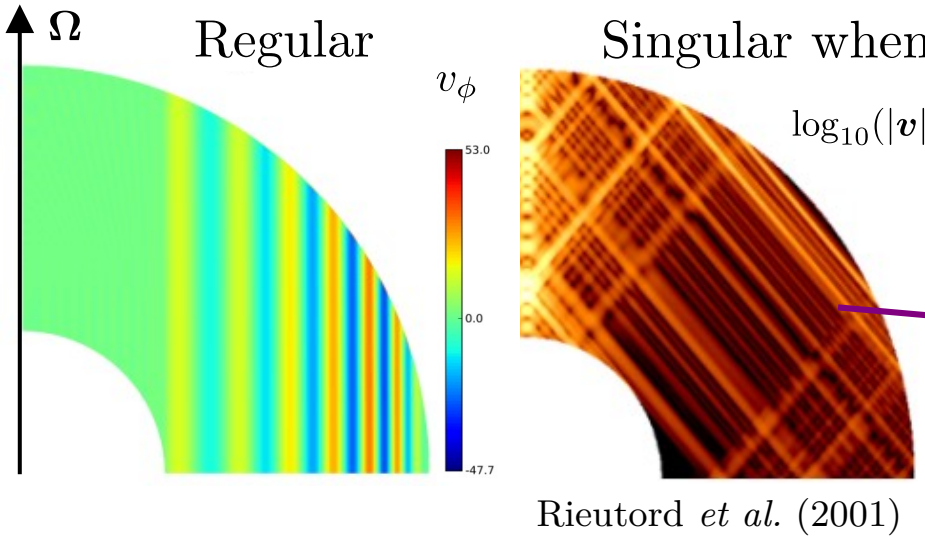
INERTIAL WAVES

General case in a bounded geometry

$$0 \leq |\omega| < 2|\mathbf{\Omega}|$$

An **ill-posed (self-adjoint)** problem

- (Almost empty) **point** spectrum
- **Continuous** spectrum



Rieutord *et al.* (2001)

Attractors for Two-Dimensional Waves with Homogeneous Hamiltonians of Degree 0

YVES COLIN DE VERDIÈRE
Université Grenoble-Alpes, Institut Fourier

LAURE SAINT-RAYMOND
Ecole Normale Supérieure de Lyon, UMPA

Abstract

In domains with topography, inertial and internal waves exhibit interesting features. In particular, numerical and lab experiments show that, in two dimensions, for generic forcing frequencies, these waves concentrate on attractors. The goal of this paper is to analyze mathematically this behavior, using tools from spectral theory and microlocal analysis. © 2019 Wiley Periodicals, Inc.

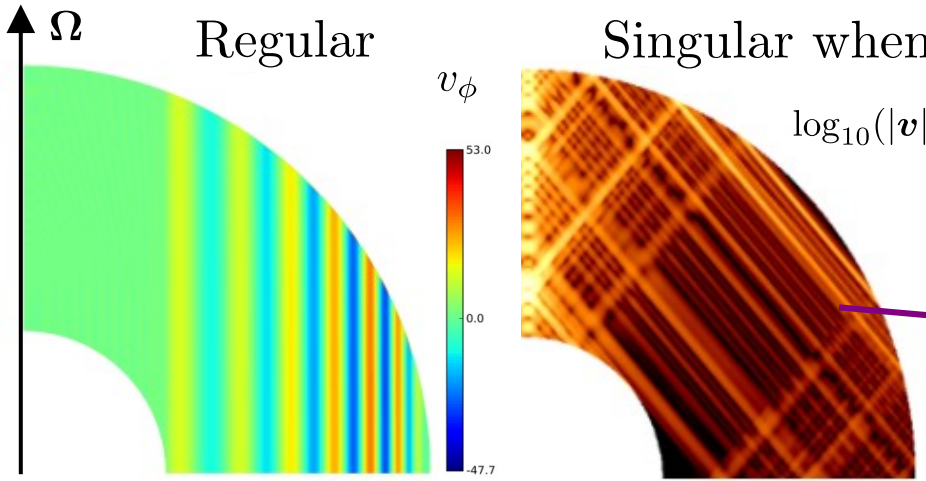
INERTIAL WAVES

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Rieutord *et al.* (2001)

! Shell volume ≠ Ball !

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$\boldsymbol{\mathcal{V}}$: Space of square-integrable vector fields

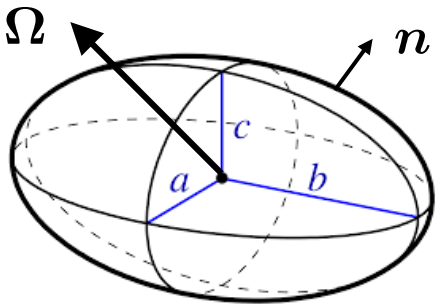
$$\mathbf{H}_{0,\mathrm{div}} := \{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}, \nabla \cdot \boldsymbol{v} = 0 \text{ in } V,\ \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial V} = 0\}$$

$$\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle := \int_V \boldsymbol{v}_1^\dagger \cdot \boldsymbol{v}_2 \, \mathrm{d}V$$

$\boldsymbol{\mathcal{V}}$: Space of square-integrable vector fields

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$$(\partial V): \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Poincaré operator (**bounded** & **self-adjoint**)

$$\mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{v}) := \mathrm{i}\mathbb{P}(2\boldsymbol{\Omega} \times \mathbb{P}\boldsymbol{v}) \qquad \mathbb{P} : \boldsymbol{\mathcal{V}} \to \mathbf{H}_{0,\mathrm{div}}$$

- Dense **pure point** spectrum
- **Complete** set of **eigenvectors**

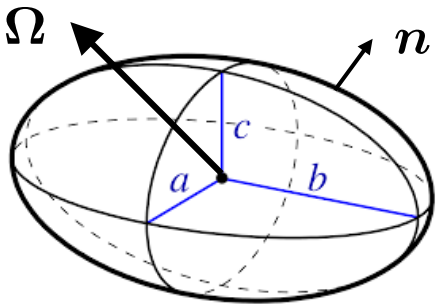
INERTIAL WAVES

Main results in ellipsoids

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$$(\partial V): \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Poincaré operator (**bounded** & **self-adjoint**)

$$\mathrm{i}\boldsymbol{\mathcal{C}}(\boldsymbol{v}) := \mathrm{i}\mathbb{P}(2\boldsymbol{\Omega} \times \mathbb{P}\boldsymbol{v}) \qquad \mathbb{P} : \boldsymbol{\mathcal{V}} \to \mathbf{H}_{0,\mathrm{div}}$$

- Dense **pure point** spectrum
- **Complete** set of **eigenvectors**

$$[\boldsymbol{\mathcal{C}}, \boldsymbol{\mathcal{L}}_E] = \mathbf{0}$$

Dense, polynomial eigenspaces in $\mathbf{H}_{0,\mathrm{div}}$

(~ Compatility theorem in QM)

EXTRA: NUMERICAL METHOD

Modes are polynomials => Bespoke numerical algorithm

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- Hilbertian **polynomial** basis $\{\mathbf{e}_j\}$ in ellipsoids in $\mathbf{H}_{0,\mathrm{div}}$

$$\boldsymbol{u} = \sum_j \alpha_i \boldsymbol{e}_j, \quad \nabla \cdot \boldsymbol{e}_j = 0, \quad \boldsymbol{e}_j \cdot \boldsymbol{n}|_{\partial V} = 0$$

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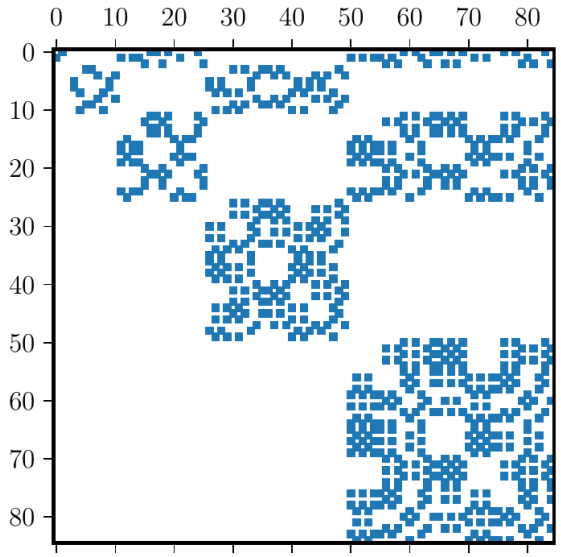
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- **Projection** method (symbolic calculus)

$$\left(-\omega^2 + \omega \mathrm{i} \boldsymbol{C}_n + \boldsymbol{K}_n\right) \boldsymbol{\alpha} = \boldsymbol{0} \qquad \boldsymbol{L}_n = \begin{pmatrix} 0 & \boldsymbol{I} \\ \boldsymbol{K}_n & \mathrm{i}\boldsymbol{C}_n \end{pmatrix}$$

- **Numerical** solutions (exact up to machine precision)



INERTIAL WAVES

Weyl asymptotic law in ellipsoids

Asymptotic distribution

$$\int_{-\infty}^{\lambda} \mathrm{d}\pi_\infty = \frac{1}{4\pi}\mathrm{Area}(\mathfrak{S}_\lambda \cap S^2)$$

$$\mathfrak{S}_\lambda := \{\tilde{\boldsymbol{k}} \in \mathbb{R}^3 \mid \omega(\tilde{\boldsymbol{k}}) \leq \lambda\}$$

$$\tilde{\boldsymbol{k}} = (k_x/a, k_y/b, k_z/c)^\top$$

Rescaled **dispersion relation** of inertial waves

Proof: Pseudo-differential operator calculus with a boundary

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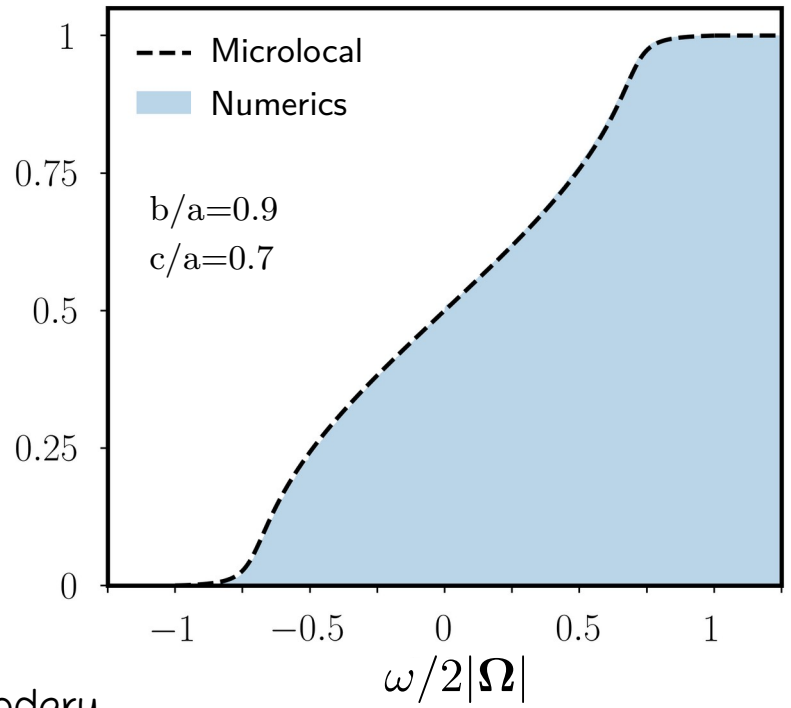
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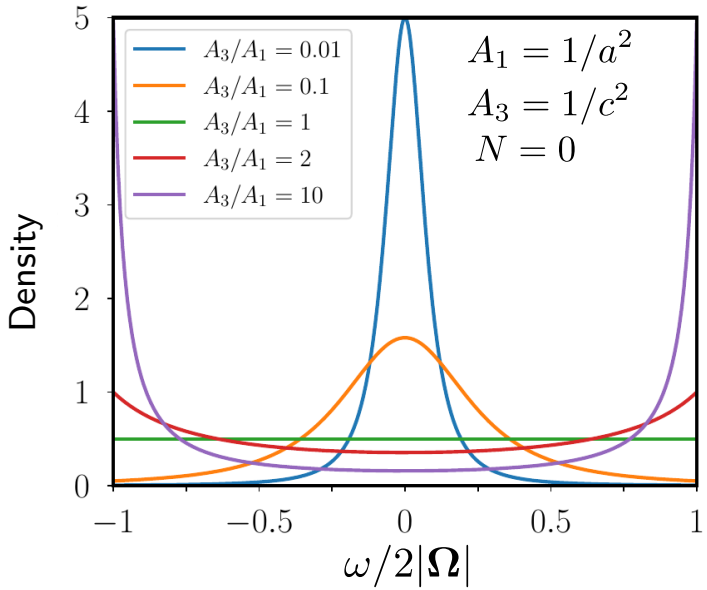
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Agreement with **numerics**

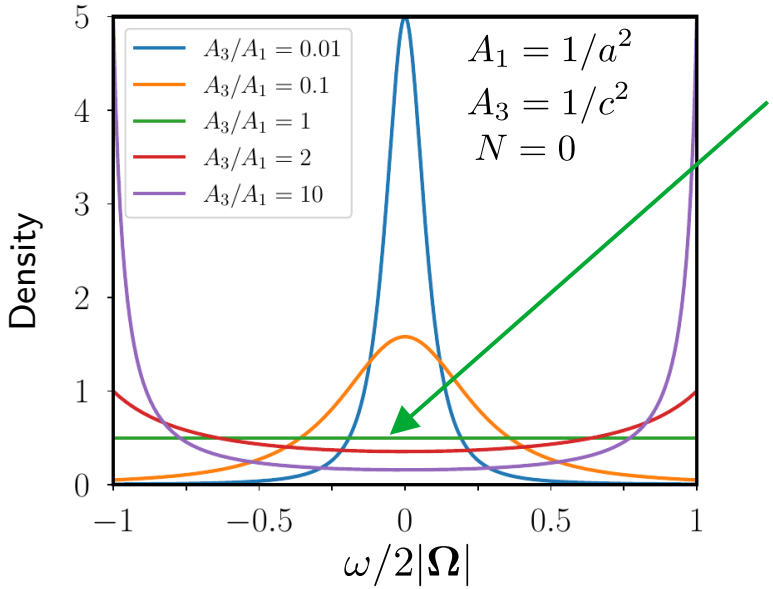
Applications for real configurations (e.g. lab. **experiments** or **DNS**)

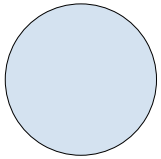


INERTIAL WAVES

Weyl asymptotic law in ellipsoids

Applications for real configurations (e.g. lab. **experiments** or **DNS)**





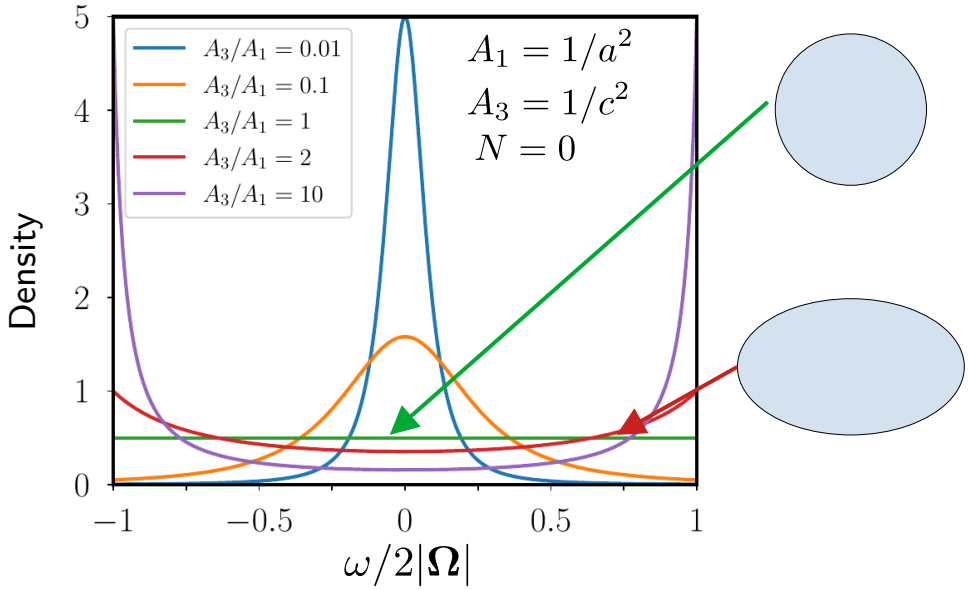
Natural bodies ($A_3/A_1 \sim 1$)



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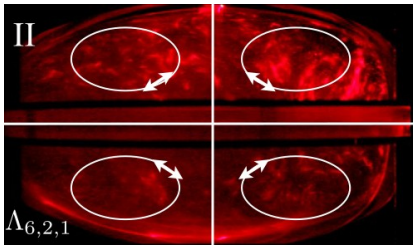


Natural bodies ($A_3/A_1 \sim 1$)



Geometry mismatch

Experiments / Simulations ($A_3/A_1 \sim 3$)

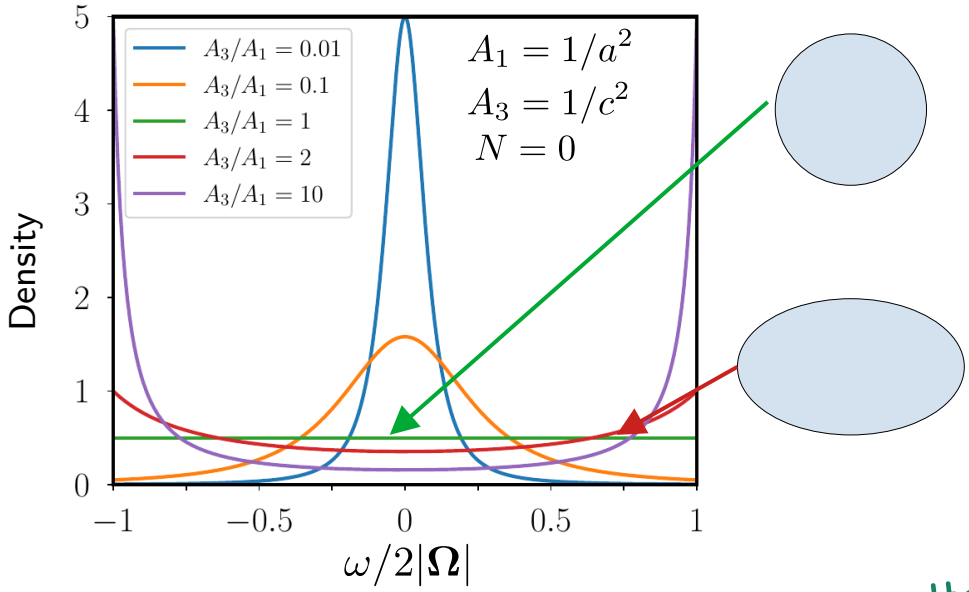


Grannan *et al.* (2014)

INERTIAL WAVES

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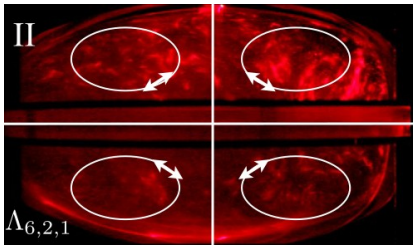


Natural bodies ($A_3/A_1 \sim 1$)



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Grannan *et al.* (2014)

Boundary effects do matter!

INERTIA-GRAVITY WAVES

A quadratic eigenvalue problem

Primitive equations

$$\mathrm{i}\omega\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} = -\nabla p - \zeta N\,\mathbf{1}_z$$

$$\mathrm{i}\omega\zeta = (\boldsymbol{v}\cdot\mathbf{1}_z)\,N$$

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Wave-like equation

$$-\omega^2 \boldsymbol{v} + 2\boldsymbol{\Omega} \times (\mathrm{i}\omega \boldsymbol{v}) + N^2 (\boldsymbol{v} \cdot \mathbf{1}_z)\mathbf{1}_z = \nabla(\dots)$$

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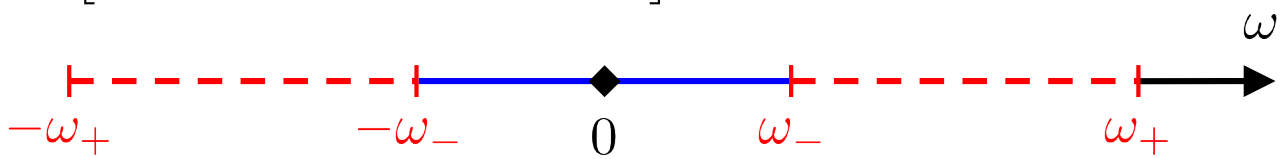
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Principal symbol

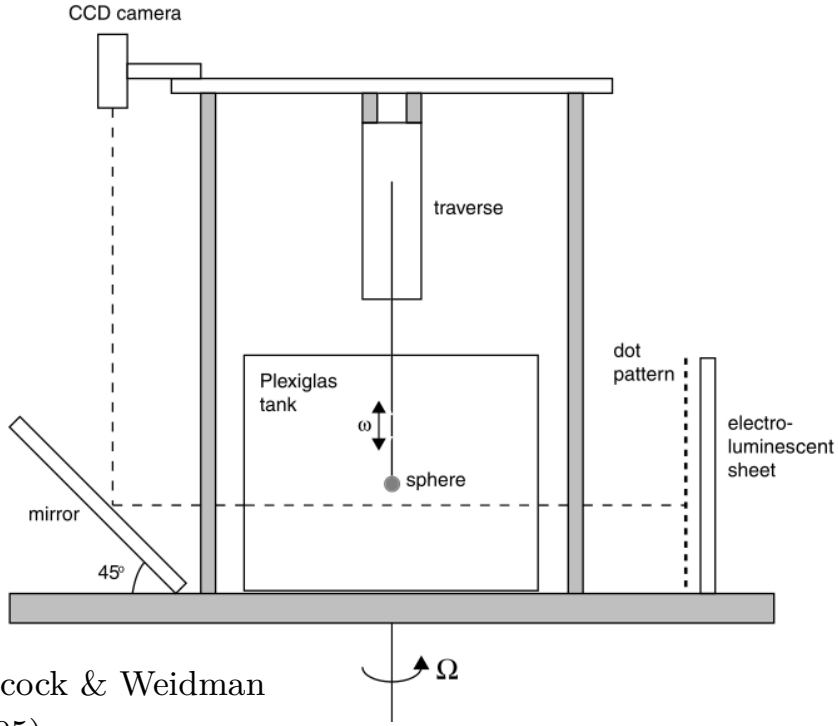
(i.e. dispersion relation)

$$\begin{cases} \omega_-^2 < \omega^2 < \omega_+^2 : & \text{Hyperbolic} \\ 0 < \omega^2 < \omega_-^2 : & \text{Elliptic} \end{cases}$$

$$\mathfrak{p} := |\boldsymbol{k}|^2 \omega^2 - \left[N^2 |\mathbf{1}_z \times \boldsymbol{k}|^2 + (2\boldsymbol{\Omega} \cdot \boldsymbol{k})^2 \right]$$

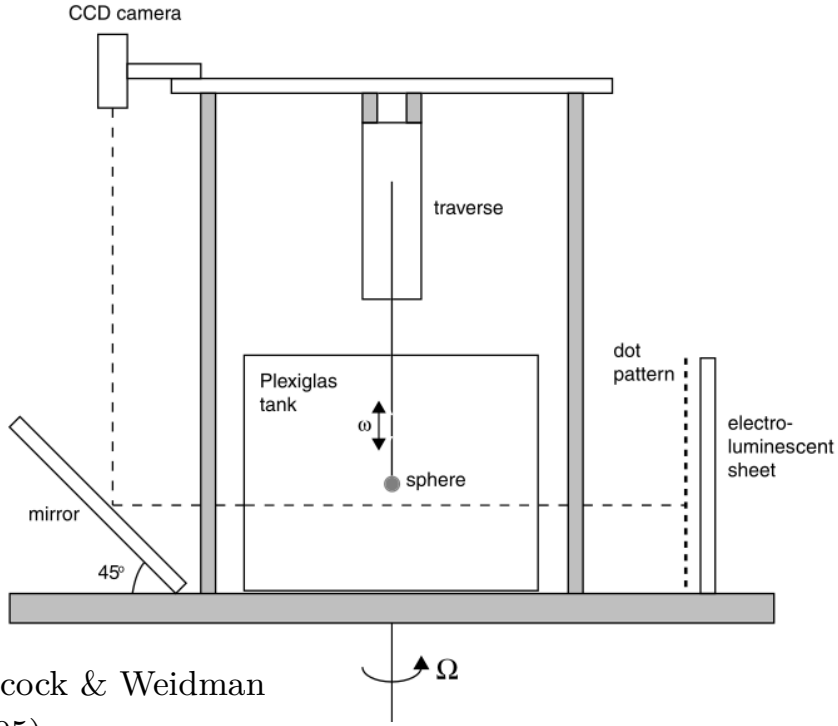


Experiment

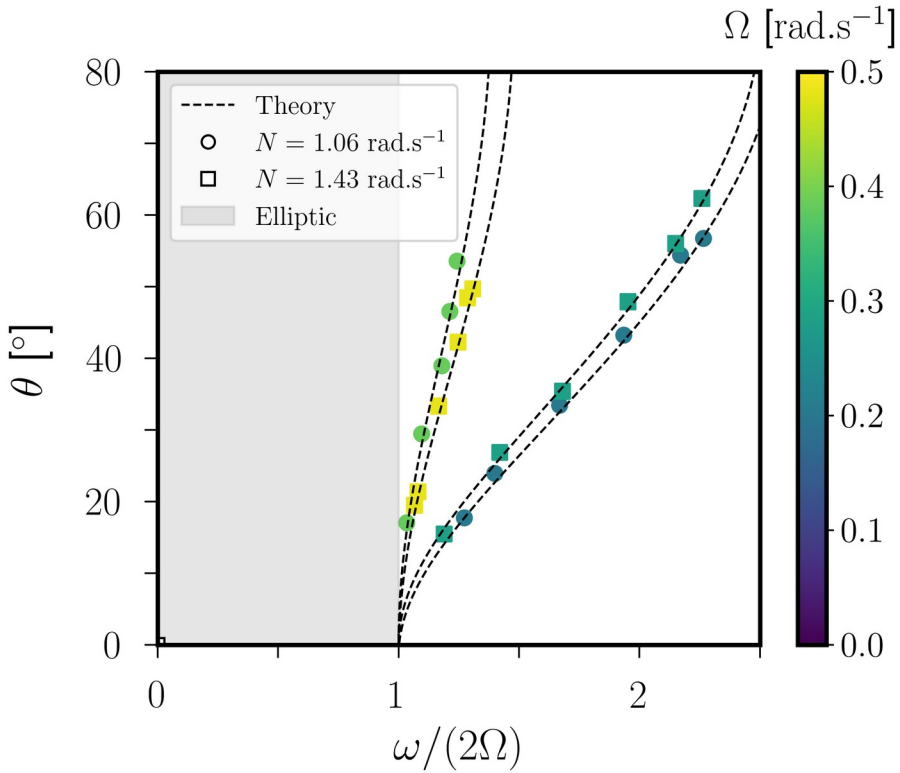


Peacock & Weidman (2005)

Experiment



Peacock & Weidman (2005)



INERTIA-GRAVITY WAVES

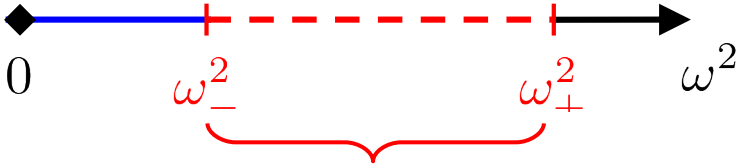
Ellipsoidal case: Hyperbolic frequency interval

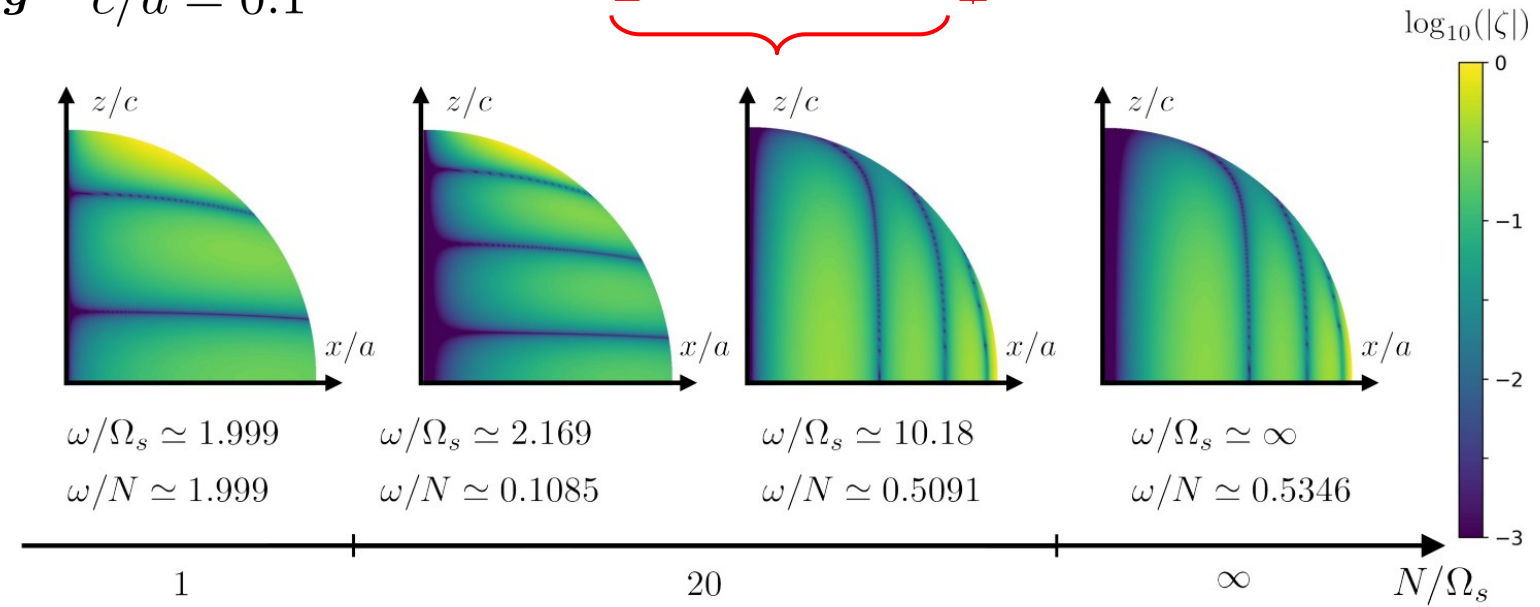
Elliptic in V

Hyperbolic in V

Ω g $\quad c/a = 0.1$

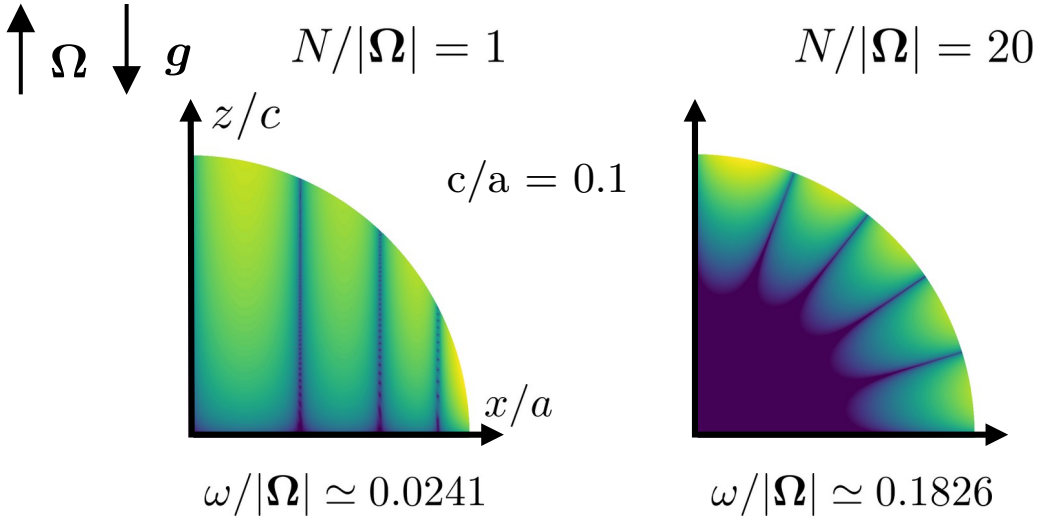
$$2\omega_\pm^2 = [N^2 + 4|\mathbf{\Omega}|^2] \pm \sqrt{[N^2 + 4|\mathbf{\Omega}|^2]^2 - 16N^2(\mathbf{\Omega}\cdot\mathbf{1}_z)^2}$$





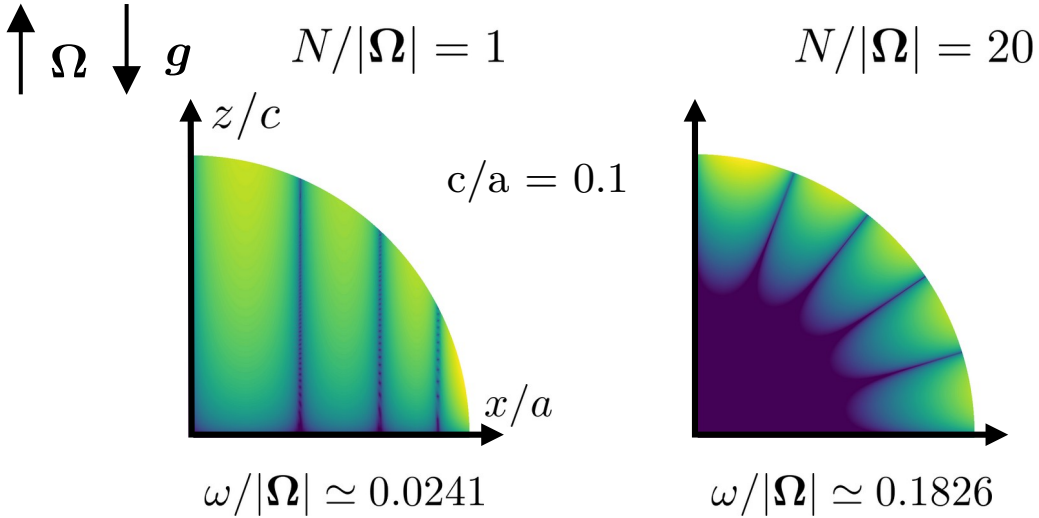
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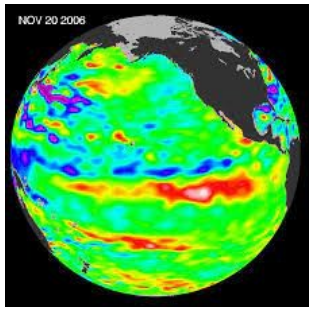
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INERTIA-GRAVITY WAVES

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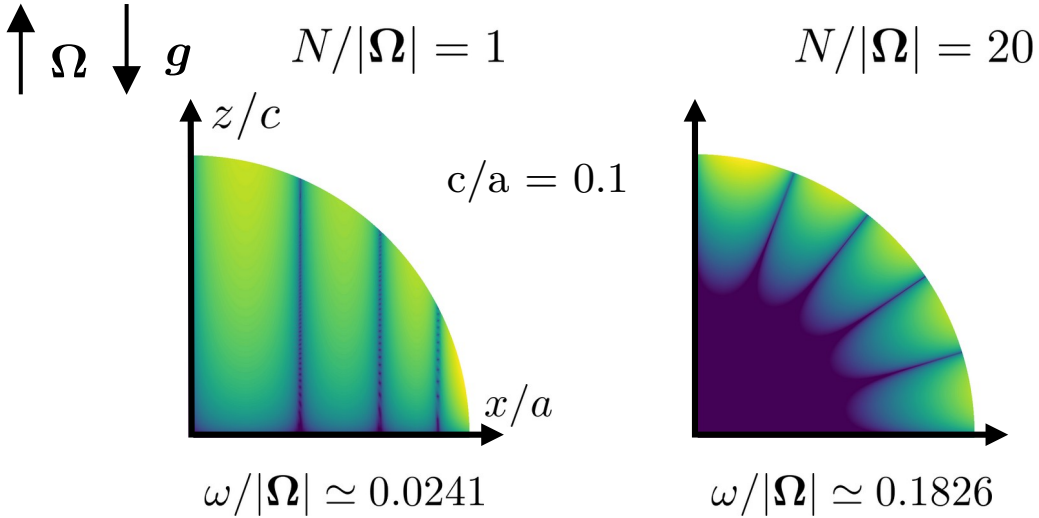


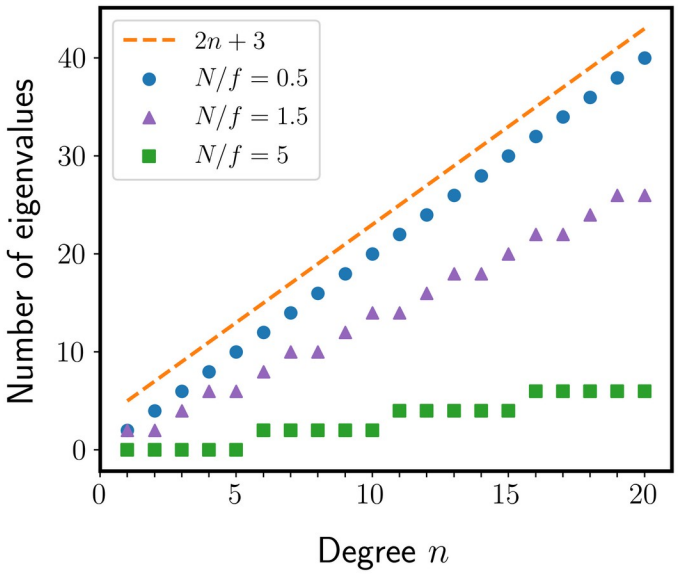


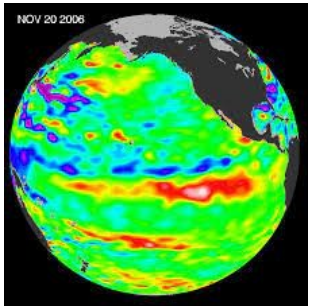
- **Similar** properties than (coastal & equatorial) **Kelvin waves**

INERTIA-GRAVITY WAVES

Ellipsoidal case: Elliptic frequency interval







- **Similar** properties than (coastal & equatorial) **Kelvin waves**
- Dense low-frequency **spectrum** in an ellipsoid

BVP for the pressure

$$-[\omega^2 - N^2]\nabla^2 p - N^2 \left(\mathbf{1}_z \cdot \nabla\right)^2 p + \left(2\boldsymbol{\Omega} \cdot \nabla\right)^2 p = 0$$

$$\boldsymbol{n} \cdot \nabla p|_{\partial V} + \mathcal{B}(\boldsymbol{n} \times \nabla p) = 0$$

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- Principal symbol defines a **metric** g on ∂V
- Conormal vector $\mathbf{m}$ w/r g

Dirichlet-Neumann operator

$$\begin{aligned} \mathcal{P}(p) &= 0 \\ \boldsymbol{m} \cdot \nabla p|_{\partial V} + \mathcal{V}(p) &= 0 \end{aligned} \quad = \quad \begin{aligned} \mathcal{N}(\Psi) &:= \boldsymbol{m} \cdot \nabla p \\ \mathcal{N}(\Psi) + \mathcal{V}(\Psi) &= 0 \end{aligned} \quad + \quad \begin{aligned} \mathcal{P}(p) &= 0 \\ p|_{\partial V} &= \Psi \end{aligned}$$

INERTIA-GRAVITY WAVES

Non-elliptic boundary value problem

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One **principal symbol** on the boundary

~ Shapiro-Lopatinski condition

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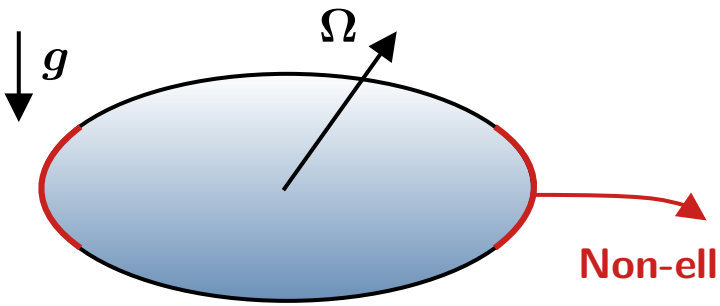
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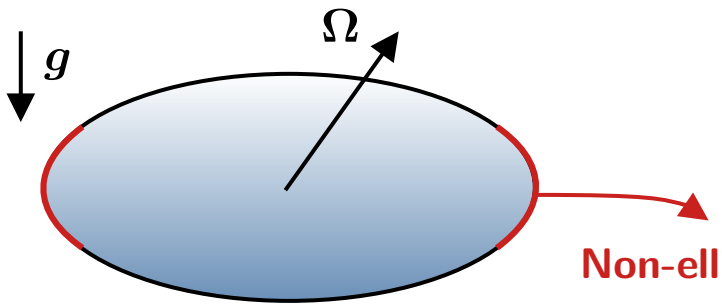
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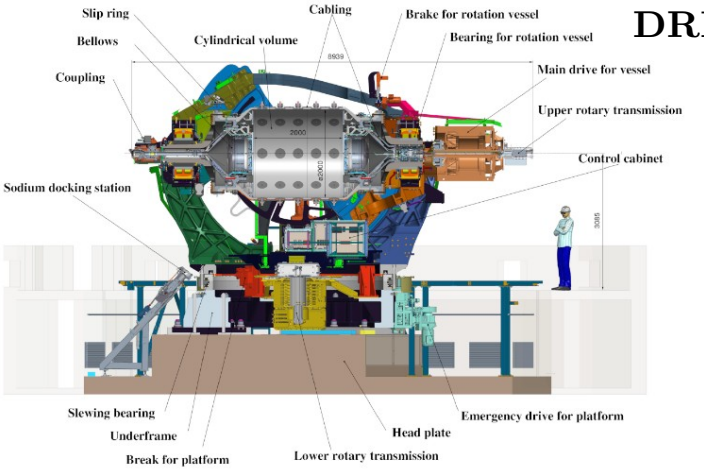


Non-elliptical

Boundary effects do matter!

INERTIA-GRAVITY WAVES

Some follow-up works?



DRESDYN experiment

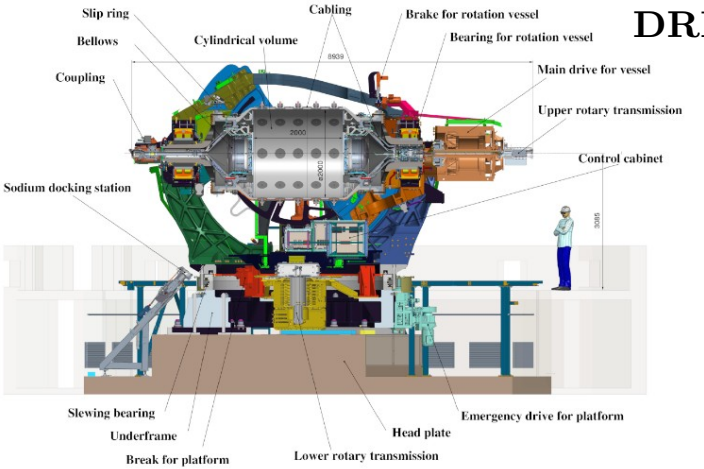
Further wave studies

- Is the modal set **complete** in other geometries?

$$[\mathcal{C}, ?] = 0$$

INERTIA-GRAVITY WAVES

Some follow-up works?

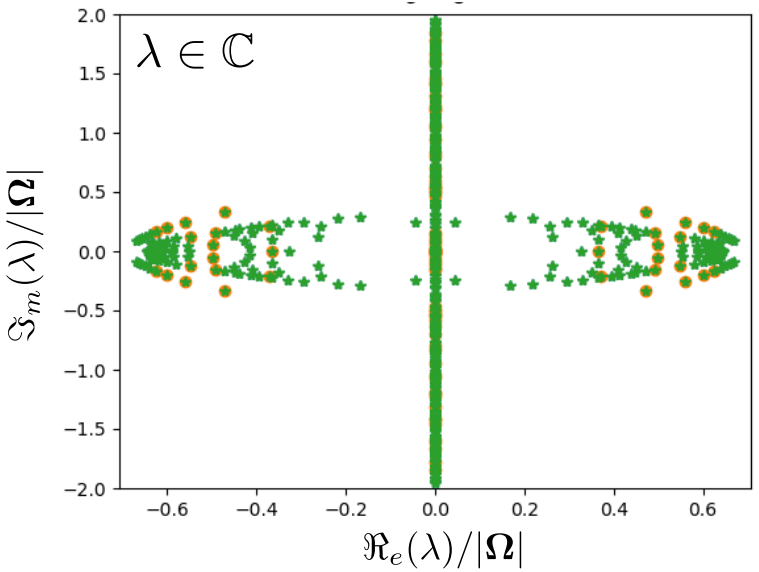


DRESDYN experiment

Further wave studies

- Is the modal set **complete** in other geometries?

$$[\mathcal{C}, ?] = \mathbf{0}$$



Non self-adjoint problem ($N^2 < 0$)

- **Essential** spectrum along the axes
- **Unknown** spectrum outside the axes

PERSPECTIVE: WAVE DRIVEN DYNAMOS?

A geological motivation: early Earth mysterious dynamo

PERSPECTIVE: WAVE DRIVEN DYNAMOS?

A geological motivation: early Earth mysterious dynamo



Tarduno *et al.* (2015, 2020)
Nichols *et al.* (2024)

Jack Hills
(Australia)

PERSPECTIVE: WAVE DRIVEN DYNAMOS?

A geological motivation: early Earth mysterious dynamo

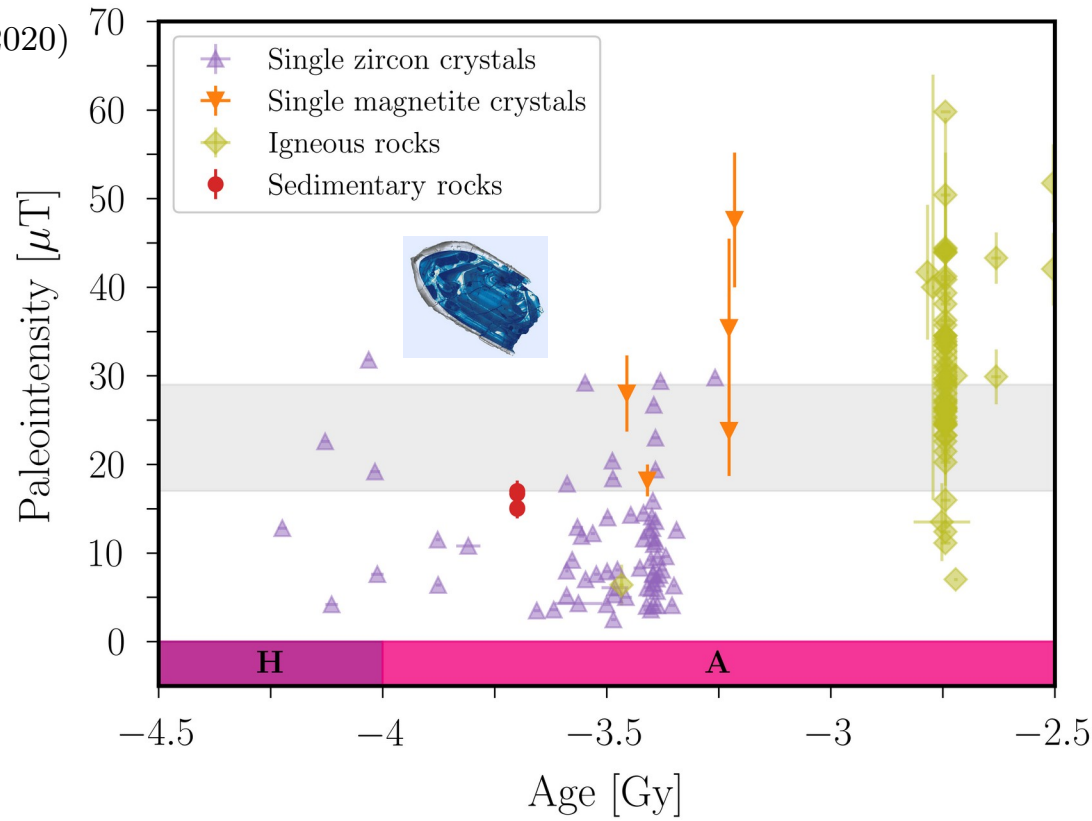


Tarduno *et al.* (2015, 2020)
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Jack Hills
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Paleomagnetism (+ rock chemistry)

- **Geodynamo** since 3 - 4 Ga
- A few mT at the core's surface
- $E(\boldsymbol{B}) > E(\boldsymbol{v})$



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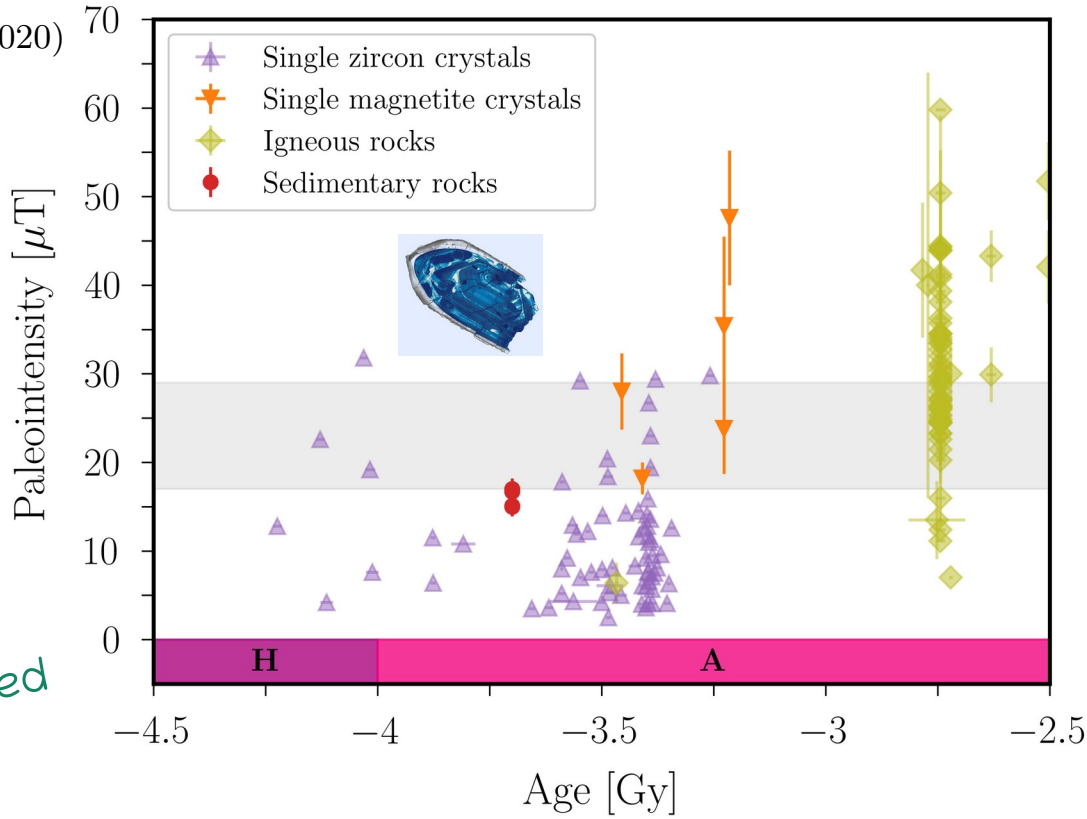
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Physical **origin** is disputed



PERSPECTIVE: WAVE DRIVEN DYNAMOS?

Some pioneering works

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Some pioneering works

J. Fluid Mech. (1970), *vol.* 44, *part* 4, *pp.* 705–719
Printed in Great Britain

Dynamo action associated with random inertial waves in a rotating conducting fluid

By H. K. MOFFATT

Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge

(Received 26 March 1970)

It is shown that a random superposition of inertial waves in a rotating conducting fluid can act as a dynamo, i.e. can systematically transfer energy to a magnetic field which has no source other than electric currents within the fluid. Dynamo action occurs provided the statistical properties of the velocity field lack reflexional symmetry, and this occurs when conditions are such that there is a net energy flux (positive or negative) in the direction of the rotation vector Ω.

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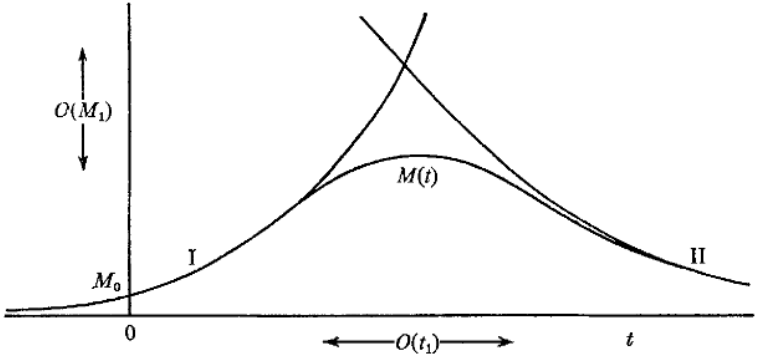
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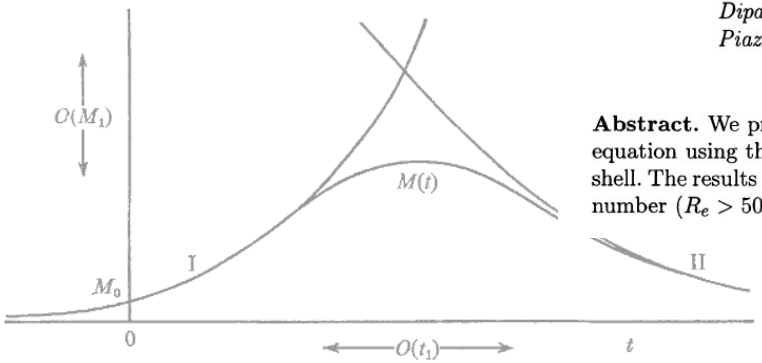
Dynamo action associated with rar waves in a rotating conductir

By H. K. MOFFATT

Department of Applied Mathematics and Theore
Silver Street, Cambridge

(Received 26 March 1970)

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fluid can act as a dynamo, i.e. can systematically transi



DYNAMO EFFECT WITH INERTIAL MODES IN A SPHERICAL SHELL ?

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Observatoire Midi-Pyrénées and Institut Universitaire de France
14 av. E. Belin, F-31400 Toulouse, France

AND

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Dipartimento di Matematica, Politecnico di Milano
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Abstract. We present preliminary results of integrations of the induction equation using the velocity field produced by inertial modes in a spherical shell. The results indicate a possible dynamo action at rather large Reynolds number ($R_e > 500$).

PERSPECTIVE: WAVE DRIVEN DYNAMOS?

Some pioneering works

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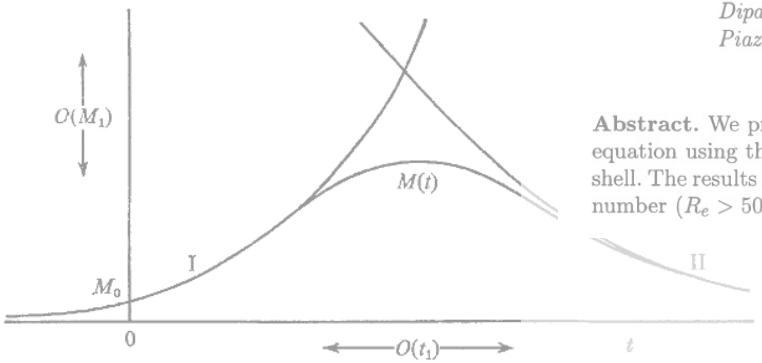
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Stokes drift dynamos

W. HERREMAN[1]† AND P. LESAFFRE[2]

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(Received 5 May 2010; revised 23 December 2010; accepted 2 March 2011;
first published online 19 April 2011)

Fluid particles can have a mean motion in time, even when the Eulerian mean flow disappears everywhere in space. In the present article, we demonstrate that this phenomenon, known as the Stokes drift, plays an essential role in the problem of magnetic field generation by fluctuation flows (kinematic dynamo) at high *Rm*. At leading order, the dynamo is generated by the Stokes drift that acts as if it were a mean flow. This result is derived from a mean-field dynamo theory in terms of time averages, which reveals how classical expressions for alpha and beta tensors actually recombine into a single Stokes drift contribution. In a test case, we find fluctuation flows that have a G. O. Roberts flow as Stokes drift and this allows to confront our model to direct integration of the induction equation. We find an excellent quantitative agreement between the prediction of our model and the results of our simulations. We finally apply our Stokes drift model to prove that a broad class of inertial waves in rapidly rotating flows cannot drive a dynamo.

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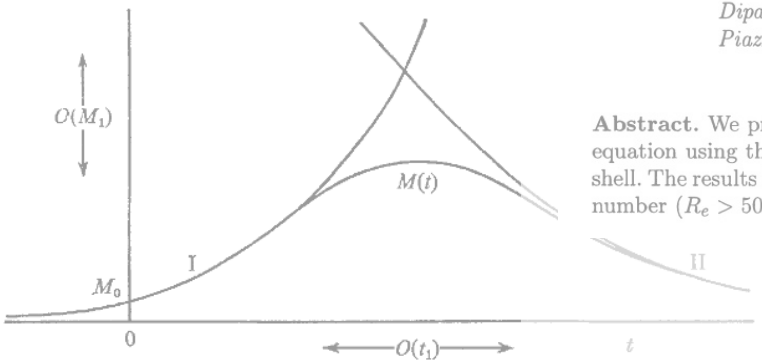
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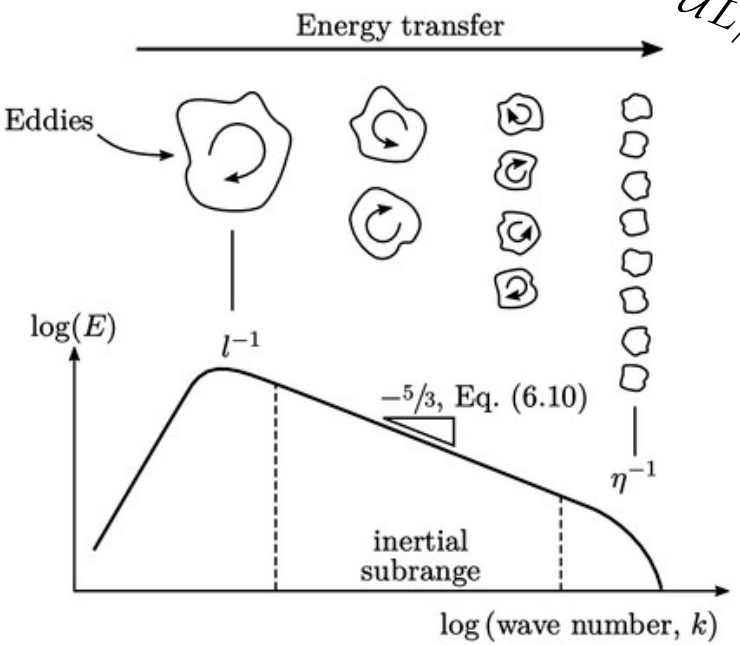
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... and what about **inertial-wave turbulence** (IWT)?

Kolmogorov turbulence (i.e. strong)

$Re = \mathcal{U}L/\nu \gg 1$

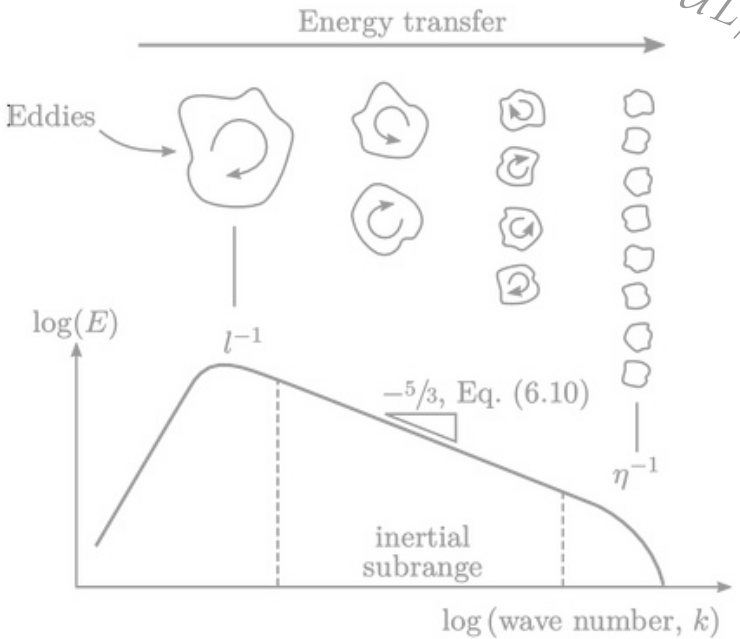


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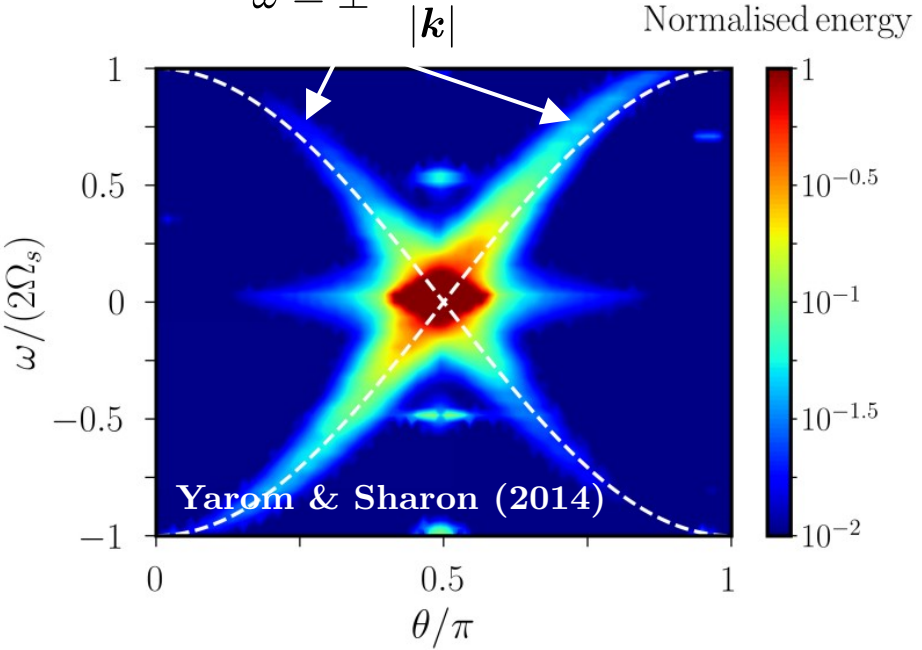
IW turbulence

Kolmogorov turbulence (i.e. strong) $\neq$ **IW turbulence** (i.e. weak)

$Re = \mathcal{U}L/\nu \gg 1$



$$\omega = \pm \frac{2\boldsymbol{\Omega} \cdot \boldsymbol{k}}{|\boldsymbol{k}|}$$



$$E(k_\perp, k_\parallel) \sim \sqrt{\epsilon |\boldsymbol{\Omega}|}\; k_\perp^{-5/2} k_\parallel^{-1/2}$$

PERSPECTIVE: WAVE DRIVEN DYNAMOS?

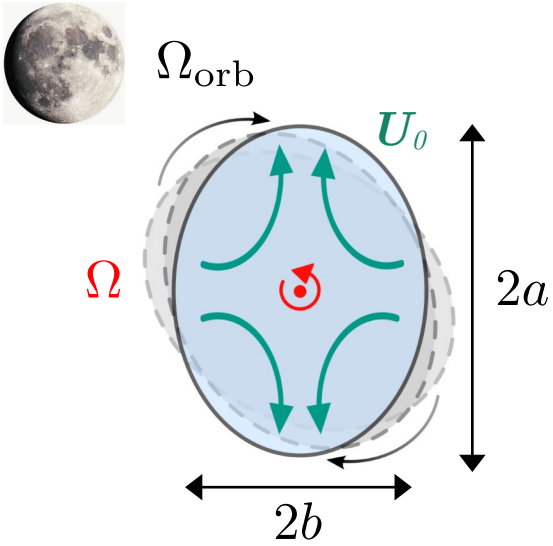
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No rigorous results yet, only **heuristic scenarios** so far...

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Tidal forcing

(e.g. Landeau *et al.*, 2023)



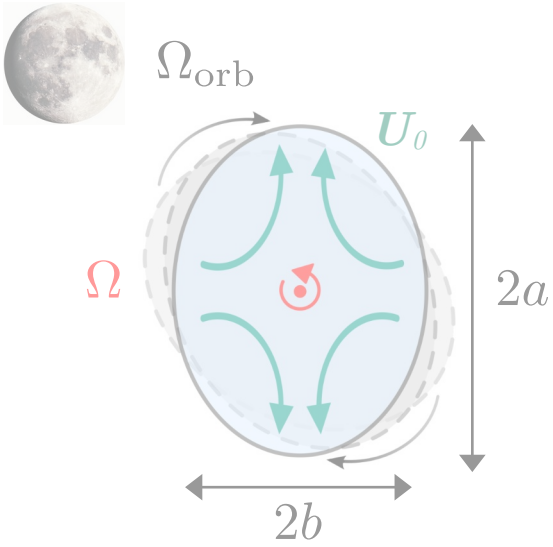
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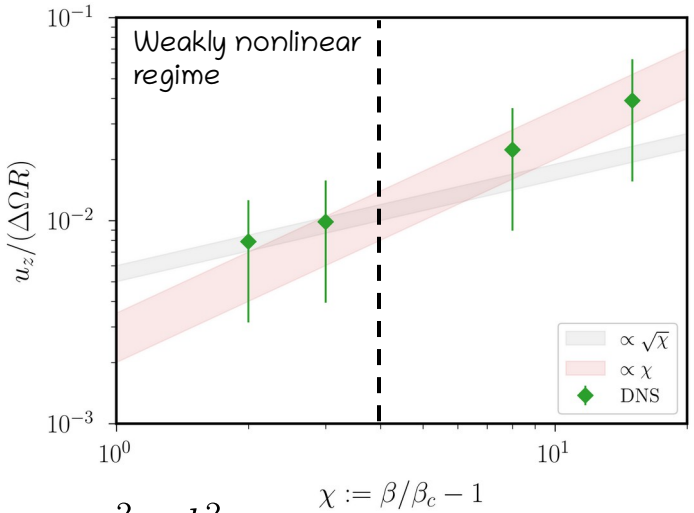
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Elliptical instability

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$$\beta = \frac{a^2 - b^2}{a^2 + b^2}$$

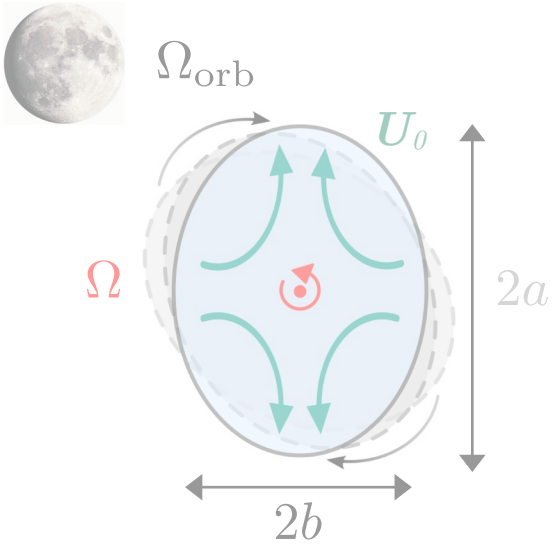
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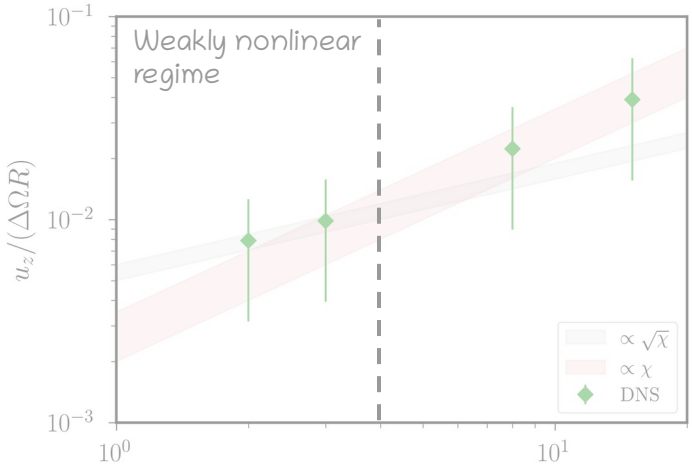
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IWT regime?

(e.g. Le Reun *et al.*, 2020)

$$\epsilon \sim \frac{u_\ell^4 l_\parallel}{|\boldsymbol{\Omega}| l_\perp^3}$$

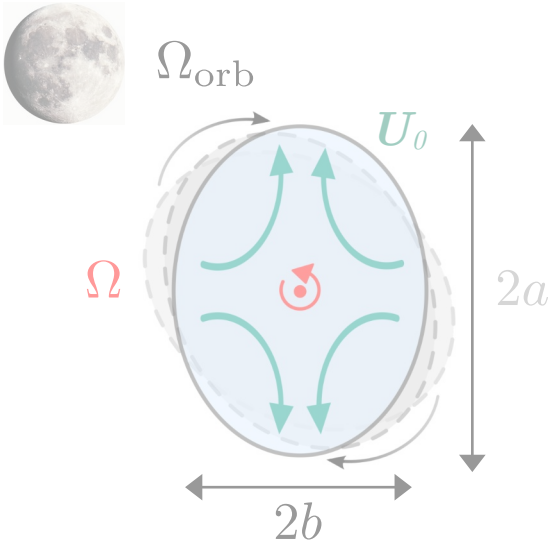
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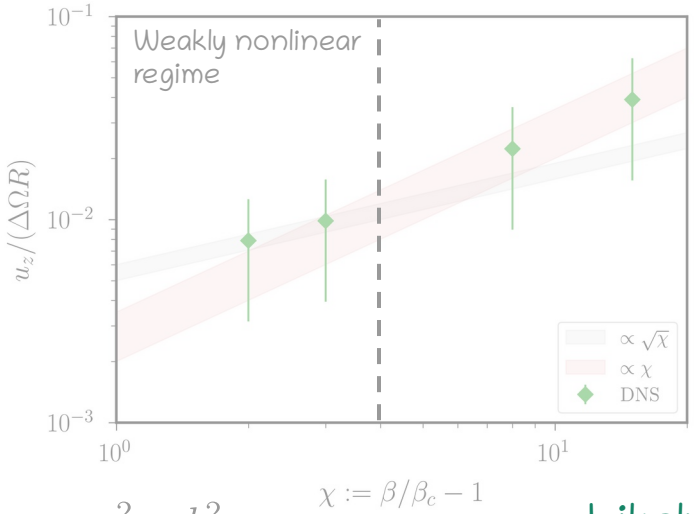
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Diffusion-free regime?

(Vidal *et al.*, 2024)

$$B \sim \sqrt{\rho\mu}(R\mathcal{P})^{1/3}$$

$$\mathcal{P} \sim \epsilon$$

Likely too **weak dynamo** with IWT, but other scenarios are possible

CONCLUSION

Pure ⊕ Applied

Differences with respect to $\mathbb{R}^3$

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Differences with respect to $\mathbb{R}^3$



Further **extension** of the **rigid** ellipsoidal model

- **Spectrum with** $\mathrm{N}^2 < 0$, and magnetic fields
- Hydrodynamic **instabilities**

$$\frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -(\nabla \boldsymbol{U}_0)^\top \boldsymbol{k} \qquad [\boldsymbol{u}, \zeta] = [\boldsymbol{a}, \theta](t) \exp(\mathrm{i}\boldsymbol{k}(t) \cdot \boldsymbol{r})$$

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = \left[\left(2\frac{\boldsymbol{k}\boldsymbol{k}^\top}{|\boldsymbol{k}|^2} - \boldsymbol{I} \right) \nabla \boldsymbol{U}_0 + 2 \left(\frac{\boldsymbol{k}\boldsymbol{k}^\top}{|\boldsymbol{k}|^2} - \boldsymbol{I} \right) \boldsymbol{\Omega}\times \right] \boldsymbol{a} + \theta \left(\boldsymbol{I} - \frac{\boldsymbol{k}\boldsymbol{k}^\top}{|\boldsymbol{k}|^2} \right) \boldsymbol{g}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\boldsymbol{a} \cdot \nabla \rho_0$$

CONCLUSION

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Differences with respect to $\mathbb{R}^3$



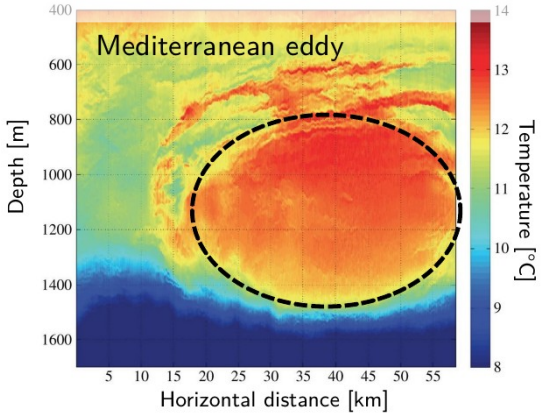
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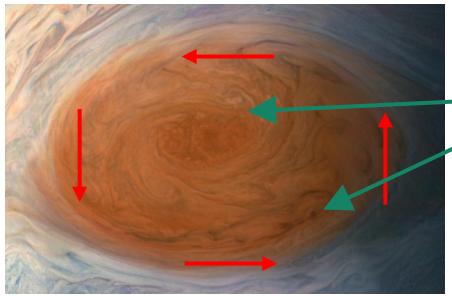
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Others **BCs** for **atmospheric/oceanic** vortices



Bulk **eddies** & **turbulence**

New collaboration starting in 2026