## 1. Around Sobolev inequalities (in relation with D. Arsenio lectures)

We star with the fundamental relation of calculus which claims that if a $C^1$ functions of one real variable which vanishes outside a bounded interval

$$f(x) - f(A) = \int_0^x f(t)dt \quad \text{which implies that} \qquad |f(x)| \leq \|f'\|_{L^1(\mathbb{R})} \tag{1}$$

if we choose $A$ such that $f(A) = 0$.

Let us apply this in the two dimensional space.

**Proposition 1.1.** *We have the following inequality.*

$$\|f\|_{L^2(\mathbb{R}^2)} \leq \|\partial_x f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)} \|\partial_y f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}. \tag{2}$$

*Proof.* We can reduce to the case of a $C^1$ function on $\mathbb{R}^2$ with vanishes outside a square. Then, applying the above inequality to each variable $x$ and $y$, we infer that

$$\forall y \in \mathbb{R},\ \|f(\cdot, y)\|_{L^\infty_x} \leq \|\partial_x f(\cdot, y)\|_{L^1_x} \quad \text{and} \quad \forall x \in \mathbb{R},\ \|f(x, \cdot)\|_{L^\infty_y} \leq \|\partial_y f(x, \cdot)\|_{L^1_y}.$$

Integrating the first equation with respect to $y$ and the second with respect to $x$ gives

$$\|f\|_{L^1_y(L^\infty_x)} \leq \|\partial_x f\|_{L^1(\mathbb{R}^2)} \quad \text{and} \quad \|f\|_{L^1_x(L^\infty_y)} \leq \|\partial_y f\|_{L^1(\mathbb{R}^2)} \tag{3}$$

Now let us apply with for proving Inequality (2). Let us write that

$$\begin{aligned}
\int_{\mathbb{R}^2} |f(x,y)|^2 dxdy &= \int_{\mathbb{R}^2} |f(x,y)|\, |f(x,y)| dxdy \\
&\leq \int_{\mathbb{R}^2} \|f(x,\cdot)\|_{L^\infty_y} \|f(\cdot,y)\|_{L^\infty_x} dxdy \\
&\leq \left( \int_{\mathbb{R}} \|f(x,\cdot)\|_{L^\infty_y} dx \right) \left( \int_{\mathbb{R}} \|f(\cdot,y)\|_{L^\infty_x} dy \right).
\end{aligned}$$

This is exactly Inequality (2). □

Now let us treat the case of dimension three. We have the following proposition.

**Proposition 1.2.** *We have*

$$\|f\|_{L^{\frac{3}{2}}} \leq \|\partial_x f\|^{\frac{1}{3}}_{L^1(\mathbb{R}^3)} \|\partial_y f\|^{\frac{1}{3}}_{L^1(\mathbb{R}^3)} \|\partial_z f\|^{\frac{1}{3}}_{L^1(\mathbb{R}^3)}. \tag{4}$$

*Proof.* It is a little bit more difficult than the case of the two dimensional case. We are going to prove that for any $C^1$ functions which vanishes outside a cube, we have

$$\int_{\mathbb{R}^3} |f(x,y,z)|^{\frac{3}{2}} dxdydz \leq \|\partial_x f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} \|\partial_y f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} \|\partial_z f\|^{\frac{1}{2}}_{L^1(\mathbb{R}^3)} \tag{5}$$

which is an other way of writing Inequality (4). Let us first observe that applying (1) to each variable $x$, $y$ and $z$, we infer that

$$\begin{aligned}
\forall (y,z) \in \mathbb{R}^2,\ \|f(\cdot,y,z)\|_{L^\infty_x} &\leq \|\partial_x f(\cdot,y,z)\|_{L^1_x}, \\
\forall (x,z) \in \mathbb{R}^2,\ \|f(x,\cdot,z)\|_{L^\infty_y} &\leq \|\partial_y f(x,\cdot,z)\|_{L^1_x} \quad \text{and} \\
\forall (x,y) \in \mathbb{R}^2,\ \|f(x,y\cdot,)\|_{L^\infty_z} &\leq \|\partial_z f(\cdot,y,z)\|_{L^1_x}
\end{aligned}$$

Then by integration on $\mathbb{R}^2$ of each above inequalities, we infer that

$$\begin{aligned}
&\|f\|_{L^1_{y,z}(L^\infty_x)} \leq \|\partial_x f\|_{L^1(\mathbb{R}^3)}, \\
&\|f\|_{L^1_{x,z}(L^\infty_y)} \leq \|\partial_y f\|_{L^1(\mathbb{R}^3)} \quad \text{and} \\
&\|f\|_{L^1_{x,y}(L^\infty_y)} \leq \|\partial_z f\|_{L^1(\mathbb{R}^3)}.
\end{aligned} \tag{6}$$

Let us write that

$$\begin{aligned}\int_{\mathbb{R}^3}|f(x,y,z)|^{\frac{3}{2}}dxdydz &\leq \int_{\mathbb{R}^3}|f(x,y,z)|^{\frac{1}{2}}|f(x,y,z)|^{\frac{1}{2}}|f(x,y,z)|^{\frac{1}{2}}dxdydz\\ &\leq \int_{\mathbb{R}^3}\|f(x,y,\cdot)\|^{\frac{1}{2}}_{L^\infty_z}\|f(x,\cdot,z)\|^{\frac{1}{2}}_{L^\infty_y}\|f(\cdot,y,z)\|^{\frac{1}{2}}_{L^\infty_z}dxdydz\\ &\leq \int_{\mathbb{R}^2}\|f(x,y,\cdot)\|^{\frac{1}{2}}_{L^\infty_z}\left(\int_{\mathbb{R}}\|f(x,\cdot,z)\|^{\frac{1}{2}}_{L^\infty_y}\|f(\cdot,y,z)\|^{\frac{1}{2}}_{L^\infty_x}dz\right)dxdy \qquad (7)\end{aligned}$$

Then Cauchy-Schwarz inequality implies that

$$\int_{\mathbb{R}}\|f(x,\cdot,z)\|^{\frac{1}{2}}_{L^\infty_y}\|f(\cdot,y,z)\|^{\frac{1}{2}}_{L^\infty_x}dz \leq \left(\int_{\mathbb{R}}\|f(x,\cdot,z)\|_{L^\infty_y}dz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\|f(\cdot,y,z)\|_{L^\infty_x}dz\right)^{\frac{1}{2}}$$

Then plugging this into the inequality (7) ensures that

$$\begin{aligned}&\int_{\mathbb{R}^3}|f(x,y,z)|^{\frac{3}{2}}dxdydz\\ &\qquad\leq \int_{\mathbb{R}^2}\|f(x,y,\cdot)\|^{\frac{1}{2}}_{L^\infty_z}\left(\int_{\mathbb{R}}\|f(x,\cdot,z)\|_{L^\infty_y}dz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\|f(\cdot,y,z)\|_{L^\infty_x}dz\right)^{\frac{1}{2}}dxdy.\end{aligned}$$

Applying Cauchy-Schwarz to the two functions $\|f(x,y,\cdot)\|^{\frac{1}{2}}_{L^\infty_z}$ and

$$\left(\int_{\mathbb{R}}\|f(x,\cdot,z)\|_{L^\infty_y}dz\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\|f(\cdot,y,z)\|_{L^\infty_x}dz\right)^{\frac{1}{2}}$$

gives

$$\begin{aligned}\int_{\mathbb{R}^3}|f(x,y,z)|^{\frac{3}{2}}dxdydz &\leq \left(\int_{\mathbb{R}^2}\|f(x,y,\cdot)\|_{L^\infty_z}dxdy\right)^{\frac{1}{2}}\\ &\qquad\times\left(\left(\int_{\mathbb{R}^2}\|f(x,\cdot,z)\|_{L^\infty_y}dz\right)\left(\int_{\mathbb{R}}\|f(\cdot,y,z)\|_{L^\infty_x}dz\right)dxdy\right)^{\frac{1}{2}}\end{aligned}$$

which can be written

$$\int_{\mathbb{R}^3}|f(x,y,z)|^{\frac{3}{2}}dxdydz \leq \|f\|^{\frac{1}{2}}_{L^1_{x,y}(L^\infty_z)}\|f\|^{\frac{1}{2}}_{L^1_{x,z}(L^\infty_y)}\|f\|^{\frac{1}{2}}_{L^1_{y,z}(L^\infty_x)}.$$

Then applying equalities (6) gives the result. □

Then we can deduce the famous Sobolev inequality in the three dimensional space when the gradient of a functuon belongs to $L^2$.

**Theorem 1.1.** *We have the following inequality*

$$\|f\|_{L^6(\mathbb{R}^3)} \leq \|\partial_x f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}\|\partial_y f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}\|\partial_z f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}.$$

*Proof.* In fact this fundamental result is a an easy consequence of Proposition 1.2. Let us compute the partial derivative of a $C^1$ function vanishing outside a cube. We have

$$\partial_x(f^4) = 4f^3\partial_x f \quad \text{and of course} \quad \partial_y(f^4) = 4f^3\partial_y f \quad \text{and} \quad \partial_z(f^4) = 4f^3\partial_z f.$$

Using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}\|\partial_x(f^4)\|_{L^1} &\leq 4\|f^3\|_{L^2}\|\partial_x f\|_{L^2} \quad \text{and of course}\\ \|\partial_y(f^4)\|_{L^1} &\leq 4\|f^3\|_{L^2}\|\partial_y f\|_{L^2} \quad \text{and}\\ \|\partial_z(f^4)\|_{L^1} &\leq 4\|f^3\|_{L^2}\|\partial_z f\|_{L^2}.\end{aligned}$$

Thus applying Proposition 1.2, and the above inequalities, we infer that

$$\begin{aligned}
\|f^4\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &\leq \|\partial_x(f^4)\|_{L^1}^{\frac{1}{3}}\|\partial_y(f^4)\|_{L^1}^{\frac{1}{3}}\|\partial_z(f^4)\|_{L^1}^{\frac{1}{3}} \\
&\leq \Big(4\|f^3\|_{L^2}\|\partial_x f\|_{L^2}\Big)^{\frac{1}{3}}\Big(4\|f^3\|_{L^2}\|\partial_y f\|_{L^2}\Big)^{\frac{1}{3}}\Big(4\|f^3\|_{L^2}\|\partial_z f\|_{L^2}\Big)^{\frac{1}{3}} \\
&\leq 4\|f^3\|_{L^2}\|\partial_x f\|_{L^2}^{\frac{1}{3}}\|\partial_y f\|_{L^2}^{\frac{1}{3}}\|\partial_z f\|_{L^2}^{\frac{1}{3}}. \qquad (8)
\end{aligned}$$

Let us observe that

$$\begin{aligned}
\|f^4\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} &= \left(\int_{\mathbb{R}^3}|f(x,y,z)|^{4\times\frac{3}{2}}dxdydz\right)^{\frac{2}{3}} = \|f\|_{L^6}^4 \quad \text{and} \\
\|f^3\|_{L^2} &= \left(\int_{\mathbb{R}^3}|f(x,y,z)|^{3\times 2}dxdydz\right)^{\frac{1}{2}} = \|f\|_{L^6}^3.
\end{aligned}$$

Then Inequality (8) becomes

$$\|f\|_{L^6}^4 \leq 4\|f\|_{L^6}^3\|\partial_x f\|_{L^2}^{\frac{1}{3}}\|\partial_y f\|_{L^2}^{\frac{1}{3}}\|\partial_z f\|_{L^2}^{\frac{1}{3}}.$$

The simplification by $\|f\|_{L^6}^3$ gives the result. □

Now let us state and prove the general Sobolev inequality in the $L^2$ framework.

**Theorem 1.2** (General Sobolev embedding in $L^2$ framework)**.** *If $s$ is bigger than $d/2$, then we have*

$$\|f\|_{L^\infty} \leq C\|f\|_{H^s} \quad \textit{with} \quad \|f\|_{H^s} \stackrel{def}{=} \Big(\int_{\mathbb{R}^d}\big(1+|\xi|^2\big)^s|\widehat{f}(\xi)|^2d\xi\Big)^{\frac{1}{2}}. \qquad (9)$$

*Let $s$ a real in the interval $[0, d/2[$. Then the space $H^s(\mathbb{R}^d)$ is inclus in the space $L^{\frac{2d}{d-2s}}$ and a constant $C$ such that, if $p_s \stackrel{def}{=} \dfrac{2d}{d-2s}$, then*

$$\|f\|_{L^p} \leq C\|f\|_{\dot{H}^s} \quad \textit{with} \quad \|f\|_{\dot{H}^s} \stackrel{def}{=} \Big(\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{f}(\xi)|^2d\xi\Big)^{\frac{1}{2}}. \qquad (10)$$

*Proof of Inequality (9).* It relies on Fourier inverse theorem. Let us write that

$$|\widehat{u}(\xi)| = (1+|\xi|^2)^{-s/2}(1+|\xi|^2)^{s/2}|\widehat{u}(\xi)|.$$

As $s$ is bigger that $d/2$,the function $\xi \mapsto (1+|\xi|^2)^{-s/2}$ is integrable on $L^2(\mathbb{R}^d)$. Cauchy-Schwarz inequality ensures that the function $\widehat{u}$ is integrable and that

$$\|\widehat{u}\|_{L^1} \leq \left(\int(1+|\xi|^2)^{-s}d\xi\right)^{\frac{1}{2}}\|u\|_{H^s}.$$

This concludes the proof of Inequality (9) □

Before going technically into the proof of Inequality (10), let us observe that the value of the index $p_s$ is determined by a argument of homogeneity. Indeed, let us consider a function $v$ of $\mathcal{D}(\mathbb{R}^d)$ and let us define $v_\lambda(x) = v(\lambda x)$. We have then

$$\|v_\lambda\|_{L^p} = \lambda^{-\frac{d}{p}}\|v\|_{L^p}$$

and also

$$\begin{aligned}
\int|\xi|^{2s}|\widehat{v_\lambda}(\xi)|^2d\xi &= \lambda^{-2d}\int|\xi|^{2s}|\widehat{v}(\lambda^{-1}\xi)|^2d\xi \\
&= \lambda^{-d+2s}\|v\|_{\dot{H}^s}^2.
\end{aligned}$$

If Inequality (10) is true, then for any real strictement positif $\lambda$, we have

$$\lambda^{-\frac{d}{p}}\|v\|_{L^p} \leq C\lambda^{-\frac{d}{2}+s}\|v\|_{\dot{H}^s}.$$

We also must have

$$-\frac{d}{p} = -\frac{d}{2} + s \quad \text{which imposes} \quad p = p_s = \frac{2d}{d-2s}\cdot$$

*Proof of Inequality (10).* Fubini's theorem ensures that

$$\begin{aligned} p_s \int_0^\infty A^{p_s-1}\mu(|f| > A)dA &= \int_0^\infty \int_X p_s\lambda^{p_s-1}\mathbf{1}_{\{(A,x)/|f(x)|>A\}}(A,x)d\mu(x)d\lambda \\ &= \int_X \left(\int_0^{|f(x)|} p_s A^{p_s-1}dA\right)d\mu(x) \\ &= \int_X |f(x)|^{p_s}d\mu(x). \end{aligned} \tag{11}$$

The proof has four steps:

(1) we fist decompose the function $f$ in a high and a low frequency part;
(2) we estimate the $L^\infty$ norm of the low frequency part using Inequality (9);
(3) we ajuste the threshold of frequency cutoff with parameter $A$ to get rid of the low frequency part;
(4) then we use Bienaymé-Tchebychev and Fourier Plancherel theorem to estimate the high frequency part.

*The decomposition in a high and a low frequency part* Let us consider a function $\chi$ smooth and compactly supported in the unit ball of $\mathbb{R}^d$ and with value 1 near the ball of center 0 and radius 1/2. Let us define

$$f_{\flat,\Lambda} \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\chi(\Lambda^{-1}\xi)\widehat{f}(\xi)\big) \quad \text{and} \quad f_{\sharp,\Lambda} \stackrel{\text{def}}{=} f - f_{\flat,\Lambda}.$$

*The estimate of the low frequency part* Let us use Inequality (9) and then Cauchy-Schwarz inequality to write

$$\begin{aligned} \|f_{\flat,\Lambda}\|_{L^\infty} &\leq (2\pi)^{-d}\|\widehat{f_{\flat,\Lambda}}\|_{L^1} \\ &\leq (2\pi)^{-d}\int_{B(0,\Lambda)} |\xi|^{-s}|\xi|^s|\widehat{f}(\xi)|d\xi \\ &\leq (2\pi)^{-d}\left(\int_{B(0,\Lambda)} |\xi|^{-2s}d\xi\right)^{\frac{1}{2}} \||f\|_{\dot{H}^s} \\ &\leq \frac{V_d}{(d-2s)^{\frac{1}{2}}}\Lambda^{\frac{d}{2}-s}\|f\|_{\dot{H}^s}. \end{aligned} \tag{12}$$

*The choice of the size of the frequency cutoff* In order to simplify the writing, let us multiply the function $f$ by $\dfrac{(d-2s)^{\frac{1}{2}}}{V_d^{\frac{1}{2}}\|f\|_{\dot{H}^s}}$. Then we have that

$$\|f\|_{\dot{H}^s} = \frac{(d-2s)^{\frac{1}{2}}}{V_d} \quad \text{and thus} \quad \sup_{\Lambda>0}\frac{\|f_{\flat,\Lambda}\|_{L^\infty}}{\Lambda^{\frac{d}{2}-s}} \leq 1\,. \tag{13}$$

Now let us write the triangle inequality in terms of sets. We have

$$(|f| > A) \subset \left(|f_{\flat,\Lambda}| > \frac{A}{2}\right) \cup \left(\left(|f_{\sharp,\Lambda}| > \frac{A}{2}\right)\cdot$$

Now let us choose

$$\Lambda_A \stackrel{\text{def}}{=} \left(\frac{A}{4}\right)^{\frac{p_s}{d}} \quad \text{because} \quad \frac{d}{2} - s = \frac{d}{p_s}\cdot \tag{14}$$

Because of (13), we have

$$\mu\left(|f_{\flat,\Lambda_A}| > \frac{A}{2}\right) = 0 \quad \text{and then} \quad \|f\|_{L^{p_s}}^{p_s} \leq p_s \int_0^\infty A^{p_s-1}(\left(|f_{\sharp,\Lambda_A}| > \frac{A}{2}\right) dA. \tag{15}$$

*The estimate of the high frequency part* Bienaymé-Tchebychev inequality ensures that

$$\mu\left(|f_{2,\Lambda_A}| > \frac{A}{2}\right) \leq 4\frac{\|f_{2,\Lambda_A}\|_{L^2}^2}{A^2}\,.$$

Then Inequality (15) and Fourier-Plancherel theorem ensure that

$$\begin{aligned} \|f\|_{L^{p_s}}^{p_s} &\leq 4p_s \int_0^\infty A^{p_s-3}\|f_{2,\Lambda_A}\|_{L^2}^2 dA \\ &\leq 4p_s(2\pi)^{-d}\int_{\mathbb{R}_+\times\mathbb{R}^d} \lambda^{p_s-3}\mathbf{1}_{\{(A,\xi)\,/\,2|\xi|\geq\Lambda_A\}}(A,\xi)|\widehat{f}(\xi)|^2 d\xi d\lambda. \end{aligned} \tag{16}$$

Then Definition (15) of $\Lambda_A$ implies that

$$|\xi| \geq \frac{\Lambda_A}{2} \Longleftrightarrow A \leq 8|\xi|^{\frac{d}{p_s}}.$$

Thus Inequality (16) becomes

$$\begin{aligned} \|f\|_{L^{p_s}}^{p_s} &\leq 4p_s \int_{\mathbb{R}^d}\left(\int_0^{8|\xi|^{\frac{d}{p_s}}} A^{p-3}dA\right)|\widehat{f}(\xi)|^2 d\xi \\ &\leq \frac{4^{2p_s-3}p_s}{p_s-2}(2\pi)^{-d}\int_{\mathbb{R}^d}|\xi|^{\frac{d(p_s-2)}{p_s}}|\widehat{f}(\xi)|^2 d\xi. \end{aligned}$$

By definition of $p_s$, the ratio $\dfrac{d(p_s-2)}{p_s}$ is equal to $2s$. This concludes the proof of the theorem. □