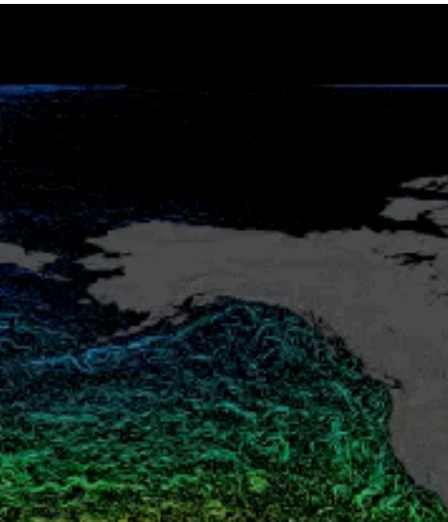
Antoine Venaille

A bit of ocean dynamics on the beta plane



Lab PHYS ENS de LYON

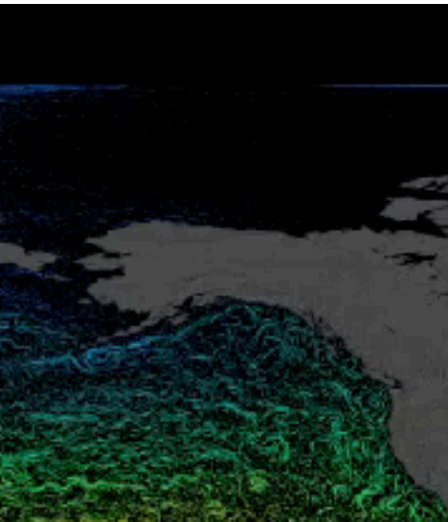
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ENS DE LYON

Cargese, April 2025

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A bit of ocean dynamics on the beta plane

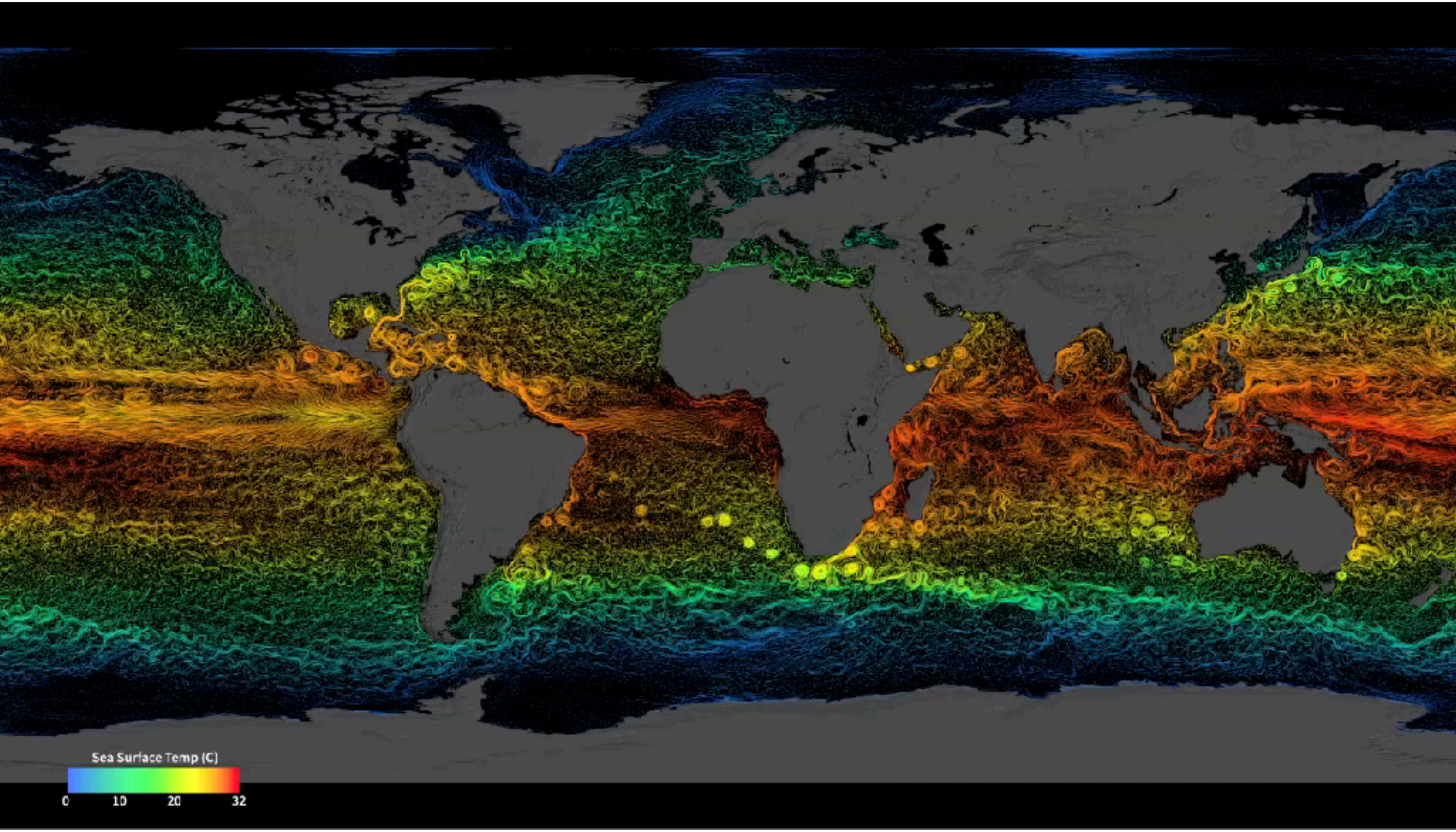


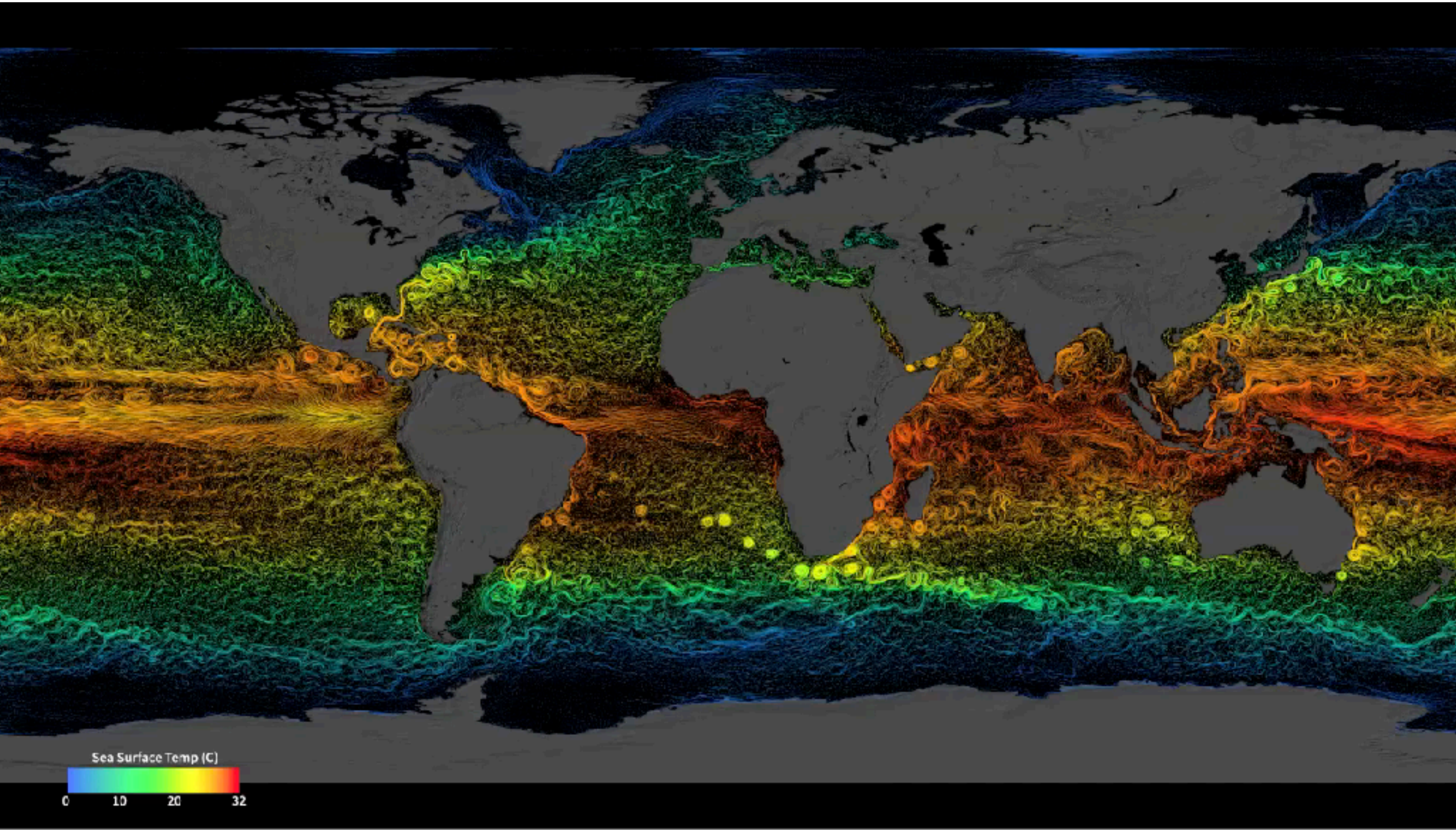
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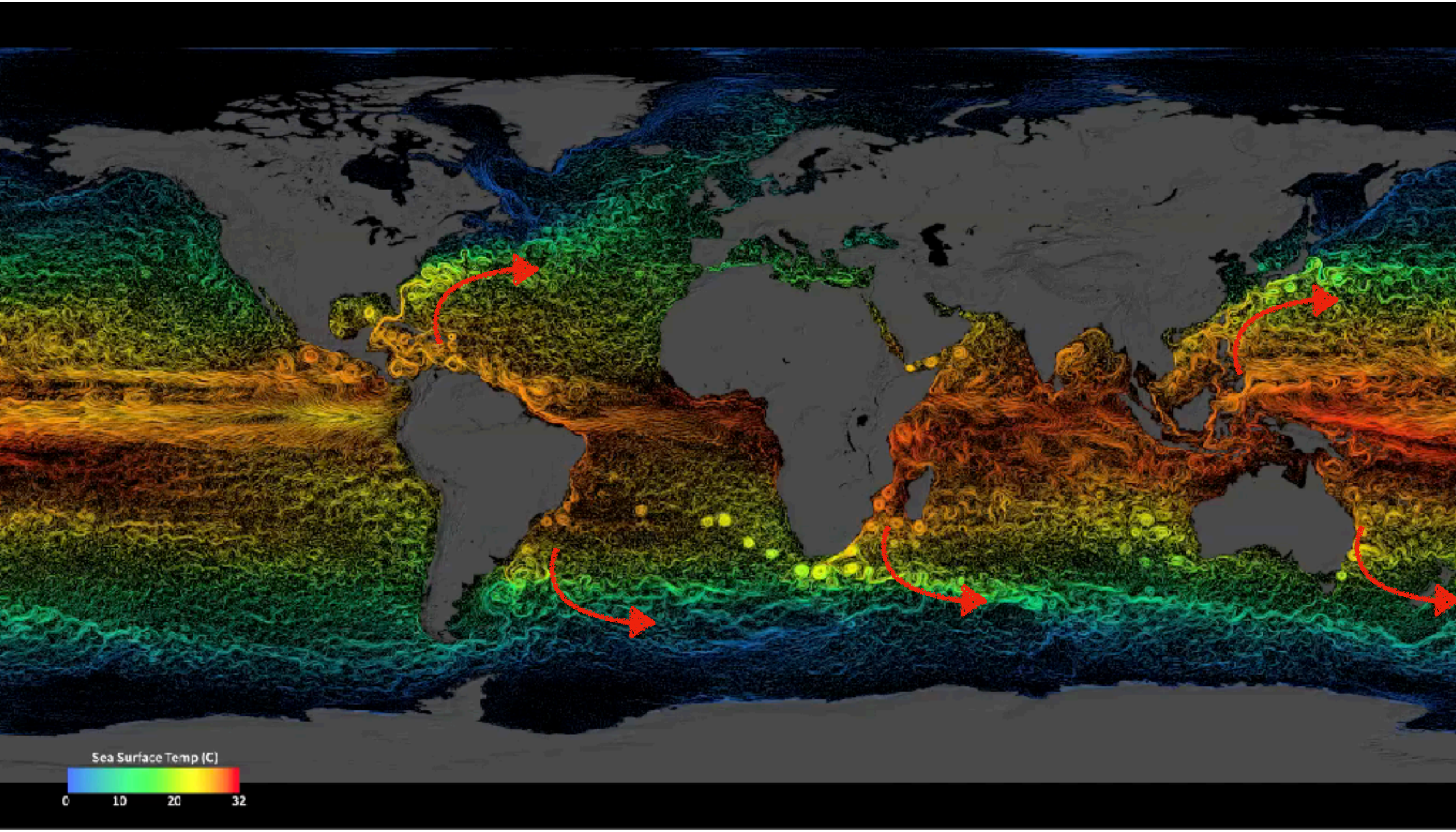
ENS DE LYON

Cargese, April 2025

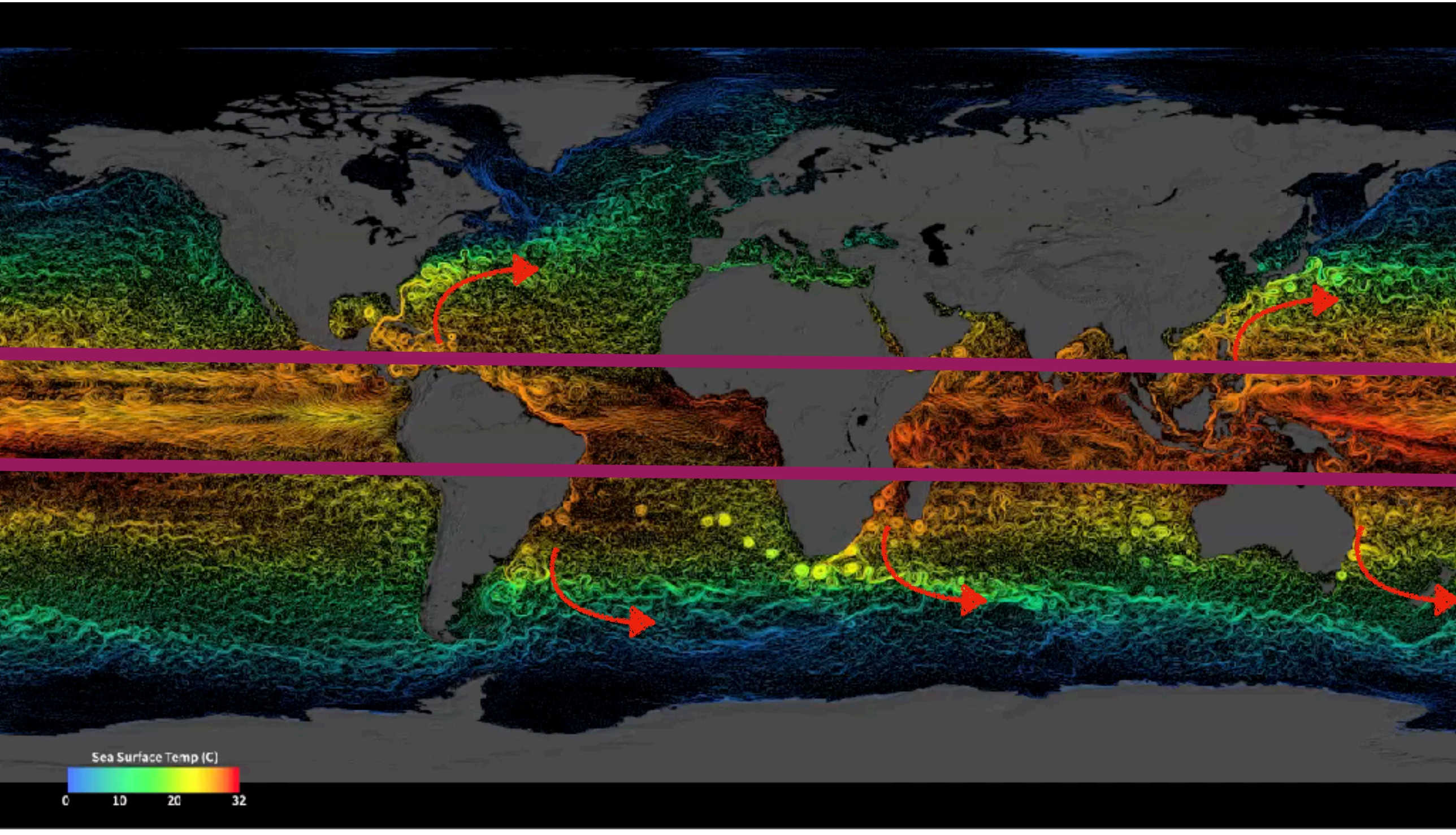

Sea Surface Temp (C)
0
10
20
32


Sea Surface Temp (C)
0
10
20
32

Western intensification of ocean currents



PART I Western intensification of ocean currents



An equatorial waveguide PART II

The beta plane

Coriolis parameter

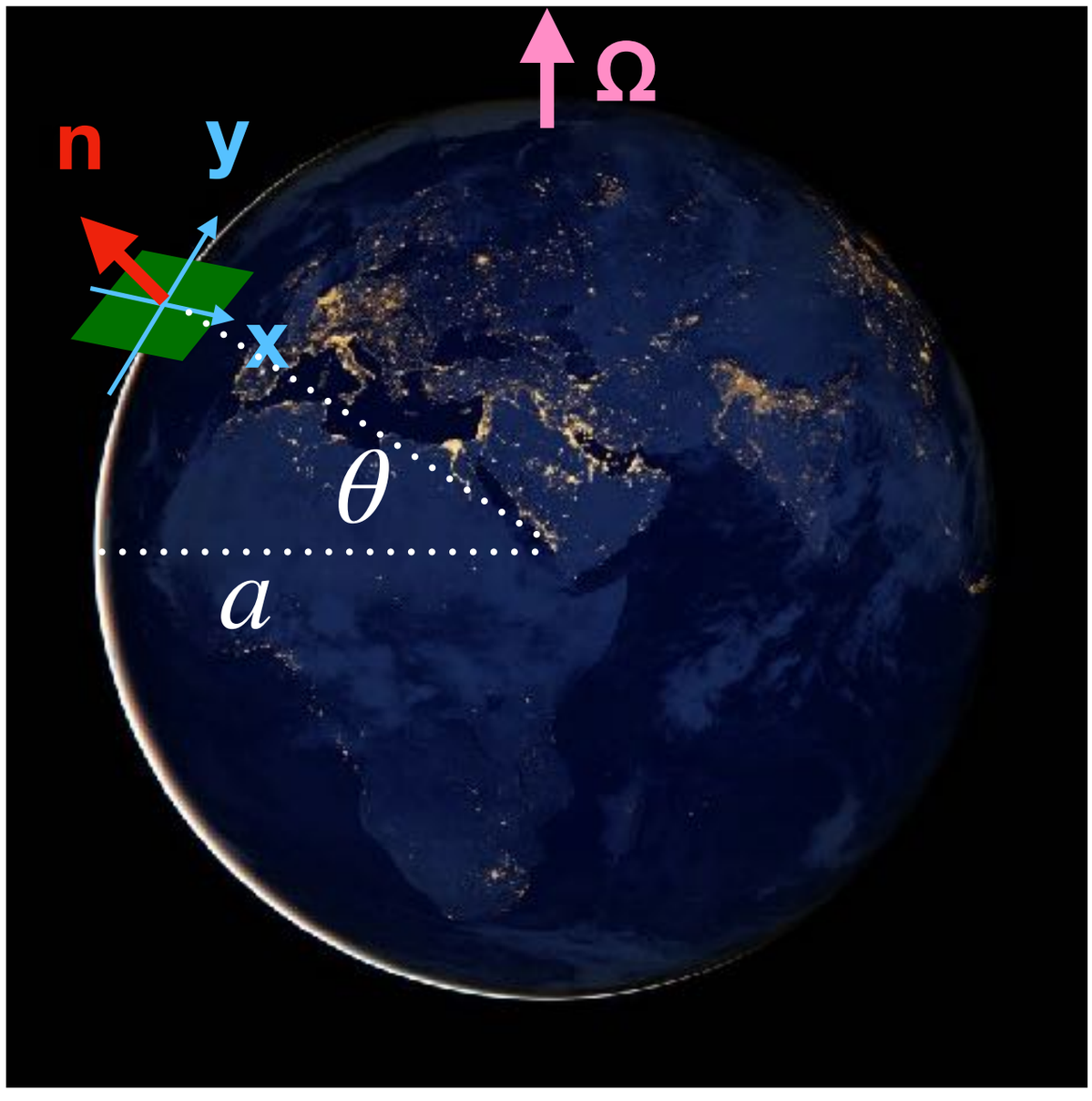
$$f = 2\boldsymbol{\Omega} \cdot \mathbf{n} = 2\Omega \sin\theta$$

Tangent plane approximation

$$y = a(\theta - \theta_0) + \ldots$$

f**-plane** $\quad f = f_0$

β**-plane** $\quad f = f_0 + \beta y$



PART I

Why is there a systematic western intensification of ocean current ?

Answered in the 40ties with linear theory

Here

numerical study of the turbulent regime

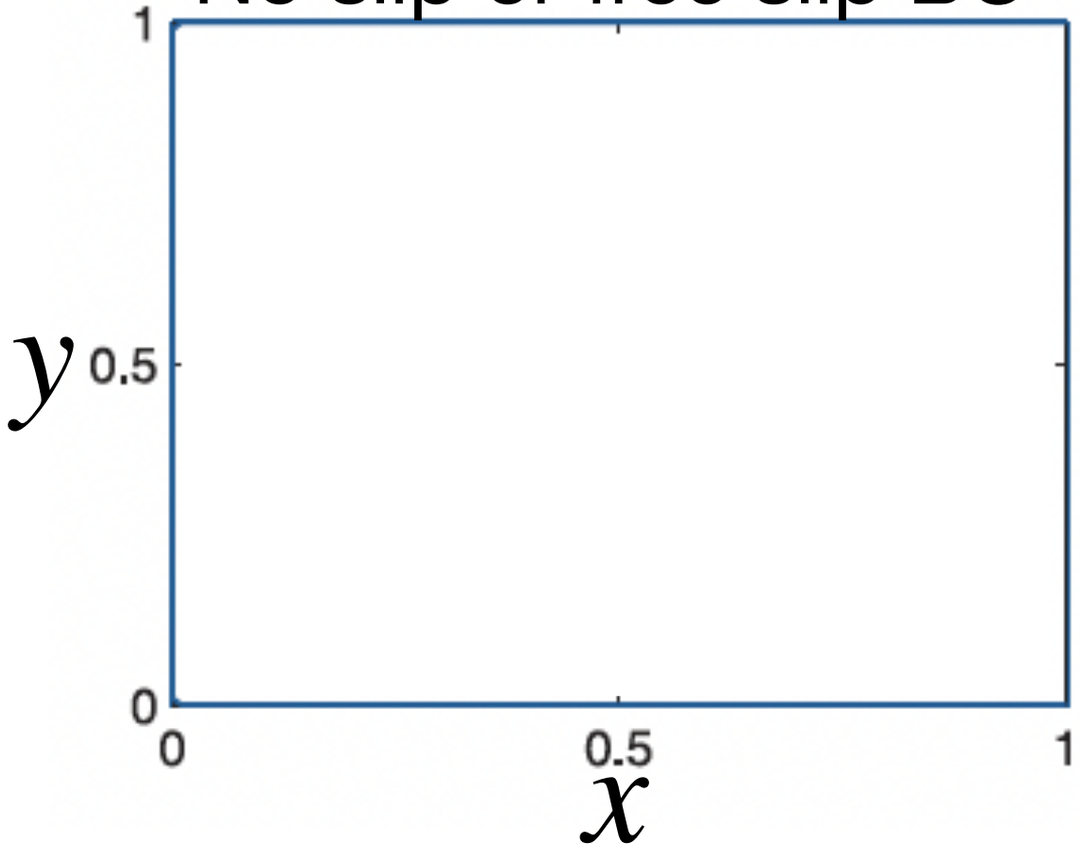
Western boundary layers

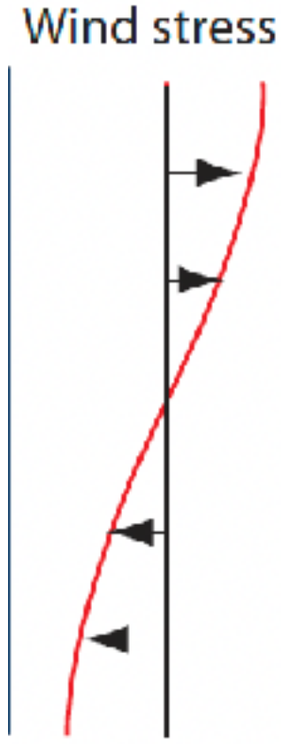
$$\mathbf{u} = (-\partial_y \psi, \partial_x \psi)$$

$$[\partial_t + \mathbf{u} \cdot \nabla]\Delta\psi + \beta\partial_x\psi = -\partial_y \tau_{wind} + \nu\Delta\Delta\psi$$

Euler 2D Coriolis Wind Dissipation

No slip or free slip BC





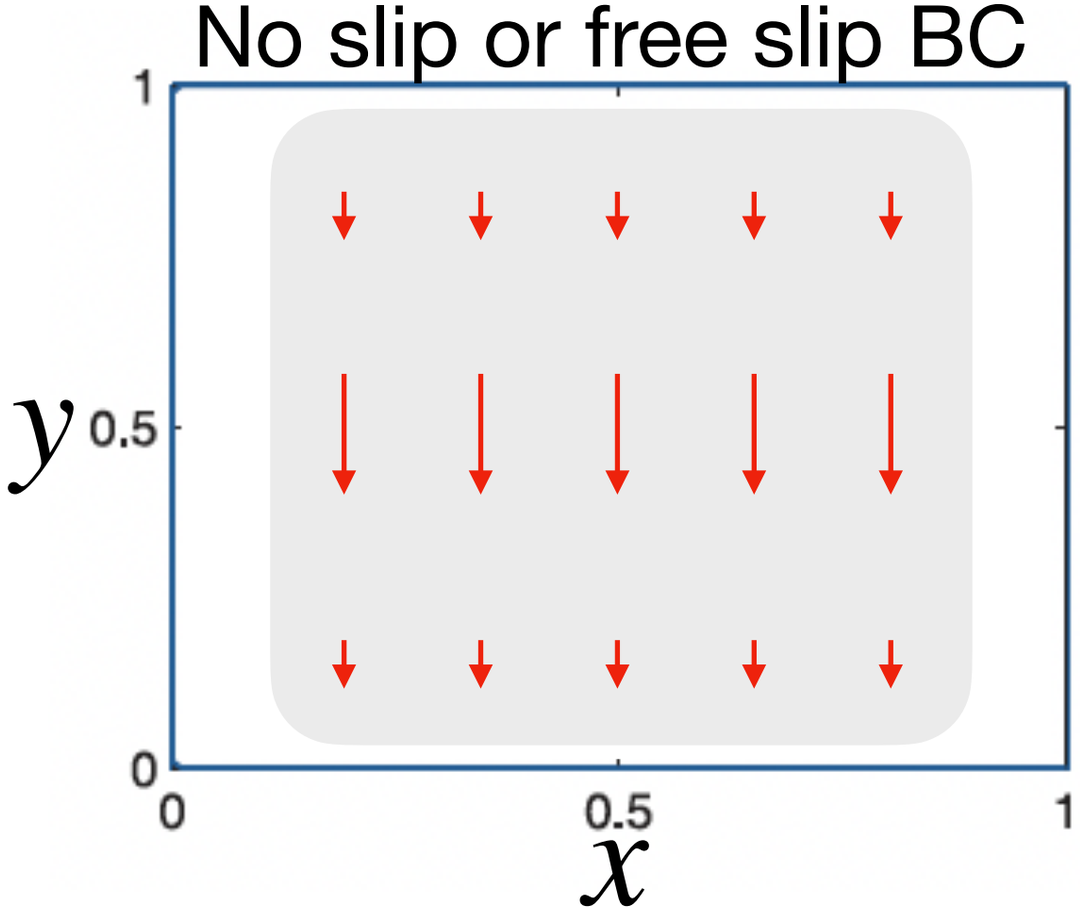
$\tau_{wind}(y)\mathbf{e}_x$

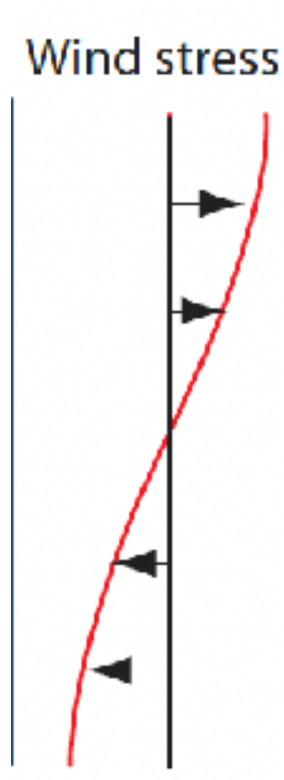
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Euler 2D Coriolis Wind Dissipation





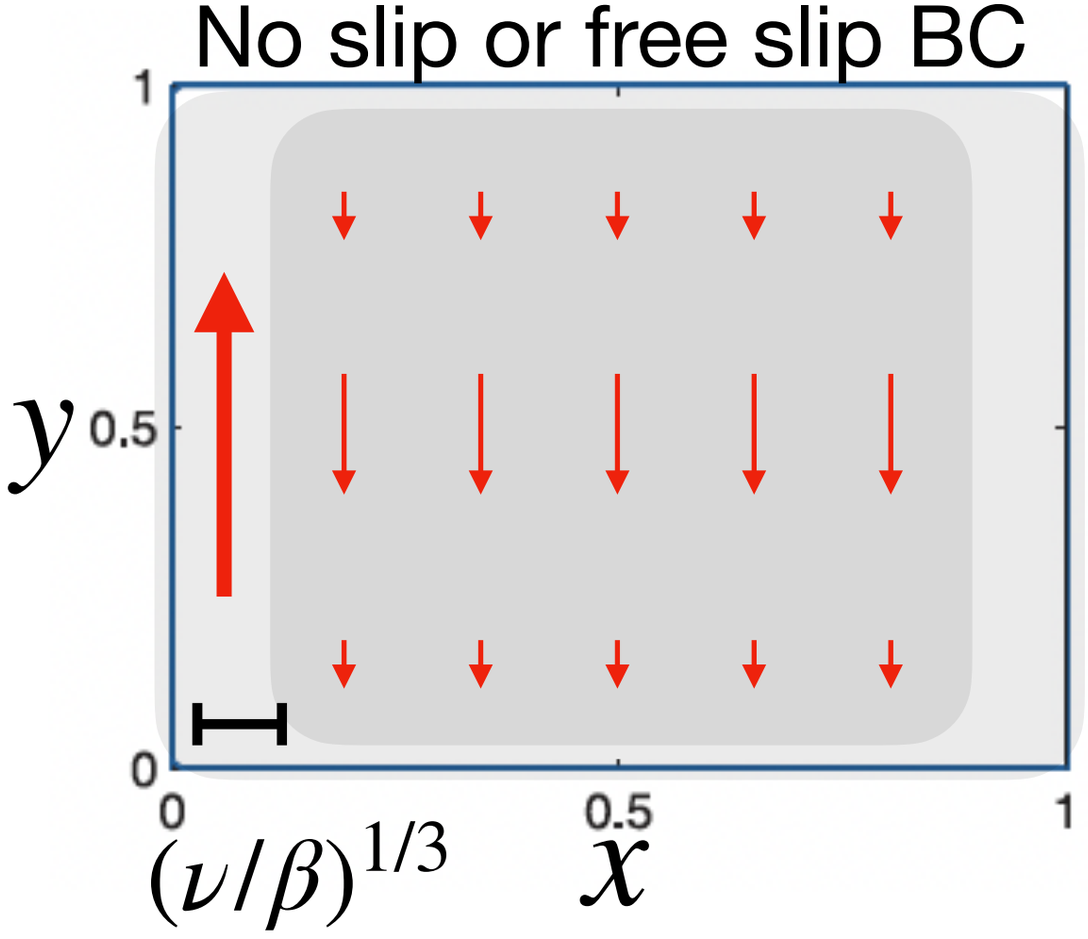
$$\tau_{wind}(y)\mathbf{e}_x$$

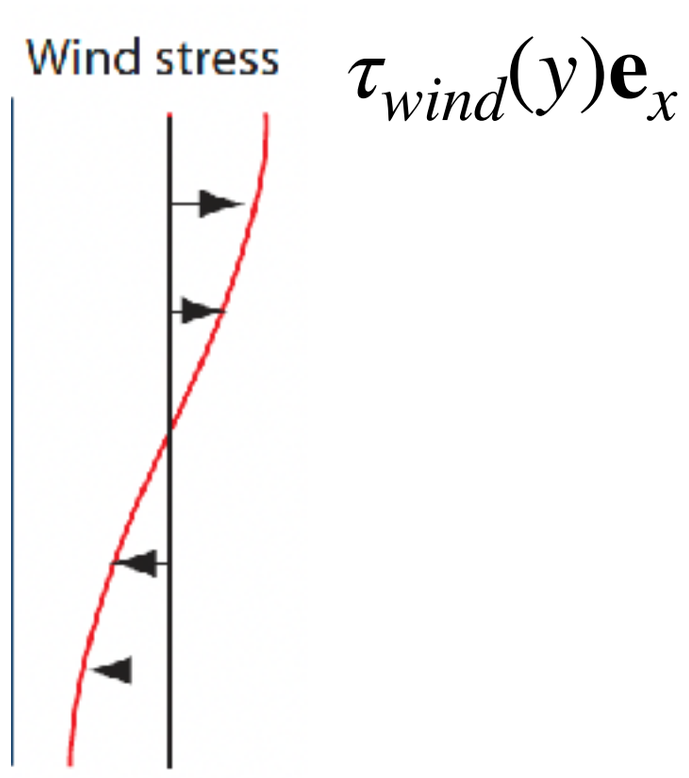
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Euler 2D Coriolis Wind Dissipation





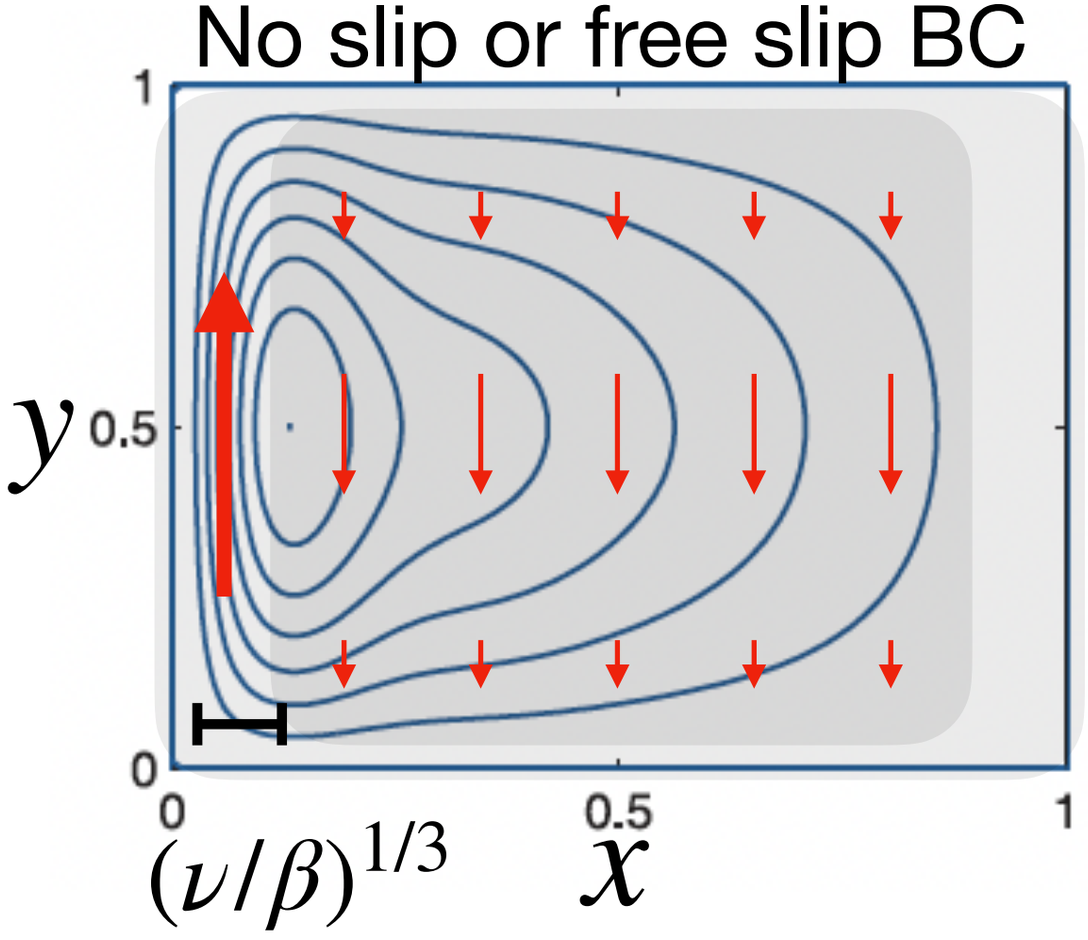
Stommel 48, Munk 52,
Dalibard & Saint-Raymond 12

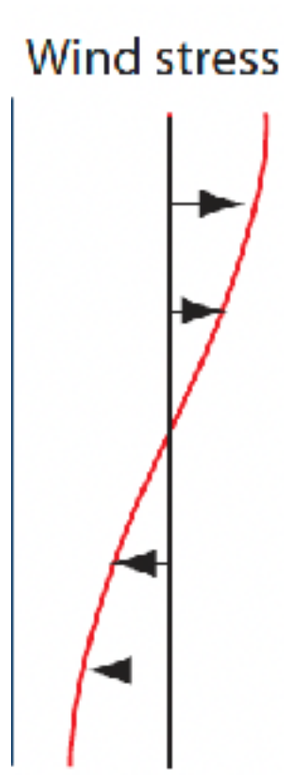
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Euler 2D Coriolis Wind Dissipation





$\tau_{wind}(y)\mathbf{e}_x$

Stommel 48, Munk 52,
Dalibard & Saint-Raymond 12

Energy budget

$$\mathbf{u} = (-\partial_y \psi, \partial_x \psi) = (u, v)$$

$$[\partial_t + \mathbf{u} \cdot \nabla]\Delta\psi + \beta\partial_x\psi = -\partial_y \tau_{wind}(y) + \nu\Delta\Delta\psi$$

$$E = \frac{1}{2}\int dxdy(\nabla\psi)^2 \qquad Z = \int dxdy(\Delta\psi)^2$$

Energy **Enstrophy**

$$\frac{dE}{dt} = \int dxdyu\tau_{wind} - \nu Z$$

Energy budget

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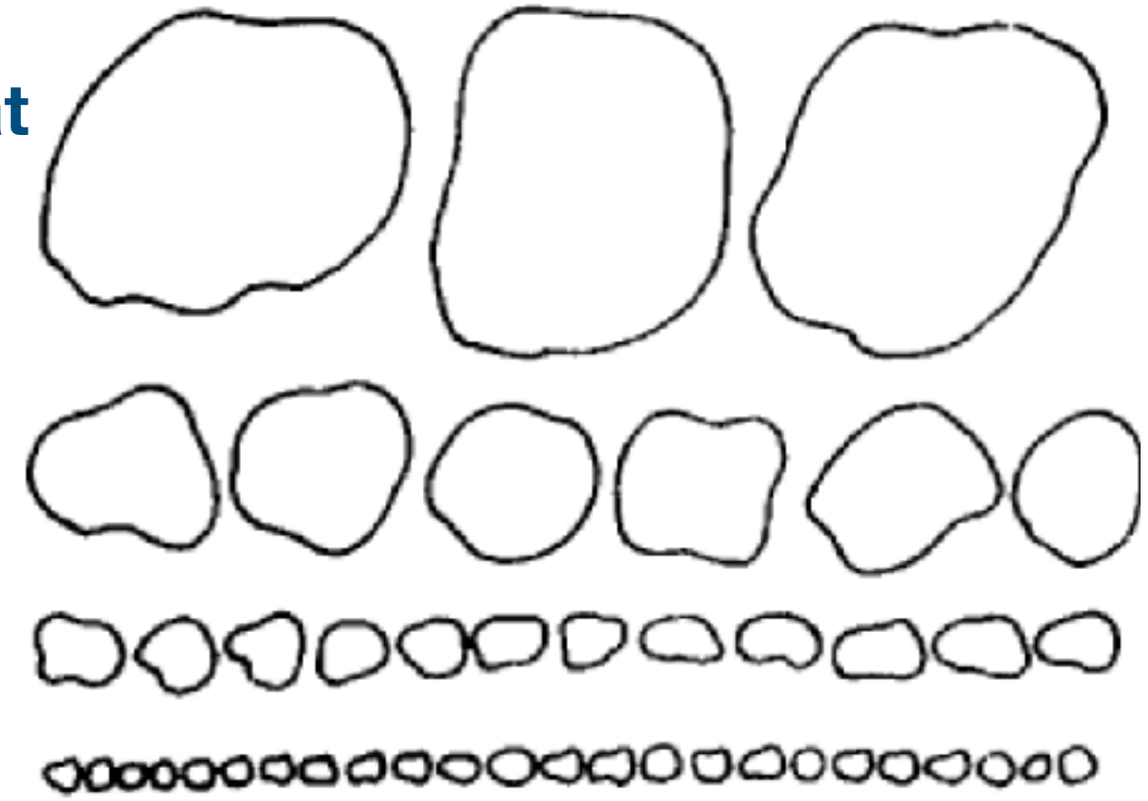
Linear steady gyre: **Injection in the bulk** $\sim$ **Dissipation et the boundary** $Z \sim 1/\nu$

What about the turbulent regime?
Take the limit of weak dissipation for a given (large) value of beta

Nonlinearities and energy cascades

Inverse cascade, self-organization at the domain scale

2D



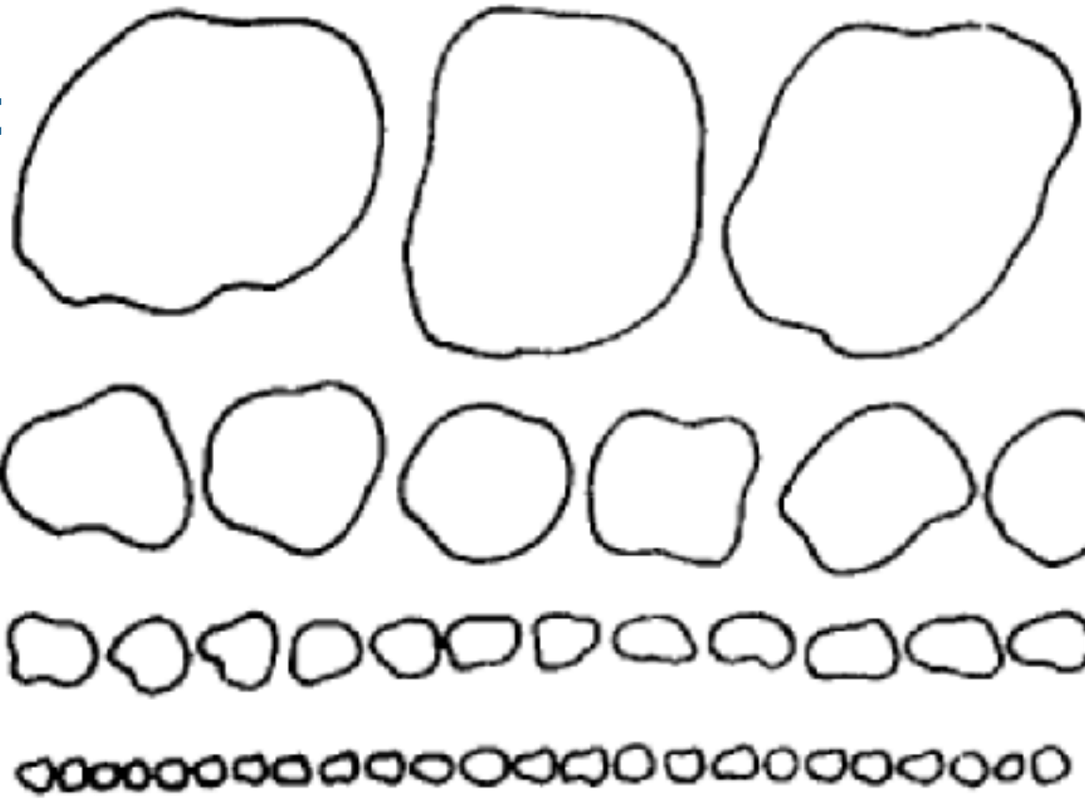
3D

Direct cascade, anomalous dissipation

Nonlinearities and energy cascades

Inverse cascade, self-organization at the domain scale

2D

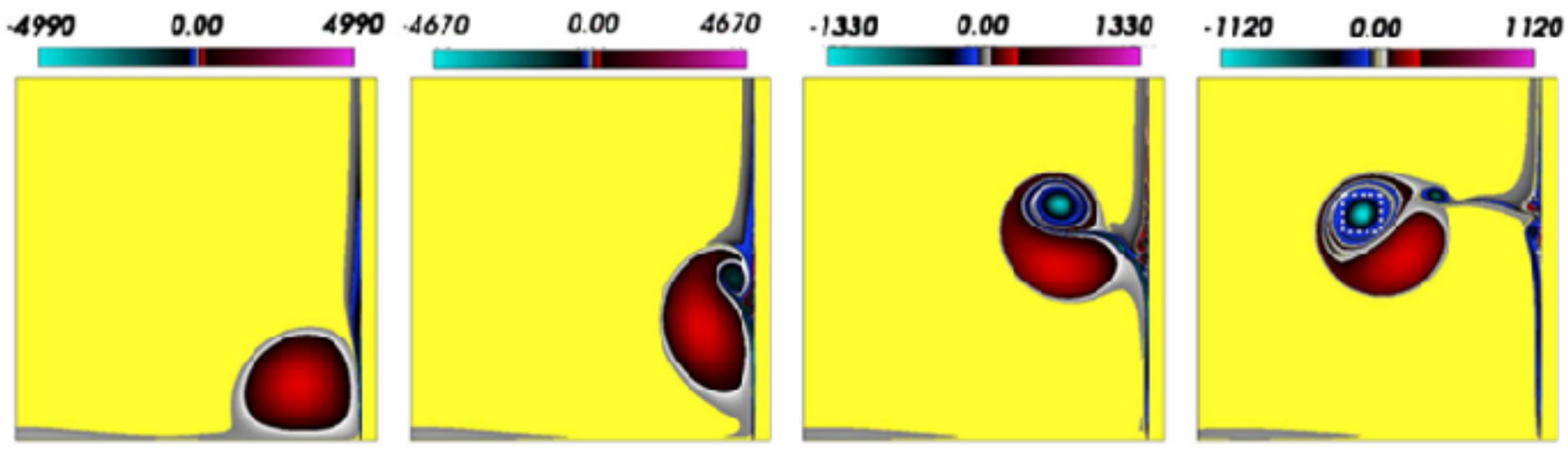


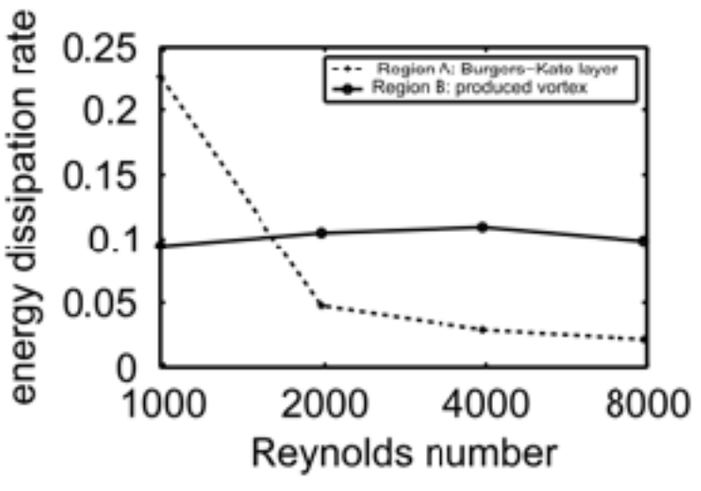
3D

Direct cascade, anomalous dissipation

What about 2D flows with walls ? Possibility for dissipative structures

Nguyen van yen, Farge, Schneider 2011





Dissipative anomaly in ocean gyres ?

(a) Inverse energy cascade and self organization at the domain scale?

(b) Formation of small scale dissipative structures with persistent low energy gyres?



Lennard Miller



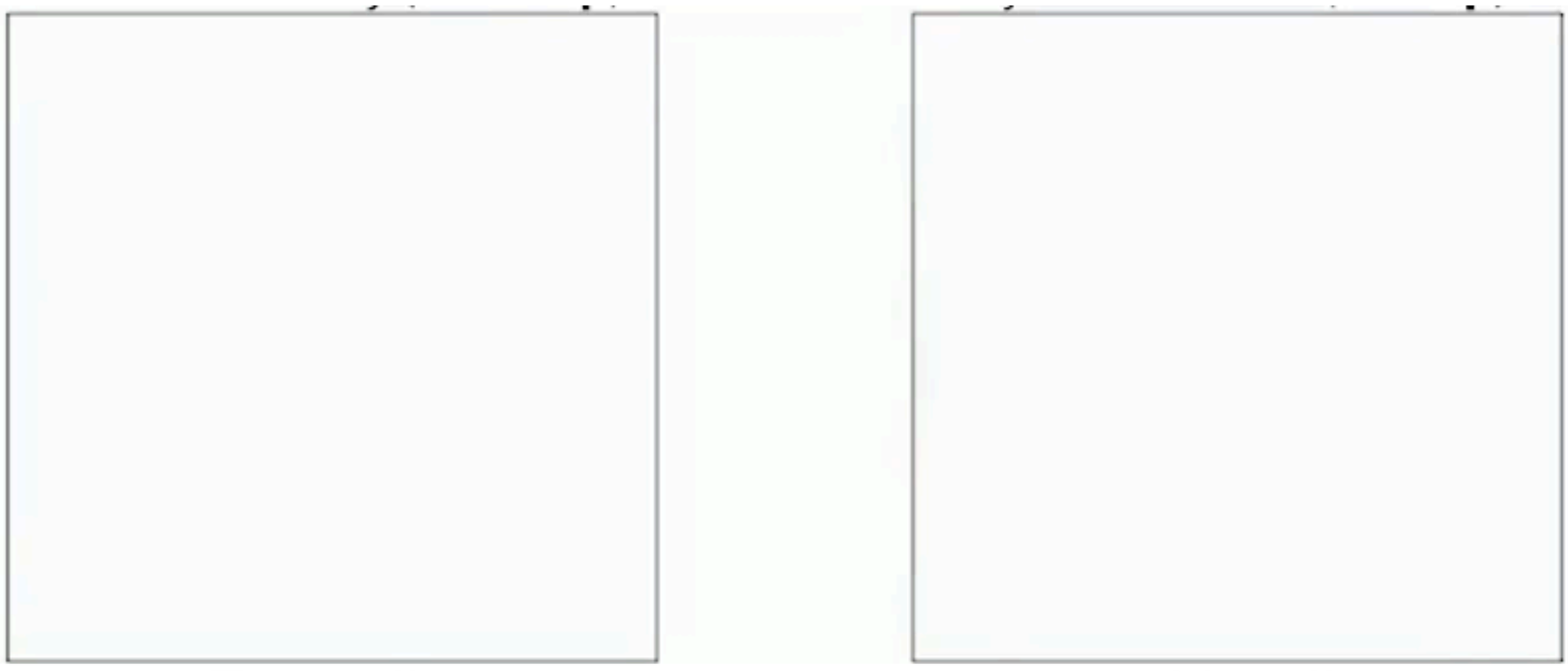
Bruno Deremble

$$[\partial_t + \mathbf{u} \cdot \nabla]\Delta\psi + \beta\partial_x\psi = \sin(\pi y/L) + \nu\Delta\Delta\psi$$

Dissipative anomaly in ocean gyres ?

(a) Inverse energy cascade and self organization at the domain scale?

(b) Formation of small scale dissipative structures with persistent low energy gyres?



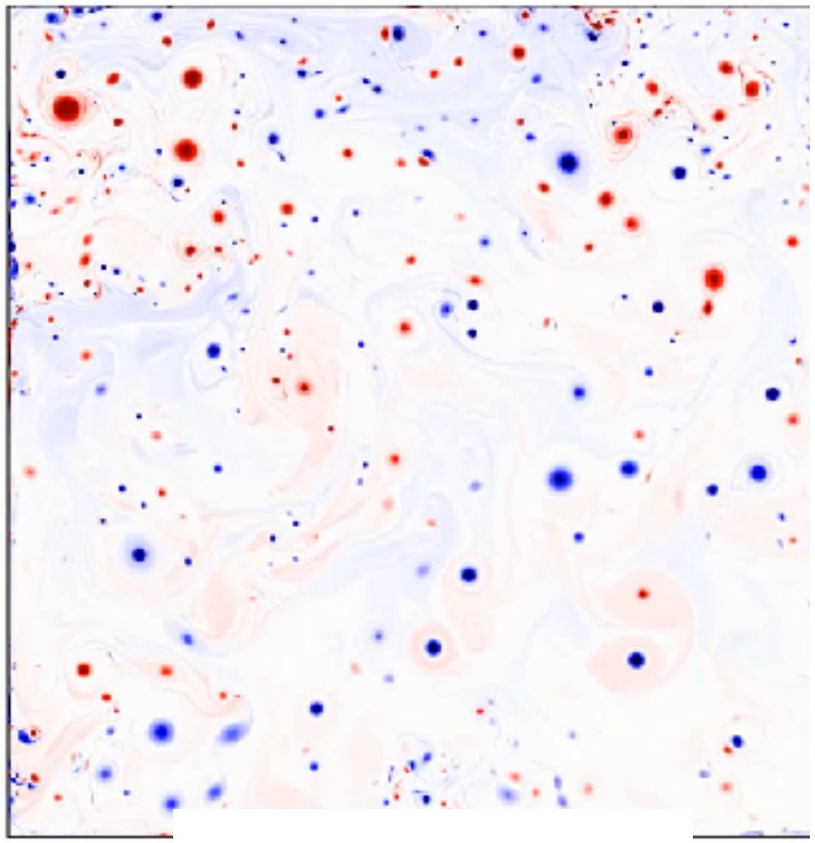
Free slip

No slip

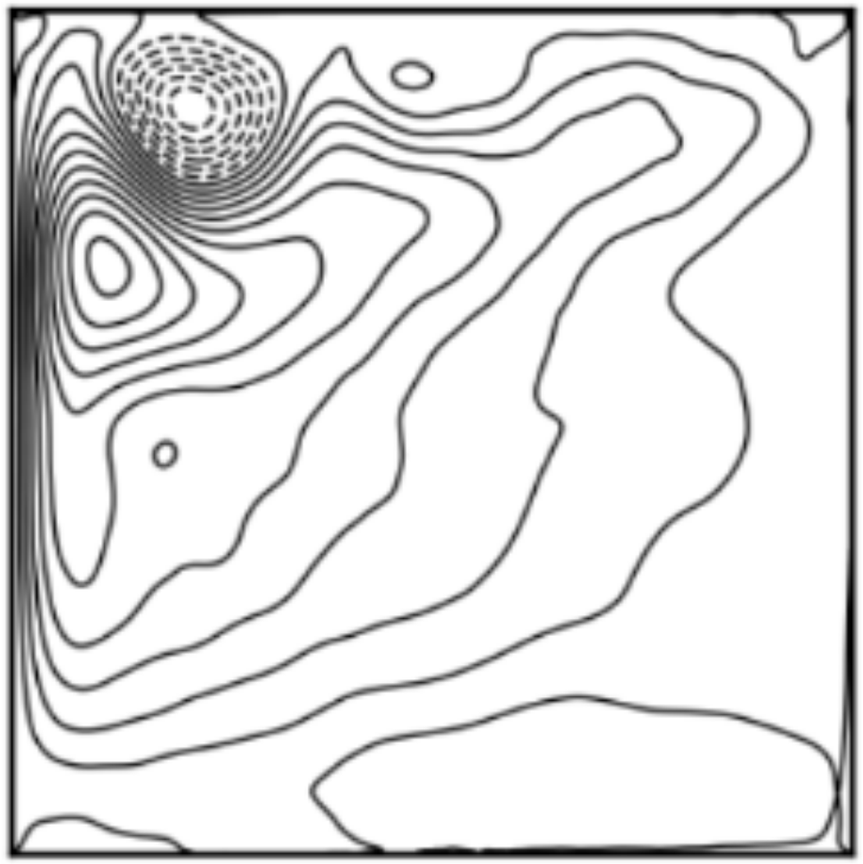
$$[\partial_t + \mathbf{u} \cdot \nabla]\Delta\psi + \beta\partial_x\psi = \sin(\pi y/L) + \nu\Delta\Delta\psi$$

Western intensified gyre patterns persist in turbulent flows

Energetic Vortex gas



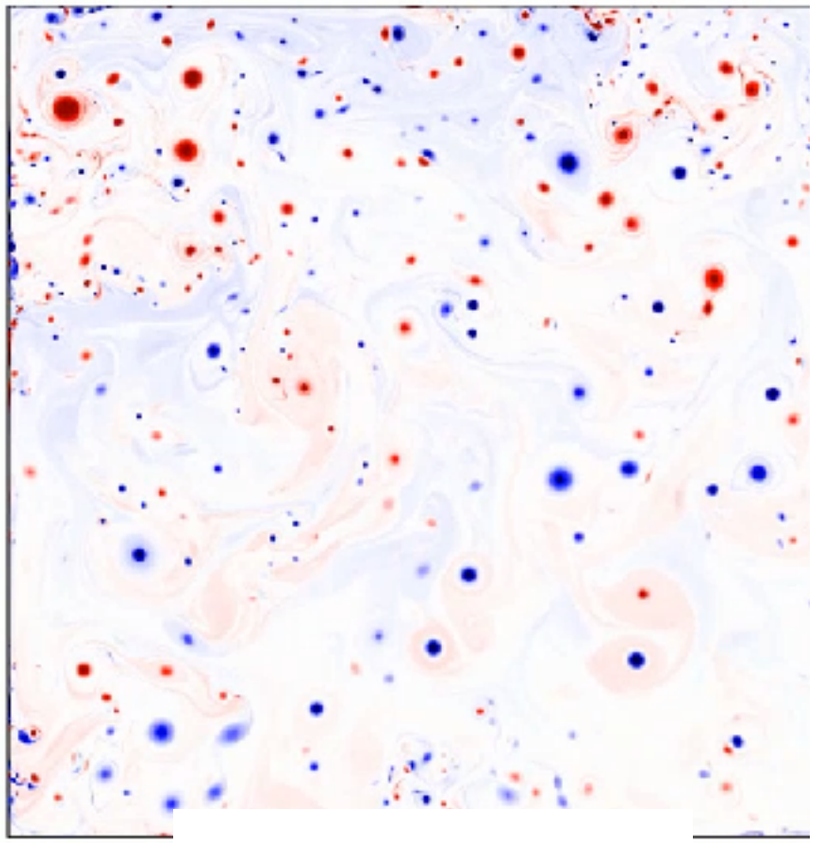
Mean flow: low energy gyre



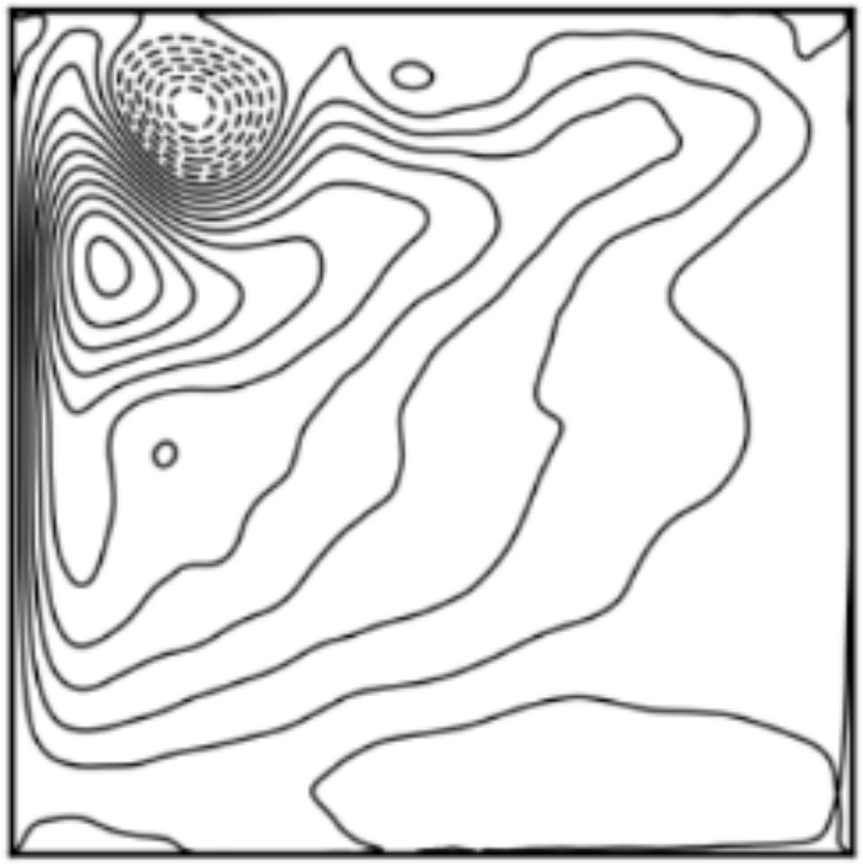
Owing the the combined effect of **walls and beta plane**, ocean 2D turbulence is a strongly out-of-equilibrium state

Western intensified gyre patterns persist in turbulent flows

Energetic Vortex gas



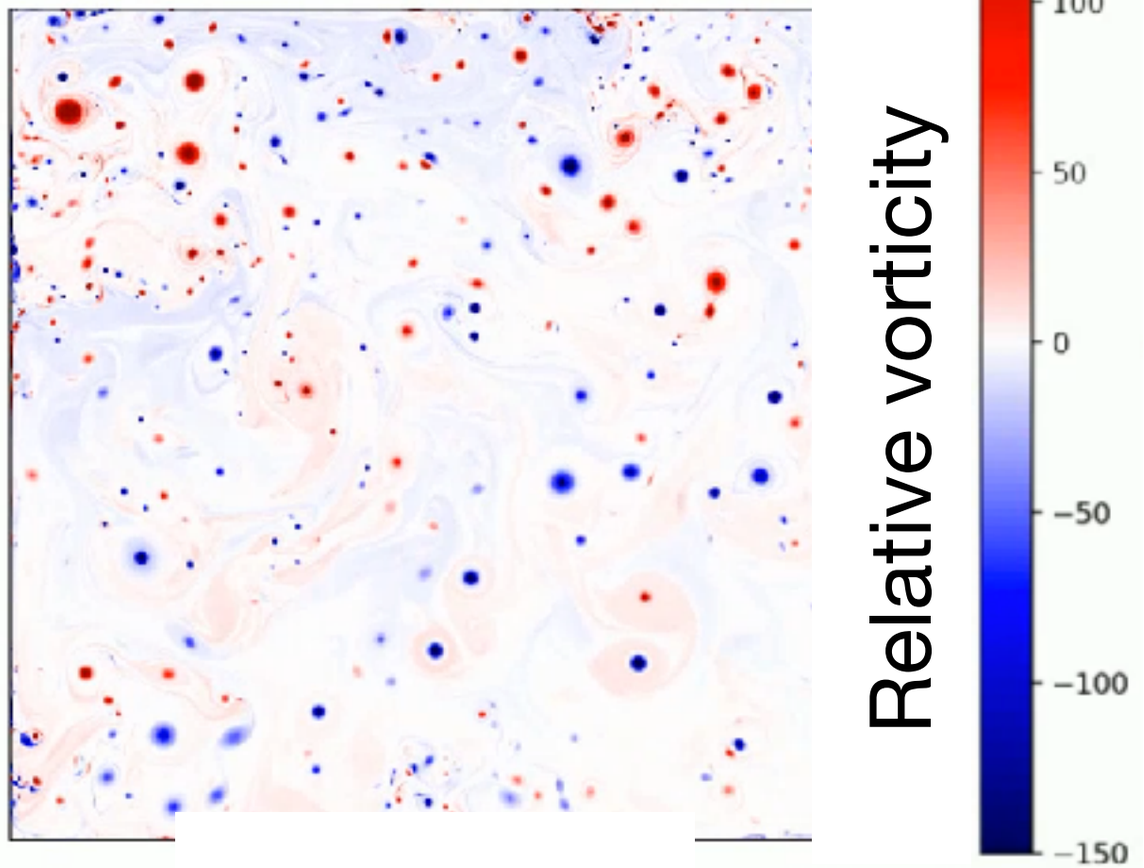
Mean flow: low energy gyre



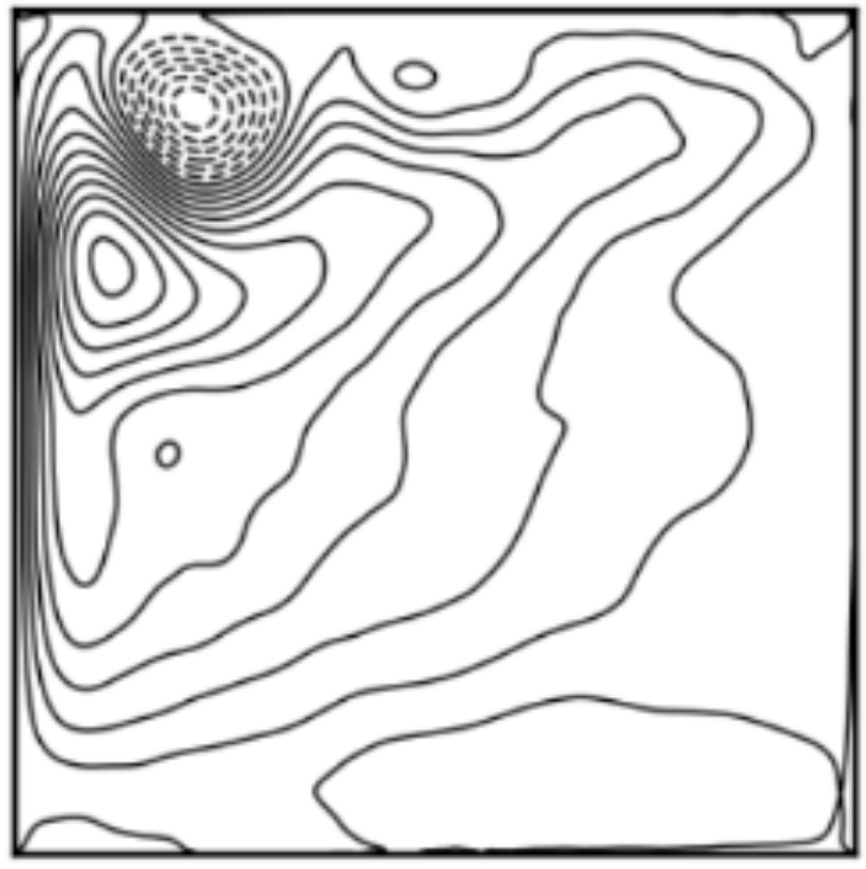
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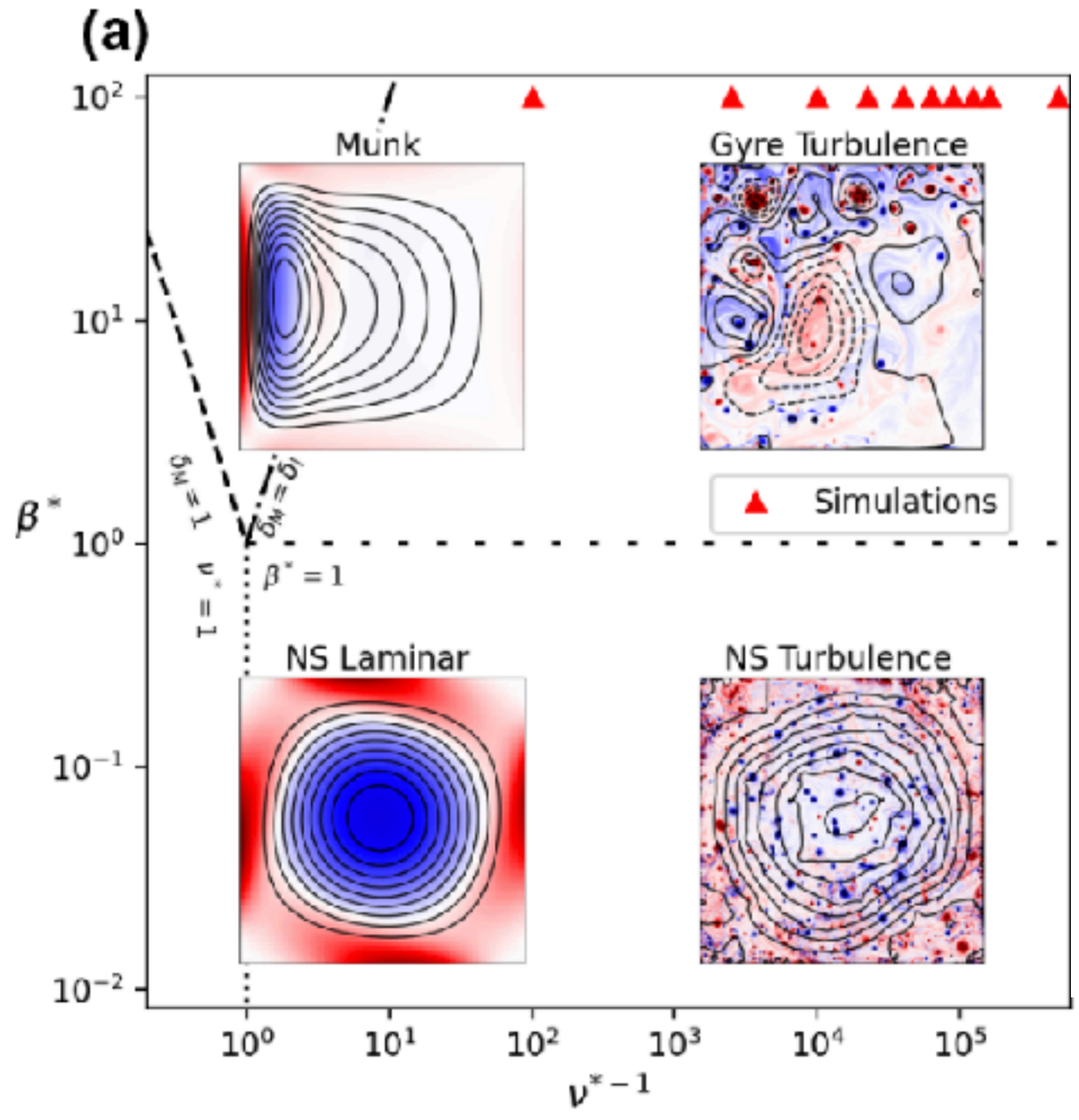


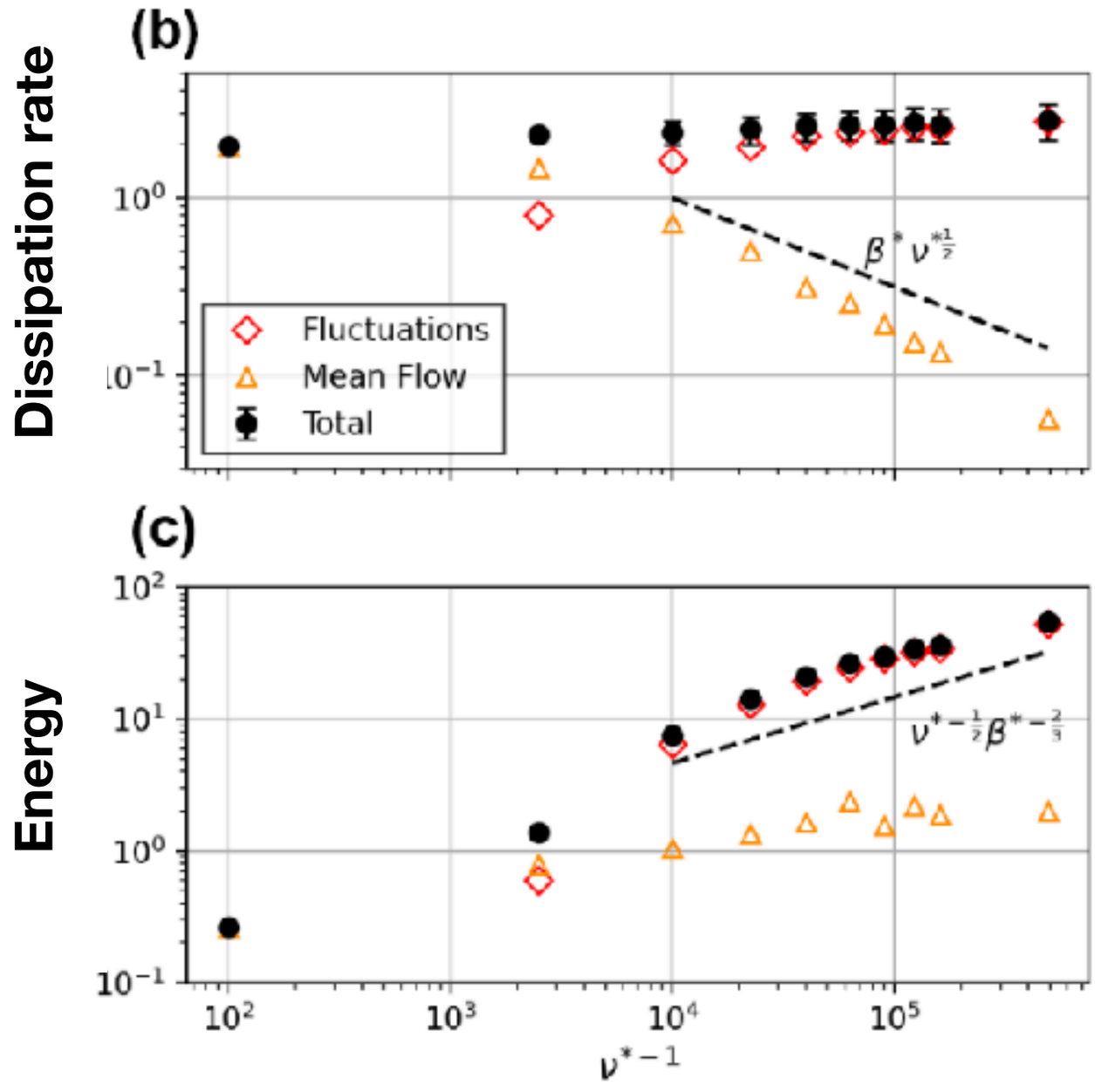
Mean flow: low energy gyre



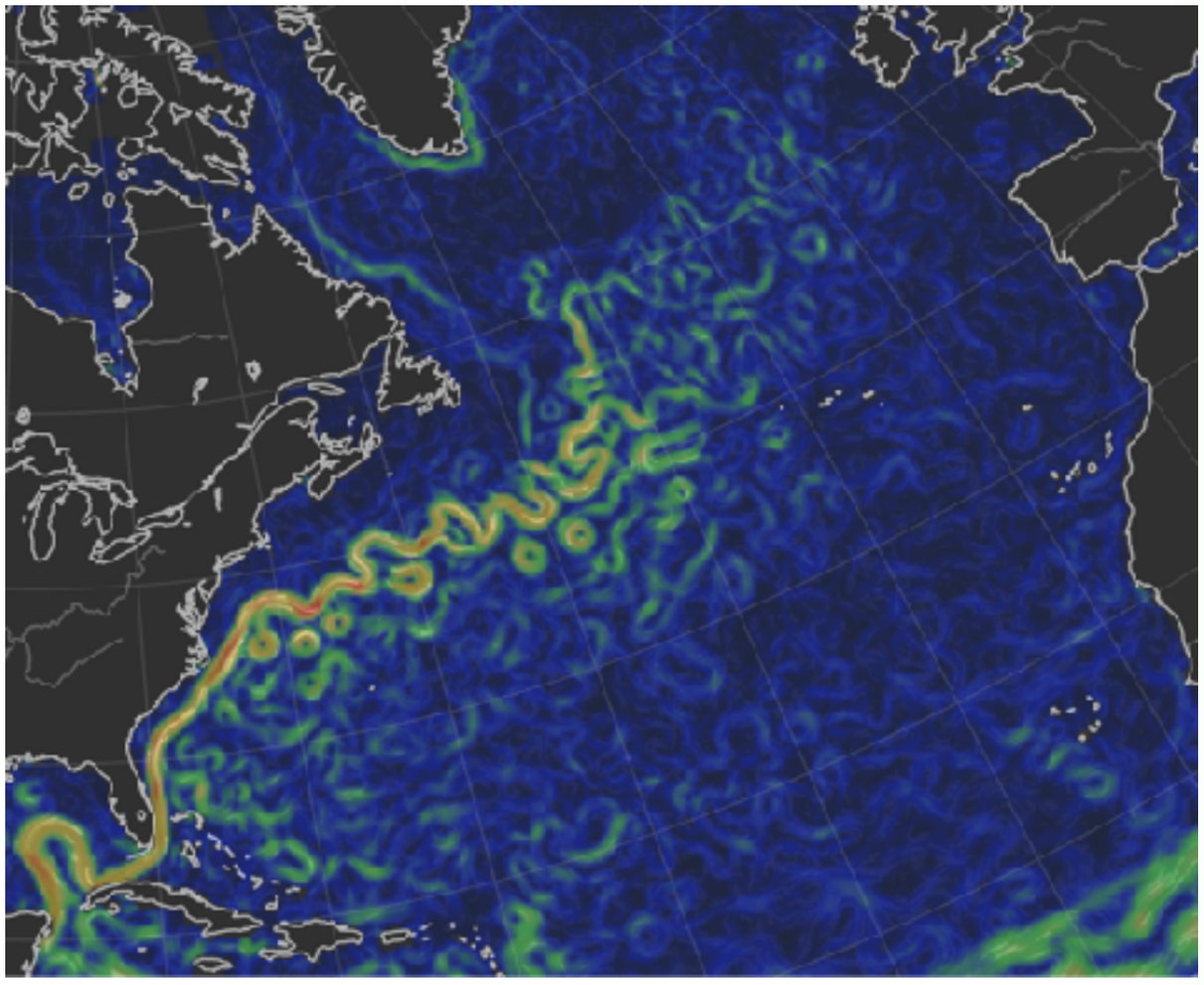
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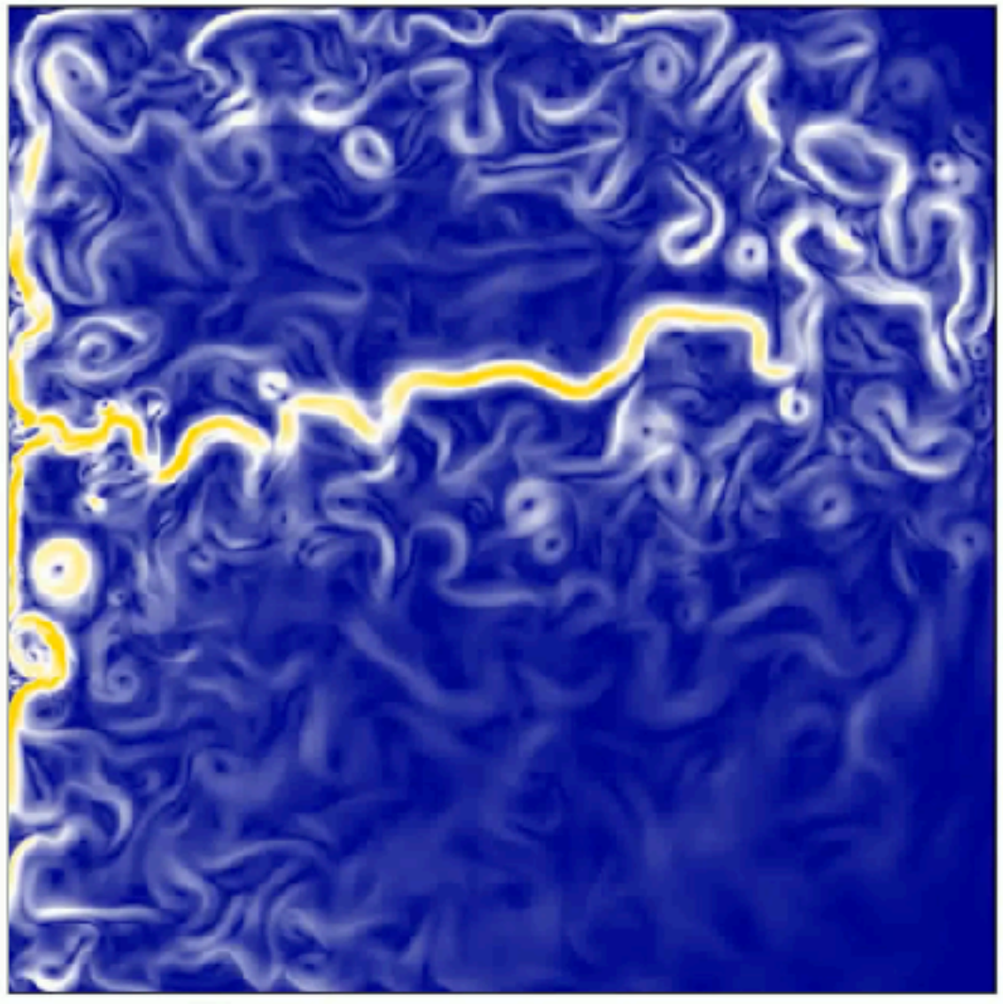
Finite dissipation rate in ocean gyres with weak dissipation





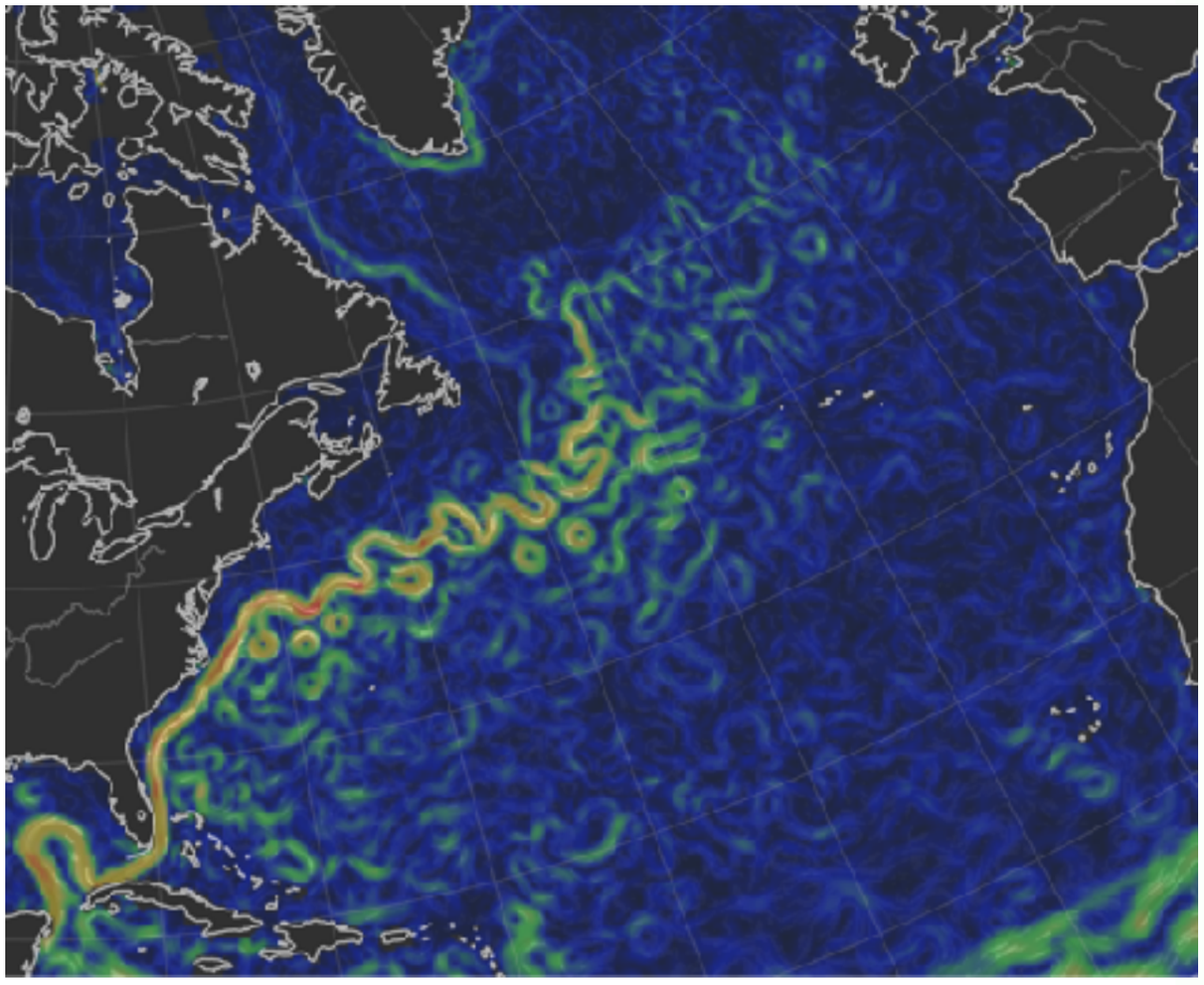
Perspectives for part I: Emergence of Gulf Stream-like jets

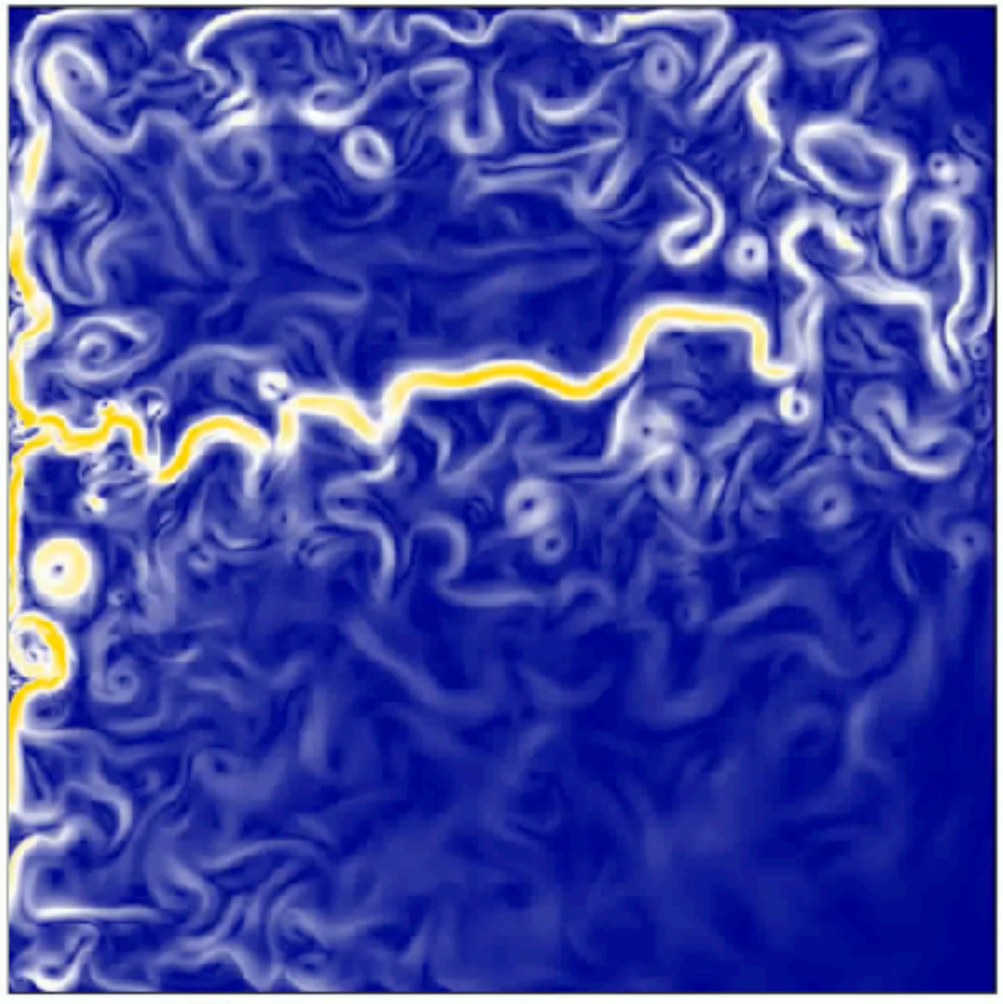




Left Surface kinetic energy in a comprehensive general circulation ocean model and **Right** In a two-layer quasigeostrophic model. **Density stratification is key.**

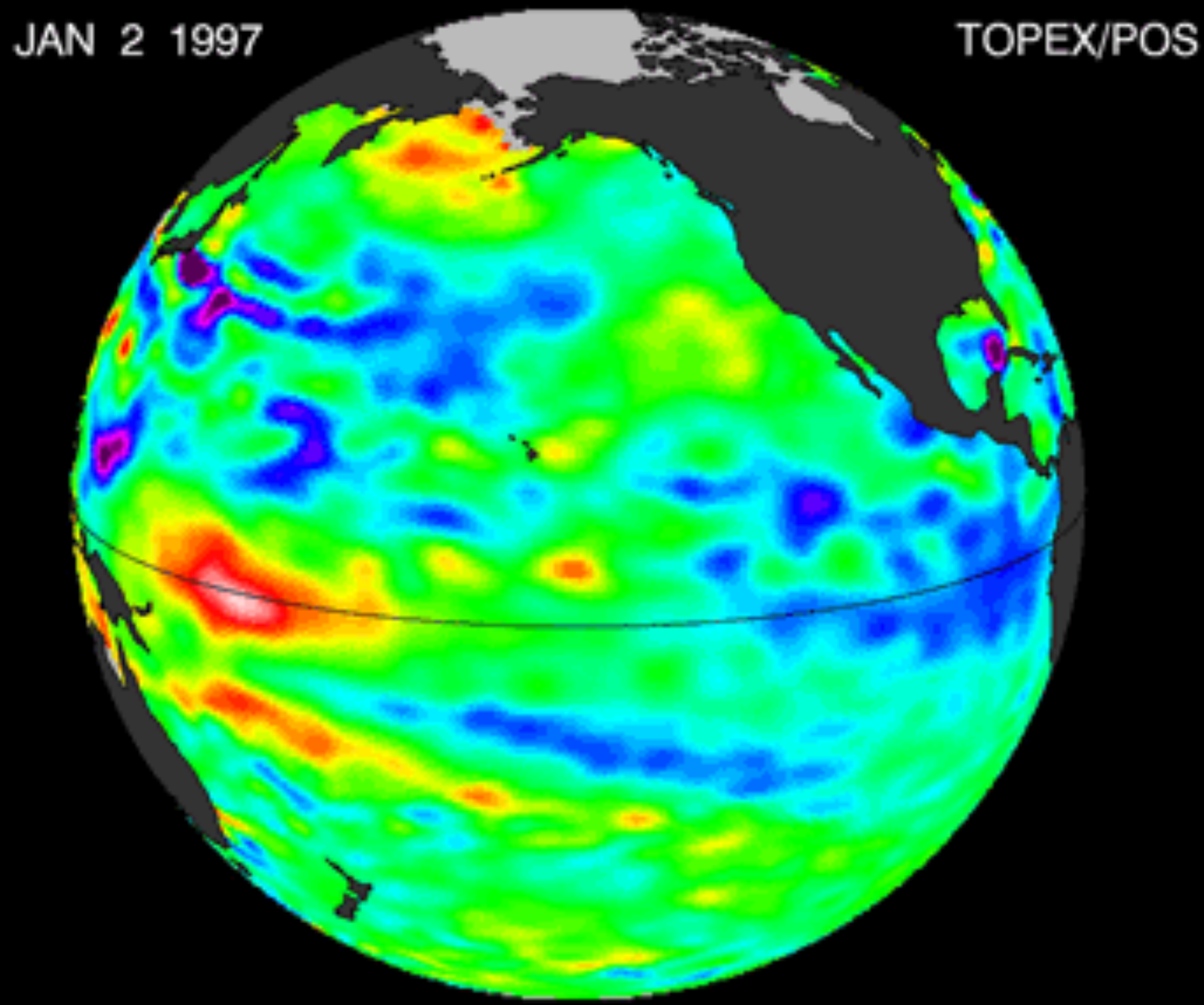
Perspectives for part I: Emergence of Gulf Stream-like jets





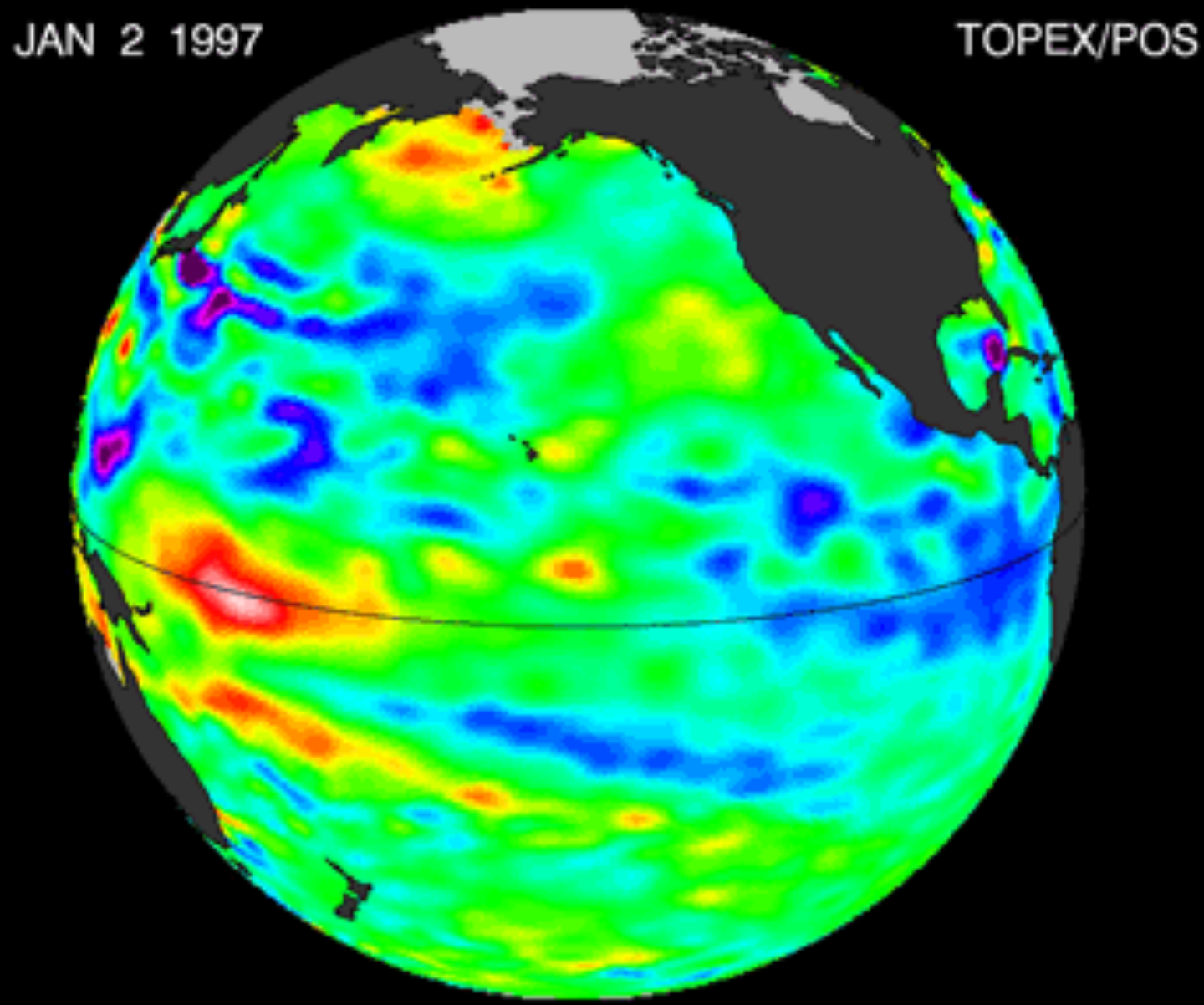
Left Surface kinetic energy in a comprehensive general circulation ocean model and **Right** In a two-layer quasigeostrophic model. **Density stratification is key.**

Part II: the equatorial wave guide



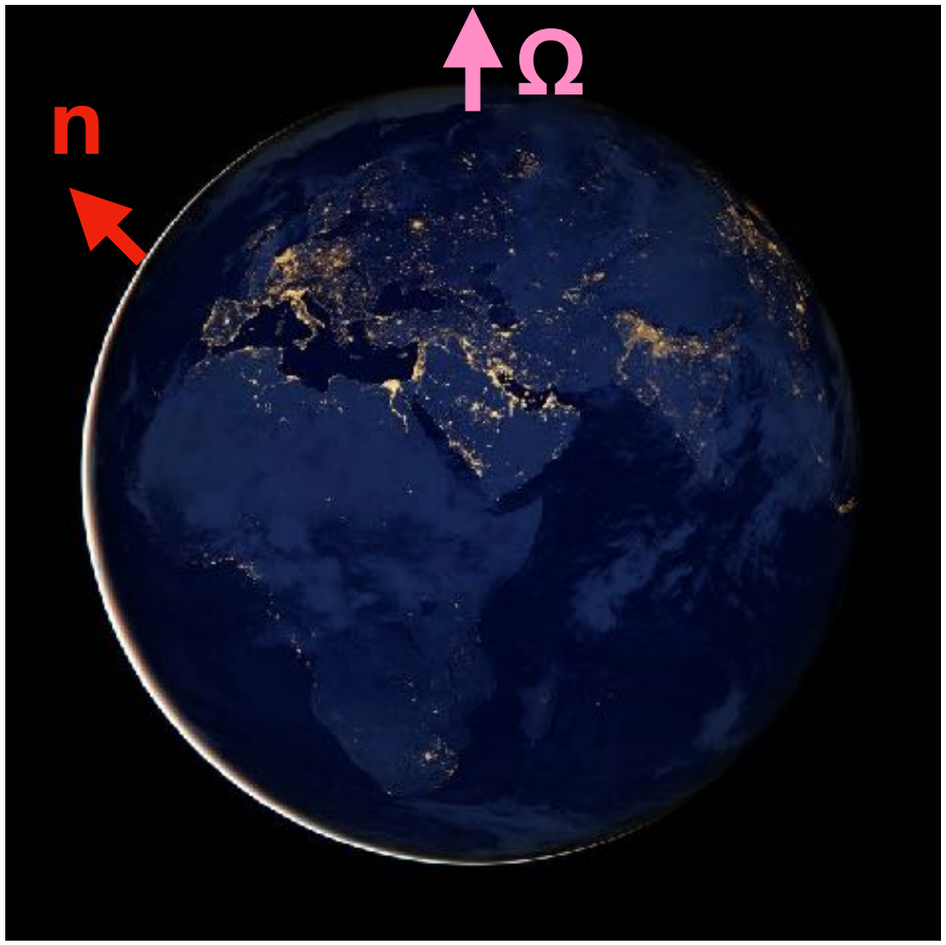
TOPEX/Poseidon 1997

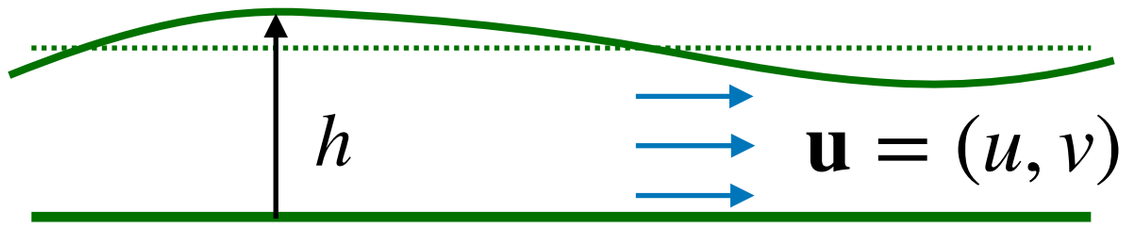
Part II: the equatorial wave guide



TOPEX/Poseidon 1997

Shallow water model



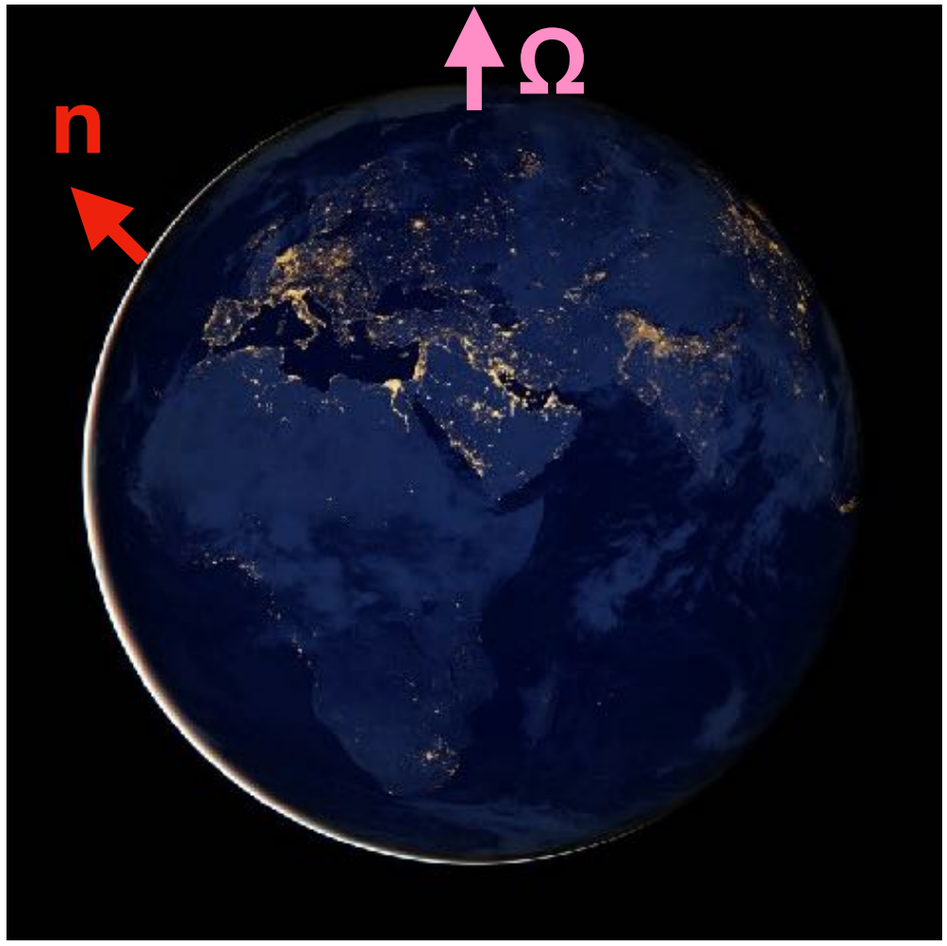


$$\partial_t h + \nabla \cdot (h\mathbf{u}) = 0,$$
$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -g\nabla h - f\mathbf{n} \times \mathbf{u}.$$

Coriolis parameter $f = 2\boldsymbol{\Omega} \cdot \mathbf{n}$

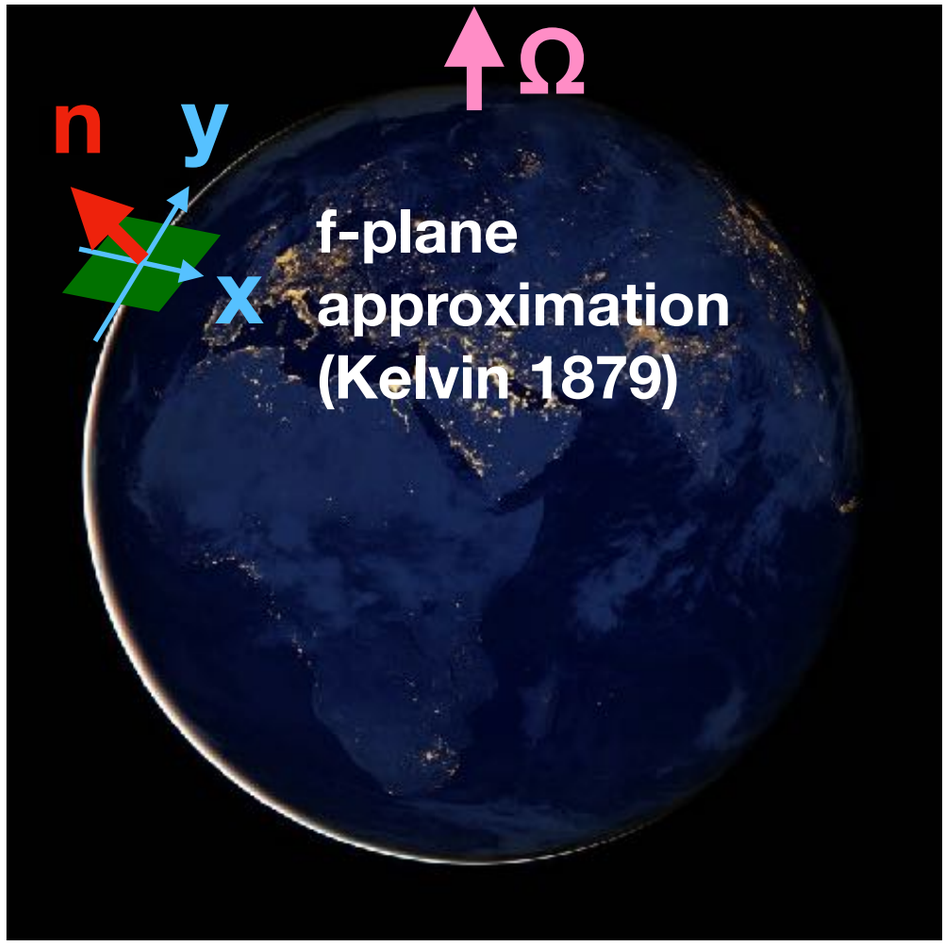
2D flow model, broken time reversal symmetry

Linear dynamics



Free modes of Laplace tidal equations.

Linear dynamics



Free modes of Laplace tidal equations.

$$h = H + \eta \qquad \mathbf{u} = (u, v)$$

$$\partial_t u = -g\partial_x \eta + fv$$
$$\partial_t v = -g\partial_y \eta - fu$$
$$\partial_t \eta = -H\partial_x u - H\partial_y v$$

f-plane waves

$$\partial_t u = -g\partial_x \eta + fv$$
$$\partial_t v = -g\partial_y \eta - fu$$
$$\partial_t \eta = -H\partial_x u - H\partial_y v$$

Kelvin 1879

$$\Psi = (u, v, \eta), \quad \Psi = \chi e^{i\omega t - ik_x x - ik_y y}$$

$$c = \sqrt{gH}$$

f-plane waves

$$\partial_t u = -g\partial_x \eta + fv$$
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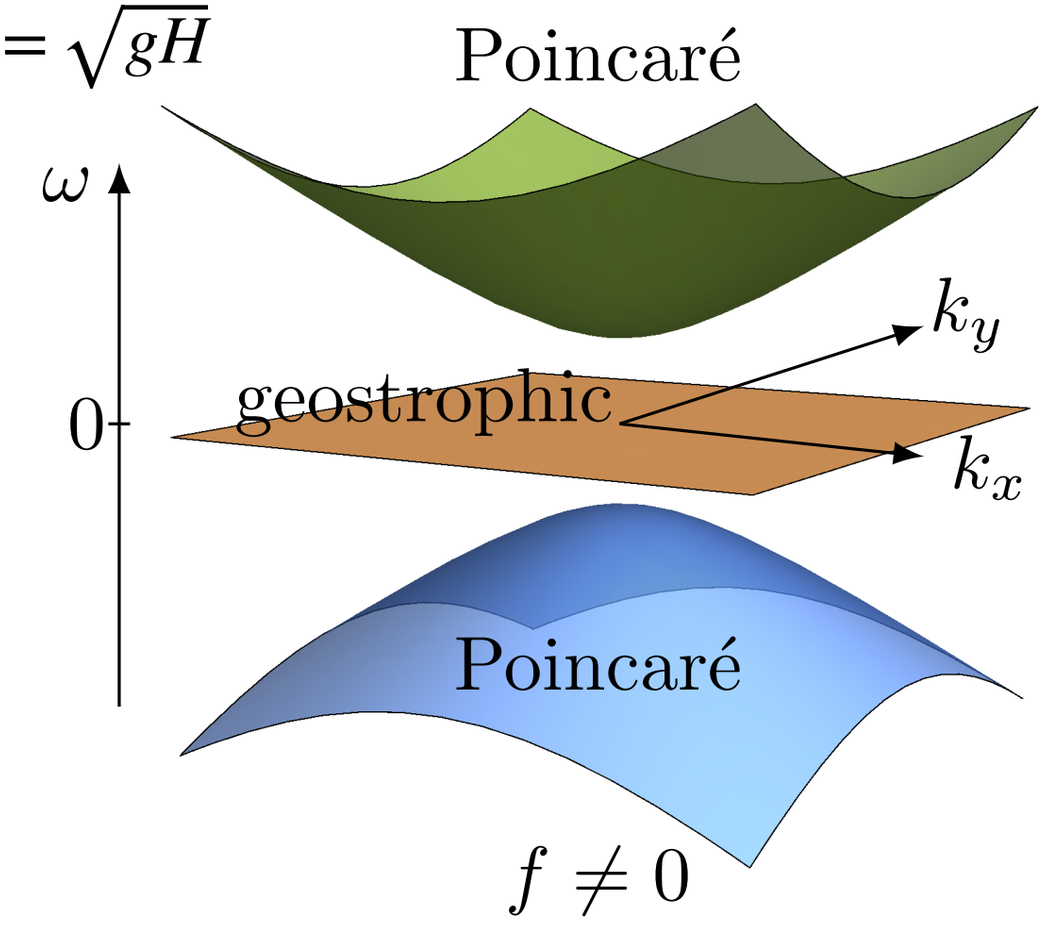
$$\omega^2 = f^2 + c^2\left(k_x^2 + k_y^2\right)$$

$$\chi(k_x, k_y, \omega) = (\hat{u}, \hat{v}, \hat{\eta})$$

Kelvin 1879

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f-plane waves

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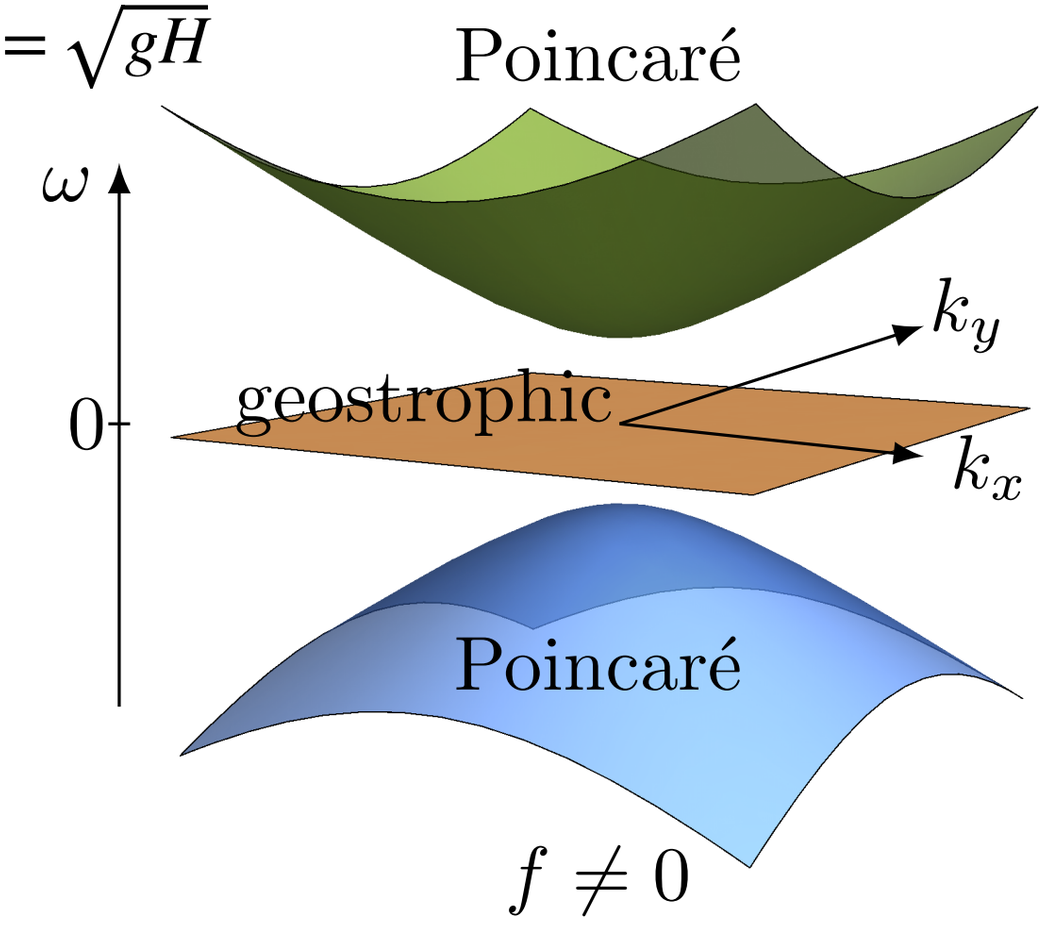
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Kelvin 1879

$$\Psi = (u, v, \eta), \quad \Psi = \chi e^{i\omega t - ik_x x - ik_y y}$$

$$c = \sqrt{gH}$$



- **Rotation opens a frequency gap**

f-plane waves

$$\partial_t u = -g\partial_x \eta + fv$$
$$\partial_t v = -g\partial_y \eta - fu$$
$$\partial_t \eta = -H\partial_x u - H\partial_y v$$

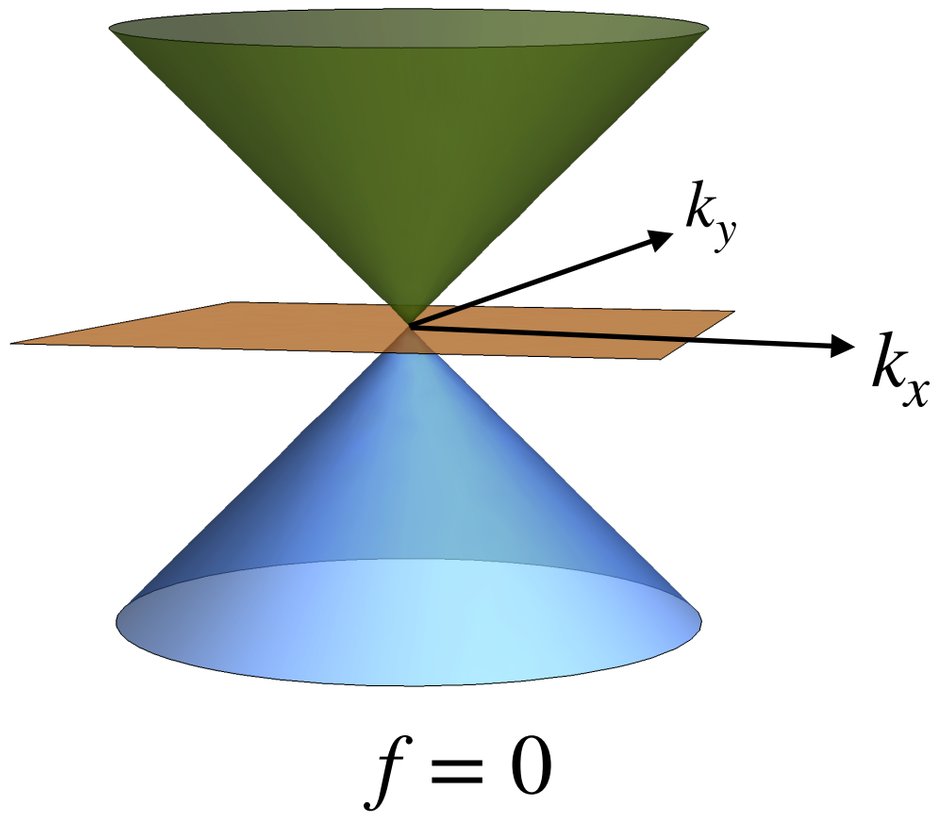
Kelvin 1879

$$\Psi = (u, v, \eta), \quad \Psi = \chi e^{i\omega t - ik_x x - ik_y y}$$

$$c = \sqrt{gH}$$

$$\omega^2 = f^2 + c^2\left(k_x^2 + k_y^2\right)$$

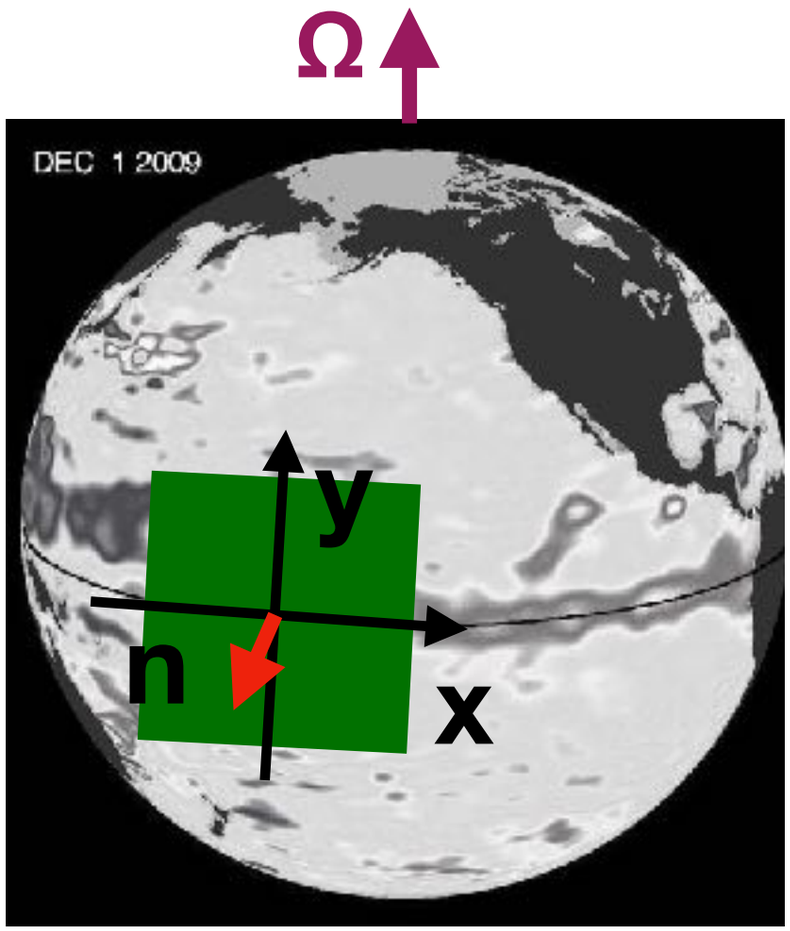
$$\chi(k_x, k_y, \omega) = (\hat{u}, \hat{v}, \hat{\eta})$$



- **Rotation opens a frequency gap**
- **A degeneracy point occurs at the equator**

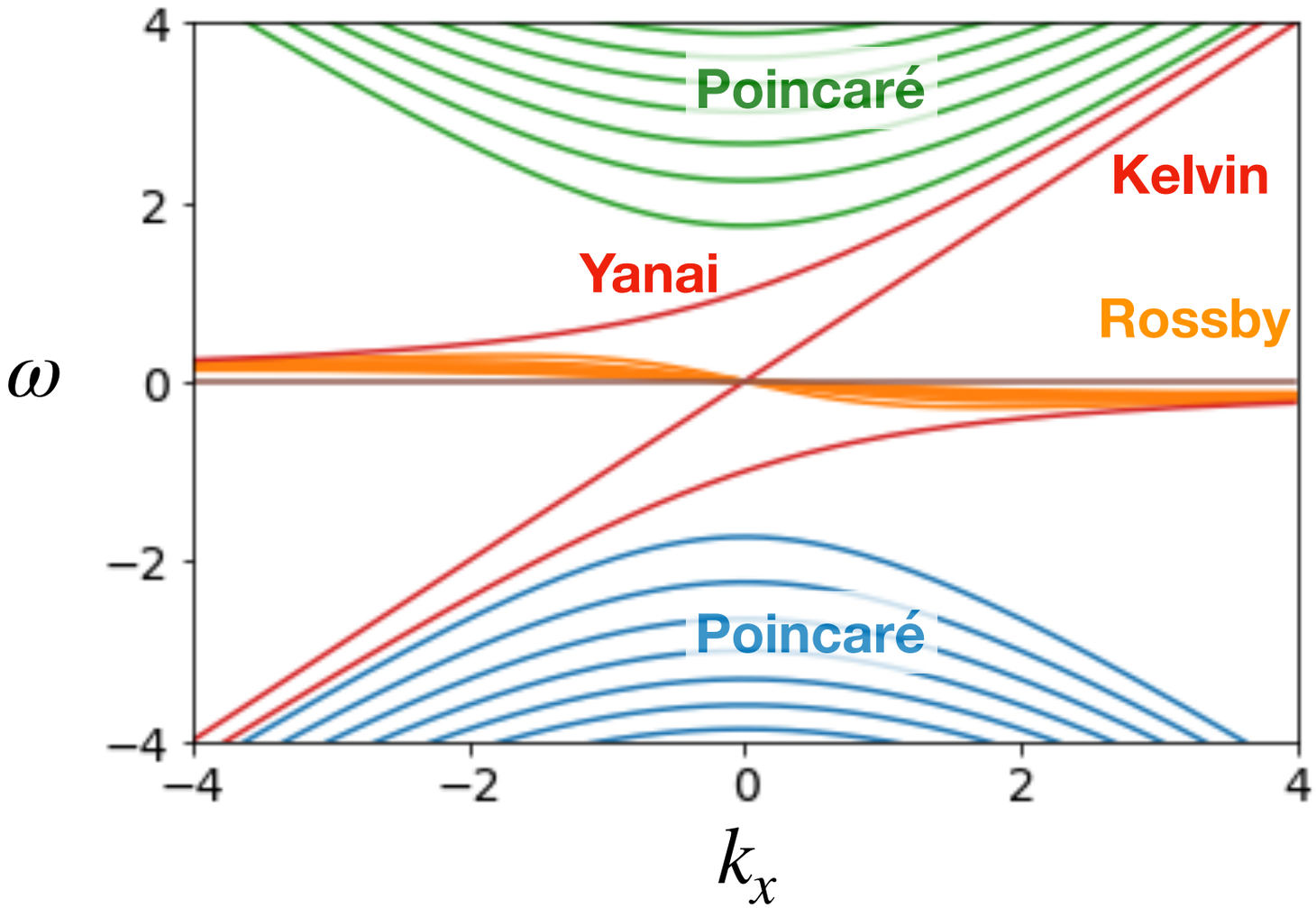
Equatorial Beta Plane

Matsuno 1966, Gallagher & Saint Raymond 2006



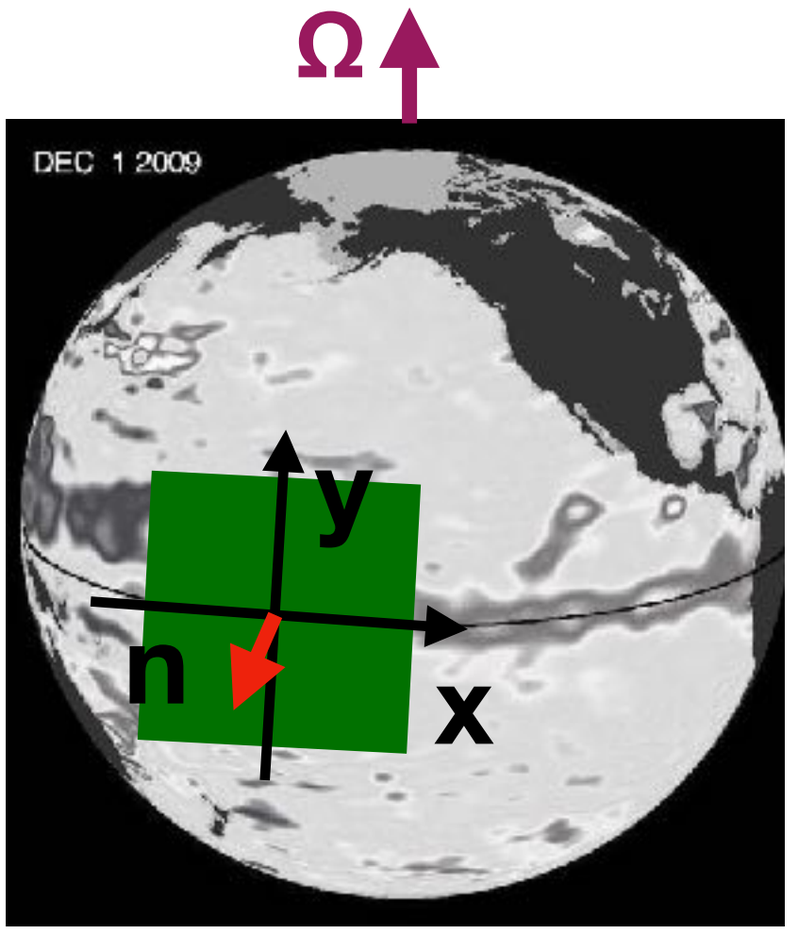
$$f = \beta y$$

$$\Psi = (u, v, \eta), \quad \Psi = \chi(y) e^{i\omega t - i k_x x}$$



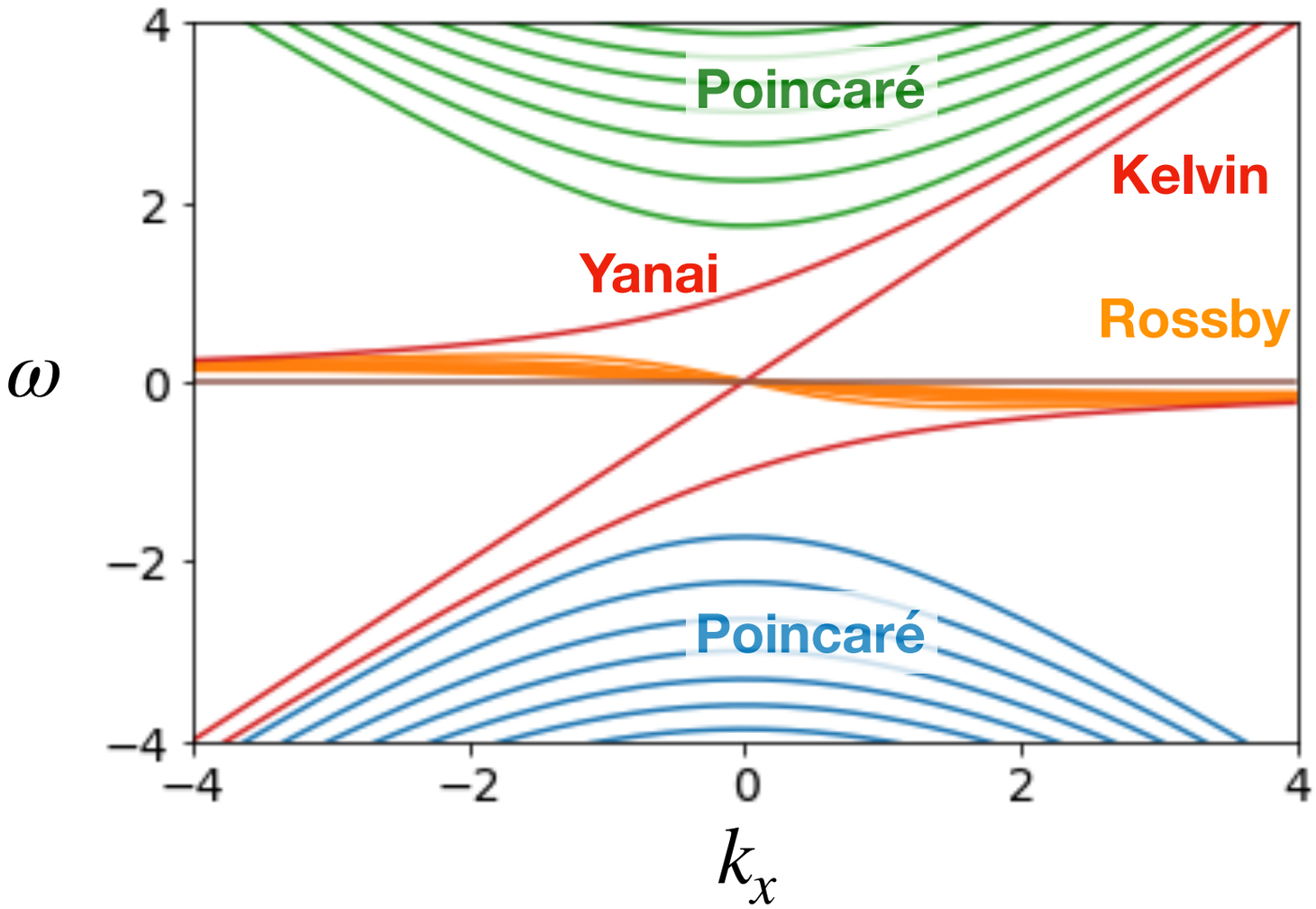
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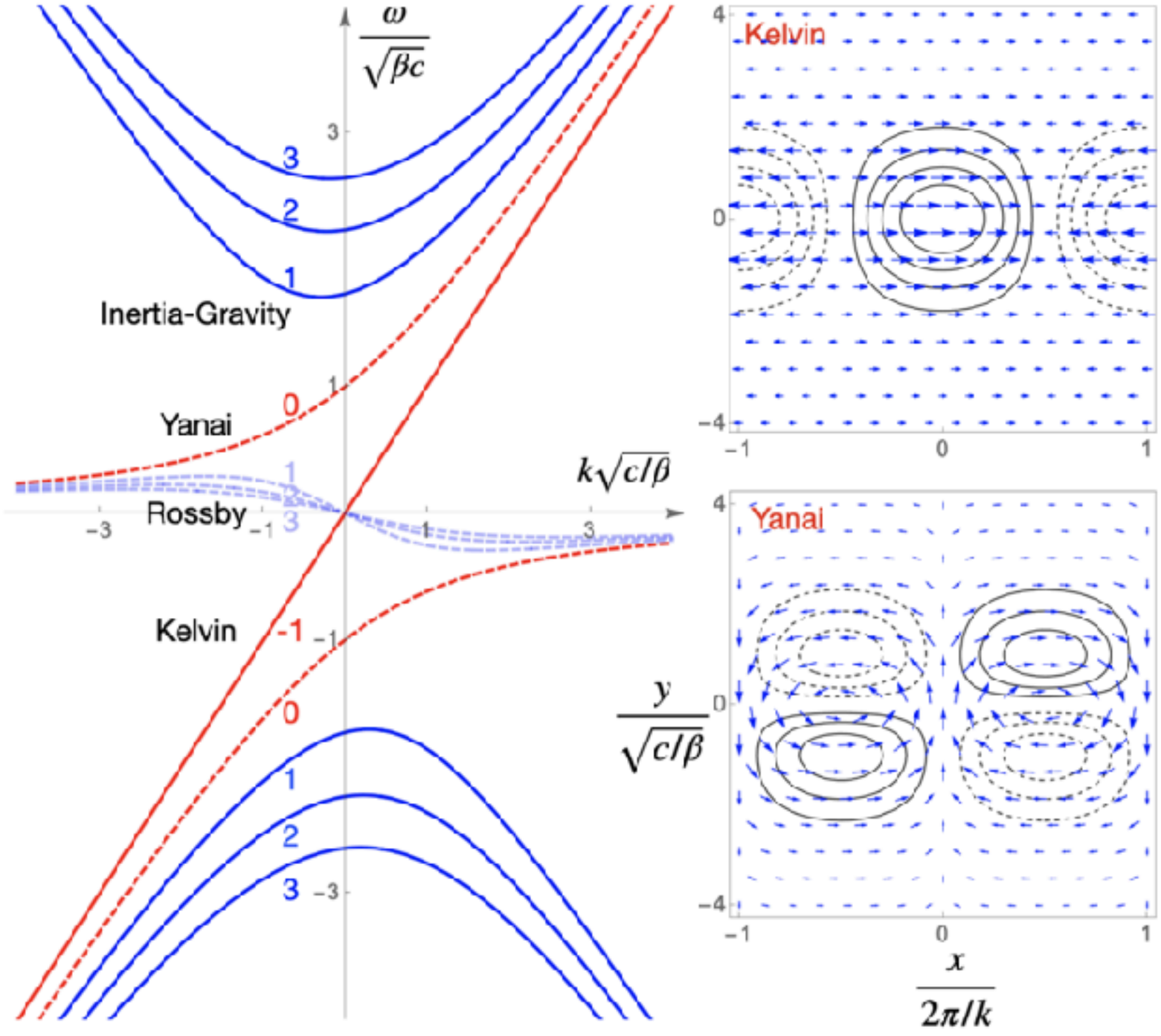
$$f = \beta y$$

$$\Psi = (u, v, \eta), \quad \Psi = \chi(y) e^{i\omega t - i k_x x}$$



- Two modes with eastward group velocity filling the frequency gap.
- Two ? Is it by chance ?

The equatorial waveguide



$$\Psi = (u, v, \eta), \quad \Psi = \chi(y) e^{i\omega t - ik_x x}$$

Insights from the physics of topological insulators



Thouless Haldane Kosterlitz

- Predict unidirectional trapped modes in complicated wave problems by computing topological invariant in dual, simpler problems
- Their existence depends on broken discrete symmetries and dimension of the system.

Message from condensed matter physicists: to understand the dispersion relation complicated problems, look at the polarization relation in simpler problems

Interpretation of Matsuno dispersion relation looking at Kelvin polarization relation

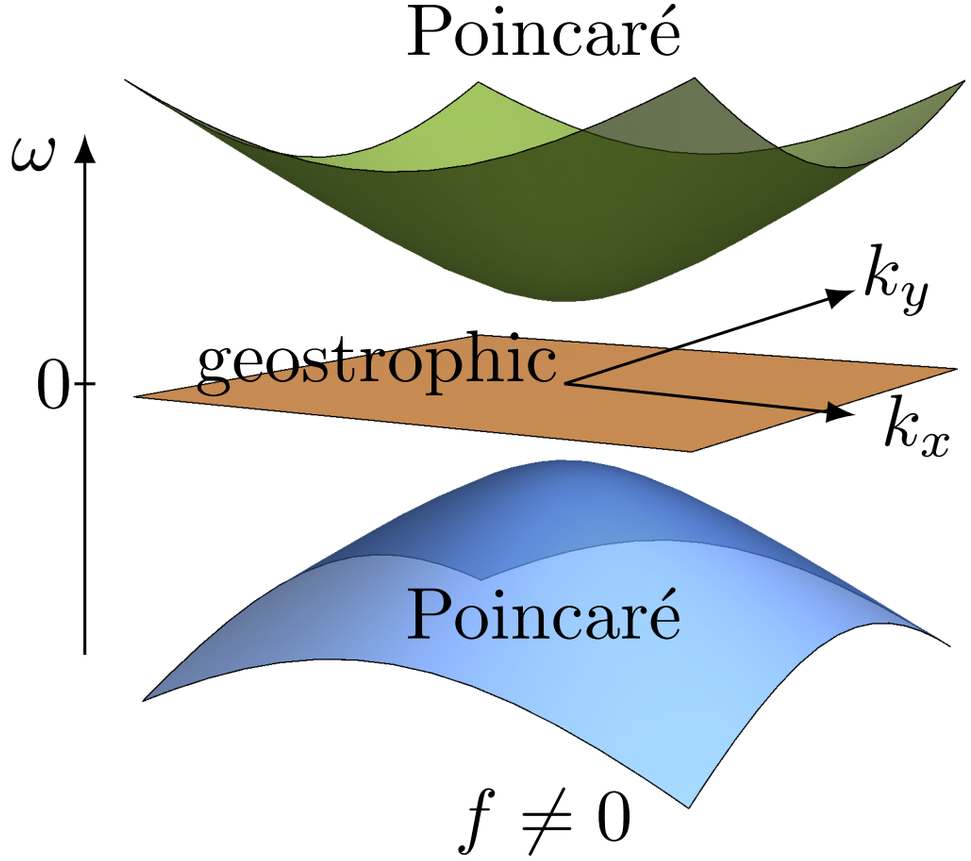




with Brad Marston and Pierre Deplace

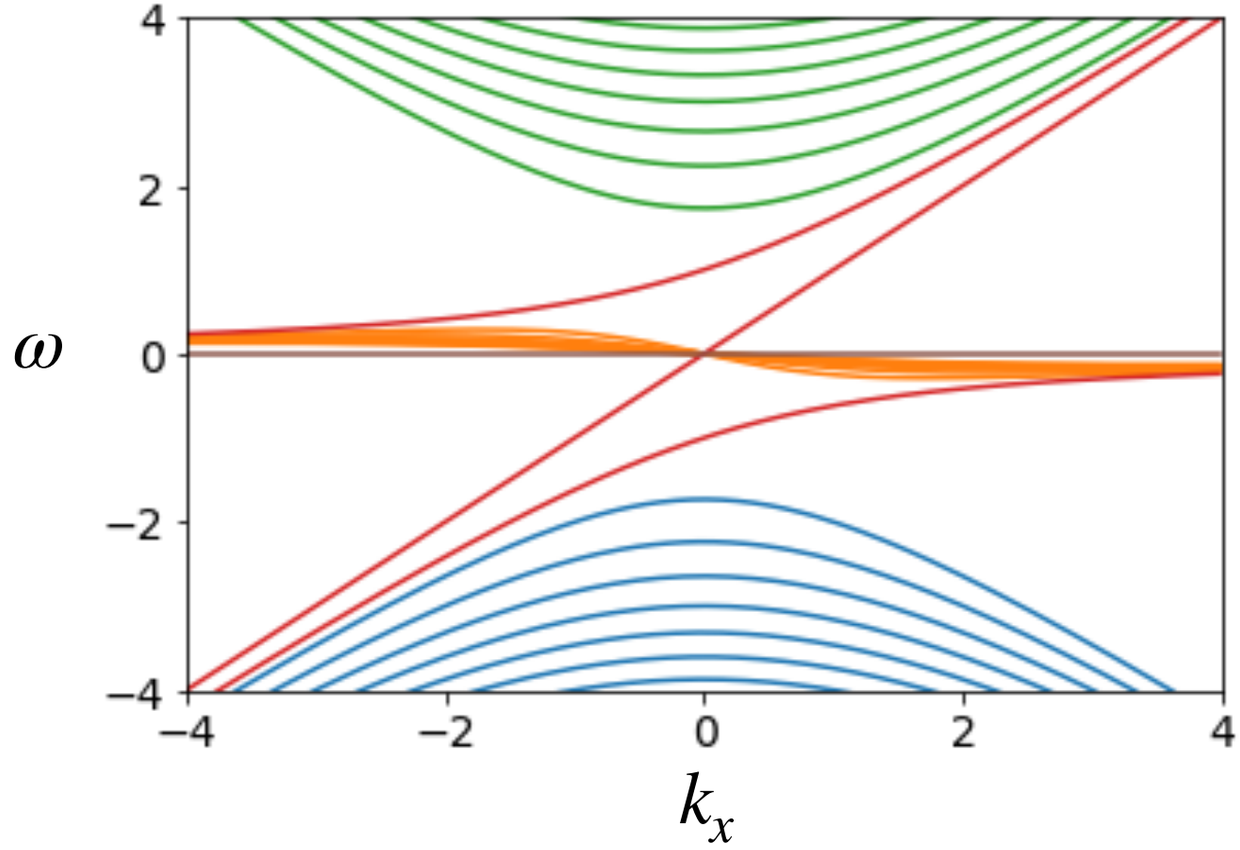
$$\Psi = (u, v, \eta), \quad \Psi = \chi e^{i\omega t - ik_x x - ik_y y}$$

Eigenmode (polarization relation)



f-plane (Kelvin)

Goal: explain why 2 modes transit from one Wave band to another by looking at χ

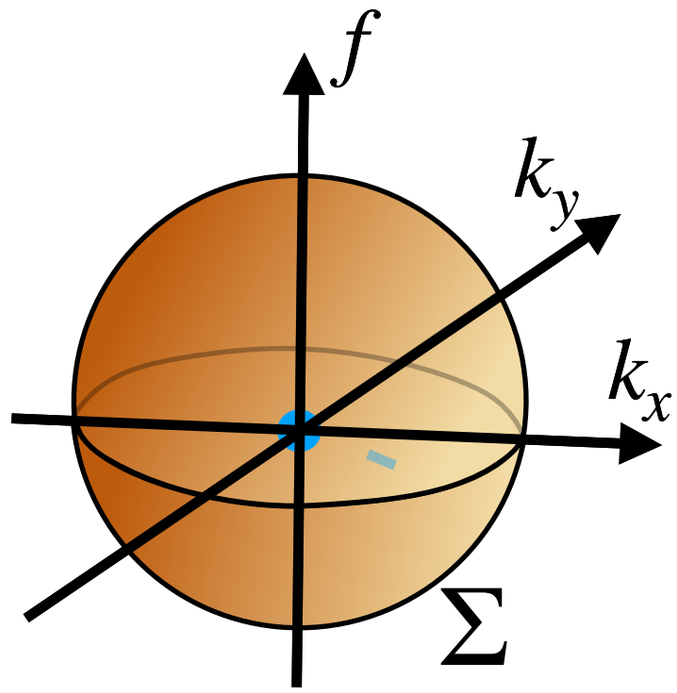


beta-plane (Matsuno)

Singularities in bundles of complex eigenmodes

f-plane shallow water eigenmode in a given waveband:

$$\chi(k_x, k_y, f) = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix} e^{i\alpha}$$

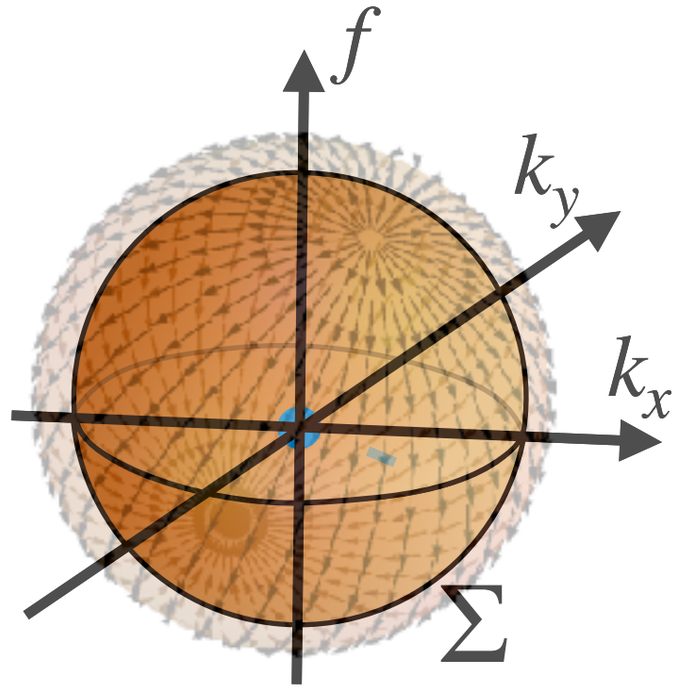


The Chern number counts the number of singularities in a bundle of vectors parameterized on a closed surface.

Singularities in bundles of complex eigenmodes

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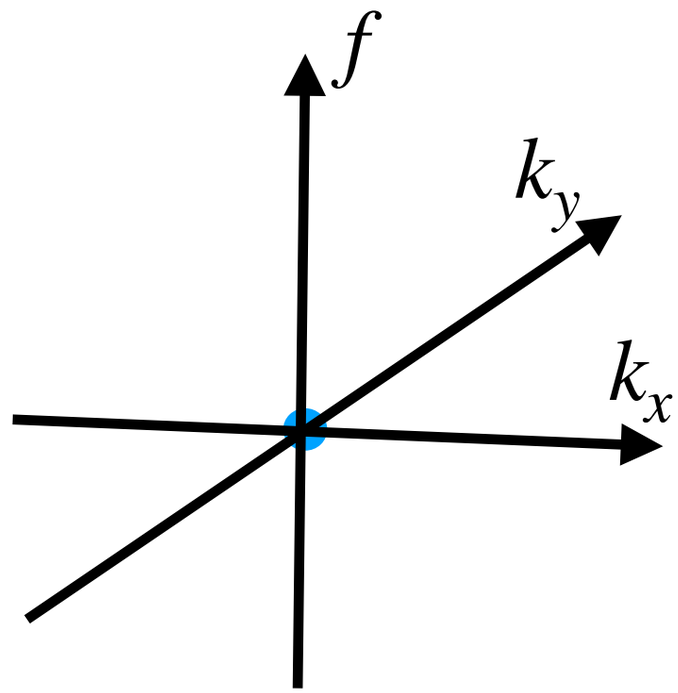


The Chern number counts the number of singularities in a bundle of vectors parameterized on a closed surface.

Chern Gauss Bonnet formula

f-plane shallow water eigenmode

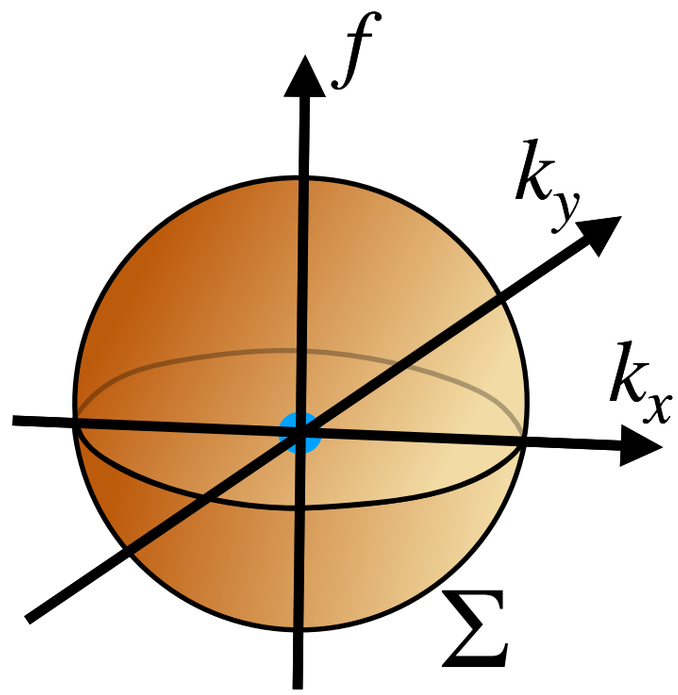
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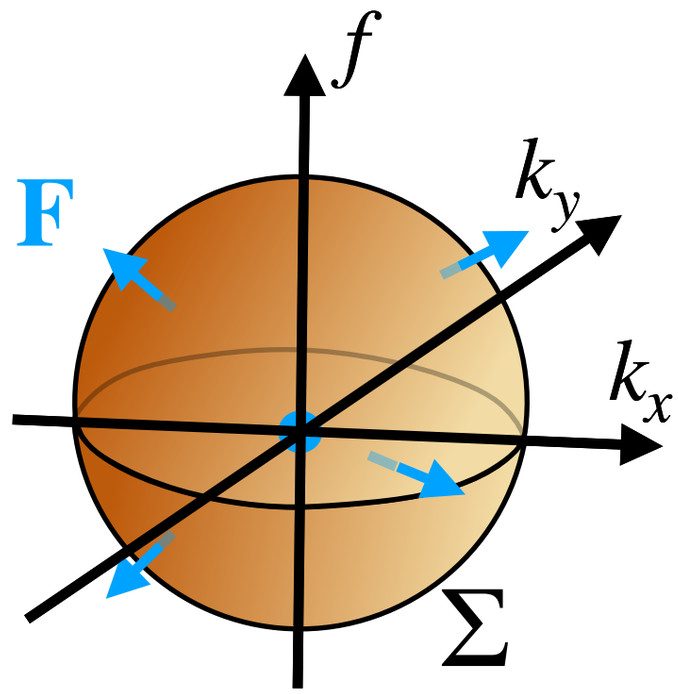


Chern Gauss Bonnet formula

f-plane shallow water eigenmode

$$\chi(k_x, k_y, f) = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\eta} \end{pmatrix} e^{i\alpha}$$

$$\nabla_p = \left(\partial_{k_x}, \partial_{k_y}, \partial_f\right) \qquad \chi^\dagger \cdot \chi = \hat{u}^*\hat{u} + \hat{v}^*\hat{v} + \hat{\eta}^*\hat{\eta}$$



A geometrical quantity

Berry Curvature

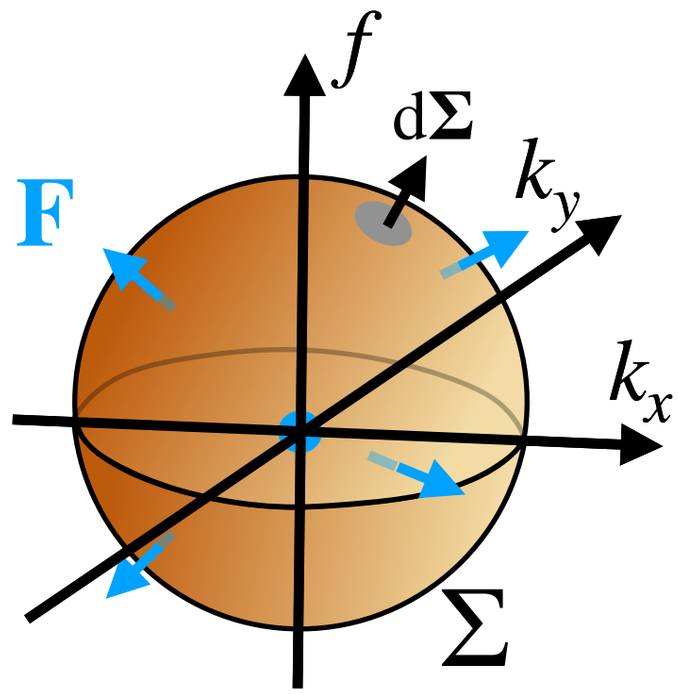
$$\mathbf{F} = i\nabla_p \times \left(\chi^\dagger \cdot \nabla_p \chi\right)$$

Chern Gauss Bonnet formula

f-plane shallow water eigenmode

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A geometrical quantity

A topological index

Berry Curvature

$$\mathbf{F} = i\nabla_p \times \left(\chi^\dagger \cdot \nabla_p \chi\right)$$

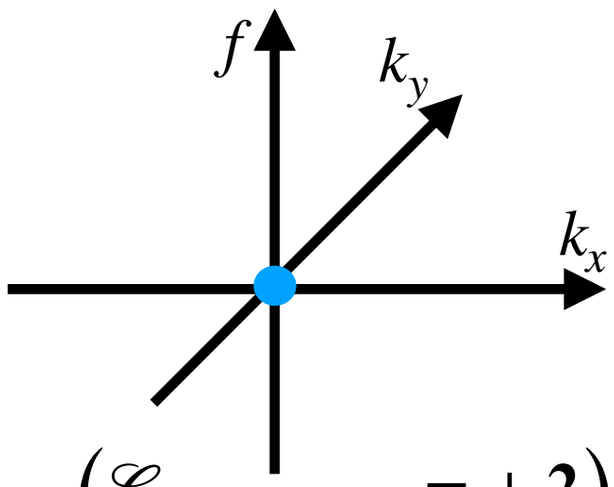
Chern number

$$\mathscr{C} = \frac{1}{2\pi}\int_\Sigma \mathbf{F} \cdot \mathrm{d}\boldsymbol{\Sigma} \in \mathbb{Z}$$

Bulk-interface correspondence

Bulk : f constant

Kelvin 1879

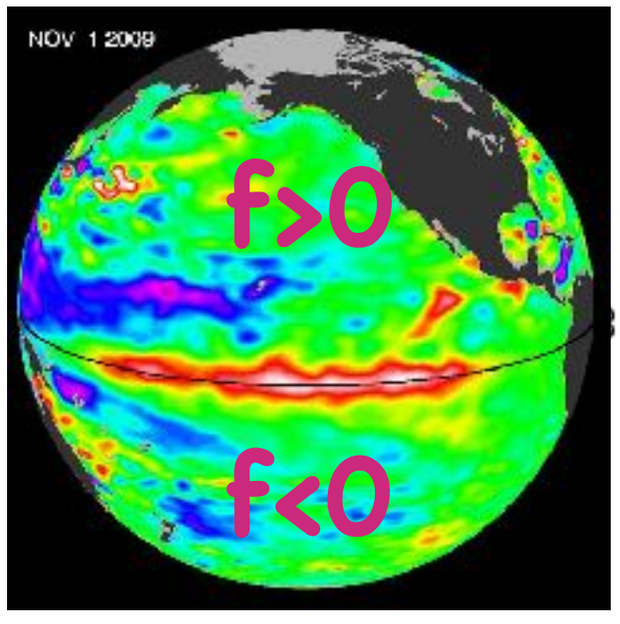


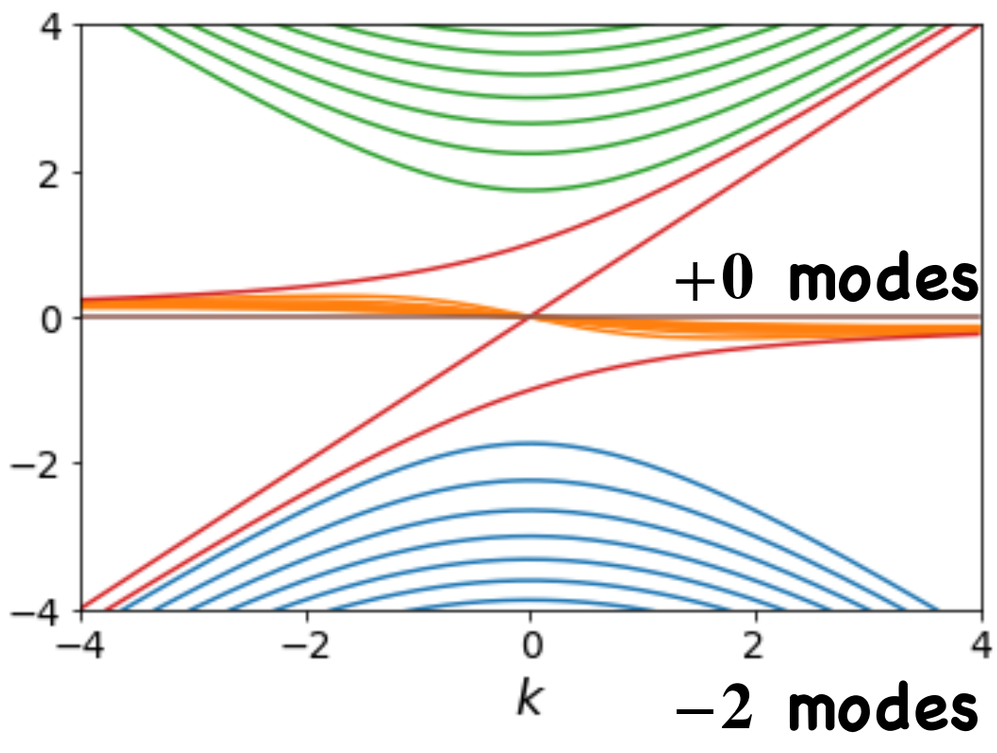
$$\begin{pmatrix} \mathcal{C}_{Poincare,(+)} = +2 \\ \mathcal{C}_{Geostrophic} = 0 \\ \mathcal{C}_{Poincare,(-)} = -2 \end{pmatrix}$$

Topological index

Interface : f(y)

Matsuno 1879





Spectral index

The correspondence is guaranteed by an index theorem Faure 2018

Useful information on global properties of the spectrum from « local » plane wave solutions

Interface problem

$$\partial_t \Psi = -i\hat{\mathscr{H}}\Psi$$

Operator

Bulk problem

$$\omega\chi = \mathscr{H}\chi$$

Symbol

Useful information on global properties of the spectrum from « local » plane wave solutions

$$\hat{\mathscr{H}}\psi(y) = \frac{1}{2\pi\epsilon}\int \mathrm{d}y'\mathrm{d}k_y \; \mathscr{H}\left(\frac{y+y'}{2}, k_y\right)\psi(\mathbf{x})e^{ik_y\cdot(y-y')/\epsilon}$$

Interface problem

$$\partial_t \Psi = -i\hat{\mathscr{H}}\Psi$$

Operator

« **Quantum mechanics** »

Beta-plane problem

Weyl

Wigner

Bulk problem

$$\omega\chi = \mathscr{H}\chi$$

Symbol

« **Classical mechanics**»

f- plane problem

Useful information on global properties of the spectrum from « local » plane wave solutions

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Interface problem — **Weyl** — **Bulk problem**

$$\partial_t \Psi = -i\hat{\mathscr{H}}\Psi$$

Operator

Wigner

$$\omega\chi = \mathscr{H}\chi$$

Symbol

« Quantum mechanics »

Beta-plane problem

« Classical mechanics»

f- plane problem

Cheverry, Gallagher, Paul, Saint-Raymond 2007-2012, for equatorial waves

Vanneste 1999, Onuki 2020 wave transport in the presence of disorder

Useful information on global properties of the spectrum from « local » plane wave solutions

$$\hat{\mathscr{H}}\psi(y) = \frac{1}{2\pi\epsilon}\int \mathrm{d}y'\mathrm{d}k_y \; \mathscr{H}\left(\frac{y+y'}{2}, k_y\right)\psi(\mathbf{x})e^{ik_y\cdot(y-y')/\epsilon}$$

Interface problem — **Weyl** — **Bulk problem**

$$\partial_t \Psi = -i\hat{\mathscr{H}}\Psi$$

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$$\omega\chi = \mathscr{H}\chi$$

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« Classical mechanics»

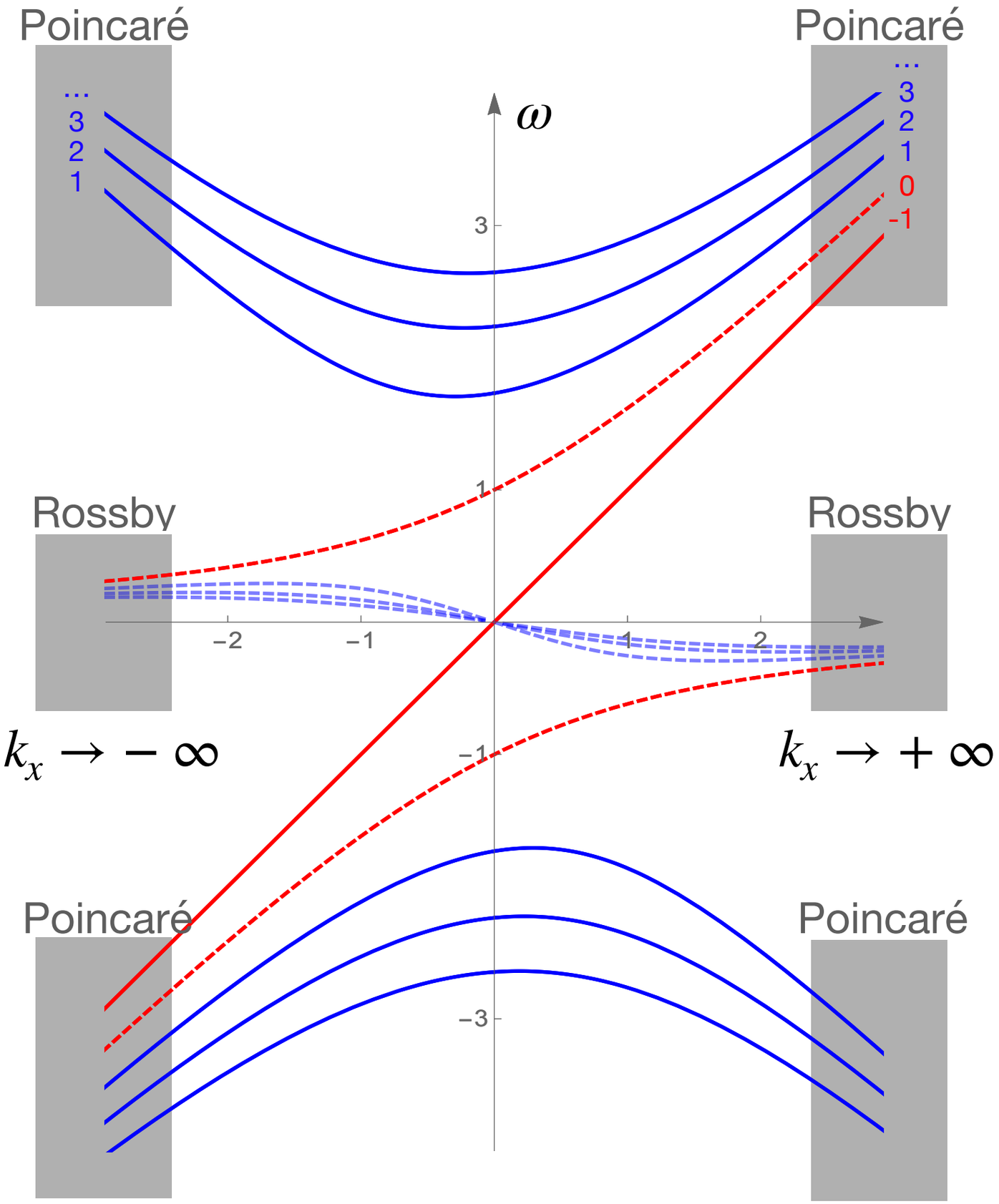
f- plane problem

Cheverry, Gallagher, Paul, Saint-Raymond 2007-2012, for equatorial waves

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Here: interpretation of bulk-interface correspondence with ray tracing

Step 1: projection on each waveband



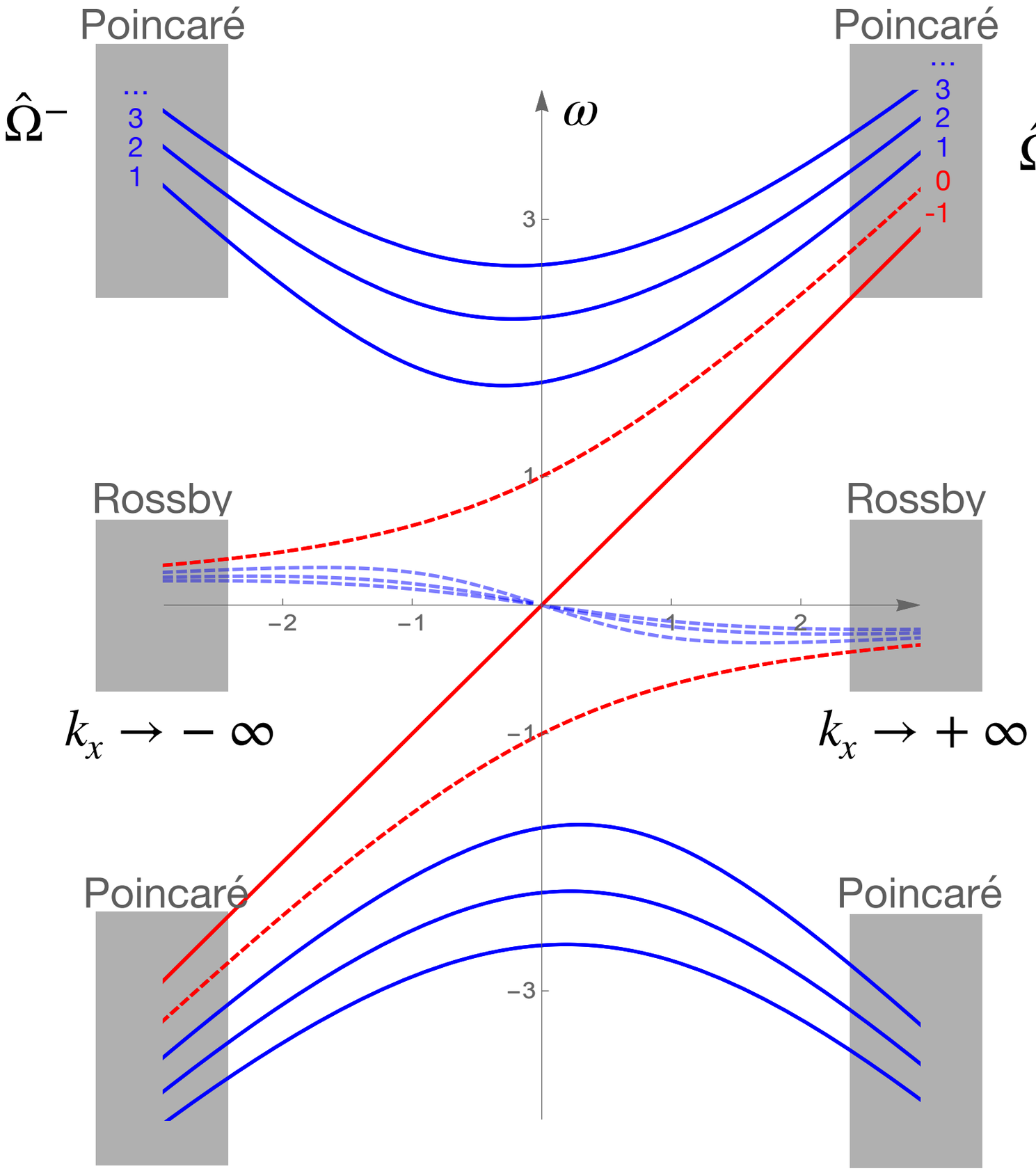
$$i\partial_t \Psi = \hat{\mathscr{H}}\Psi$$

Matsuno Operator

$$\hat{\mathscr{H}} = \begin{pmatrix} 0 & iy & 1 \\ -iy & 0 & -i\epsilon\partial_y \\ 1 & -i\epsilon\partial_y & 0 \end{pmatrix}$$

Semiclassical parameter $\epsilon = \dfrac{\beta}{ck_x^2} \to 0$

Step 1: projection on each waveband



$\hat{\Omega}^+$ Scalar operator $i\partial_t \psi = \hat{\Omega}^+ \psi$

$$i\partial_t \Psi = \hat{\mathscr{H}} \Psi$$

Matsuno Operator

$$\hat{\mathscr{H}} = \begin{pmatrix} 0 & iy & 1 \\ -iy & 0 & -i\epsilon\partial_y \\ 1 & -i\epsilon\partial_y & 0 \end{pmatrix}$$

Semiclassical parameter $\epsilon = \dfrac{\beta}{ck_x^2} \to 0$

Why $\hat{\Omega}^+$ has two more modes than $\hat{\Omega}^-$?

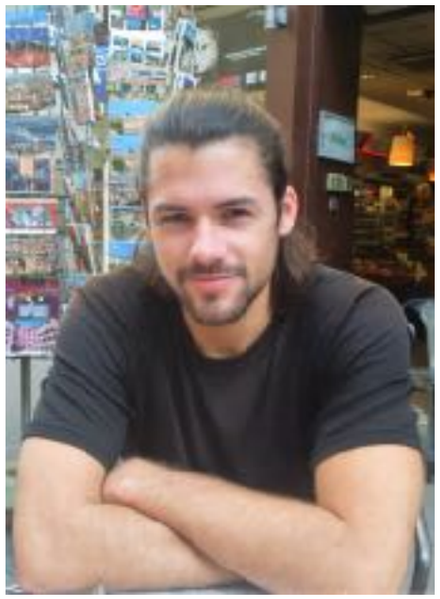
Step 2: ray tracing $\hat{\Omega}^{\pm} \rightarrow \Omega^{\pm}(y, k_y)$

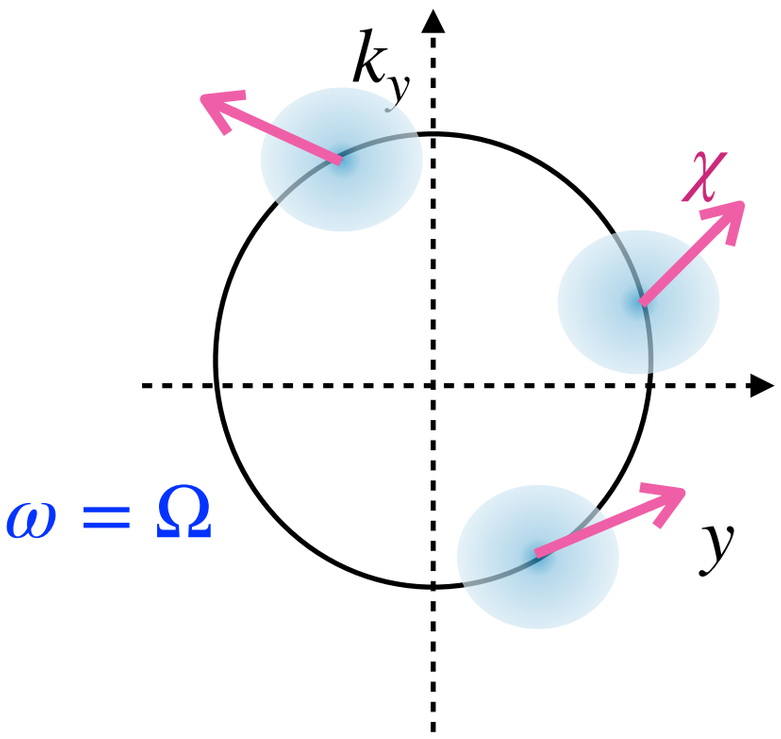
$$\dot{y} = \partial_{k_y}\Omega + \epsilon F_{k_y,y}\dot{y}$$

$$\dot{k}_y = -\partial_y\Omega + \epsilon F_{k_y,y}\dot{k}_y$$

[Littlejohn & Flynn 1991]

Nicolas Perez et al 2021
Application to shallow water waves





Ray Hamiltonian

$$\Omega(y, k_y) = \Omega_0 + \epsilon\Omega_1$$

$$\Omega_0 = \sqrt{f^2 + c^2(k_x^2 + k_y^2)}$$

Berry Curvature

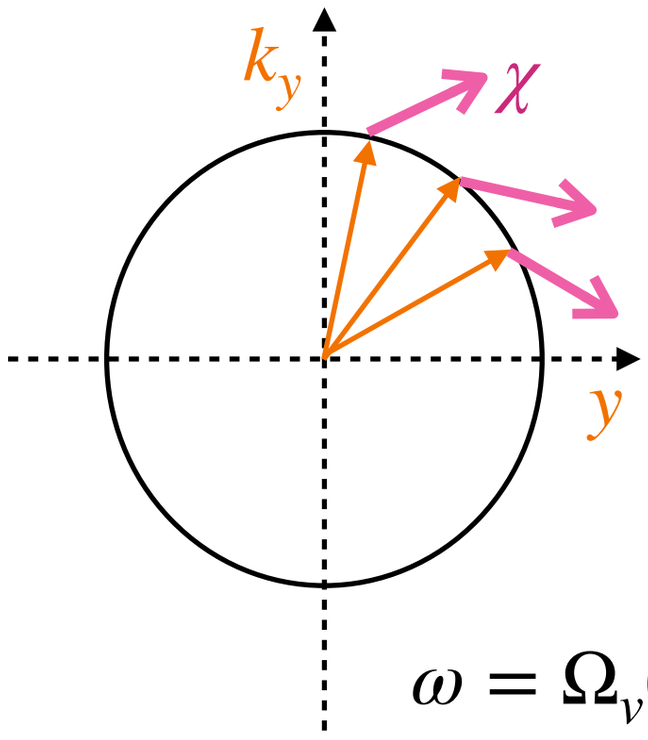
$$F_{k_y,y} = i\left(\partial_y\chi^\dagger\partial_{k_y}\chi - \partial_{k_y}\chi^\dagger\partial_y\chi\right)$$

Step 3: Bohr-Sommerfed condition

Dynamical phase **Geometric phase** **Maslov** **Number of modes below ω**

[Argyres 1965]

$$\frac{1}{2\pi\epsilon}\oint_{\omega=\Omega_v} k_y dy + \frac{i}{2\pi}\oint_{\omega=\Omega_v} \chi^\dagger \cdot d\chi - \frac{1}{2} = n$$



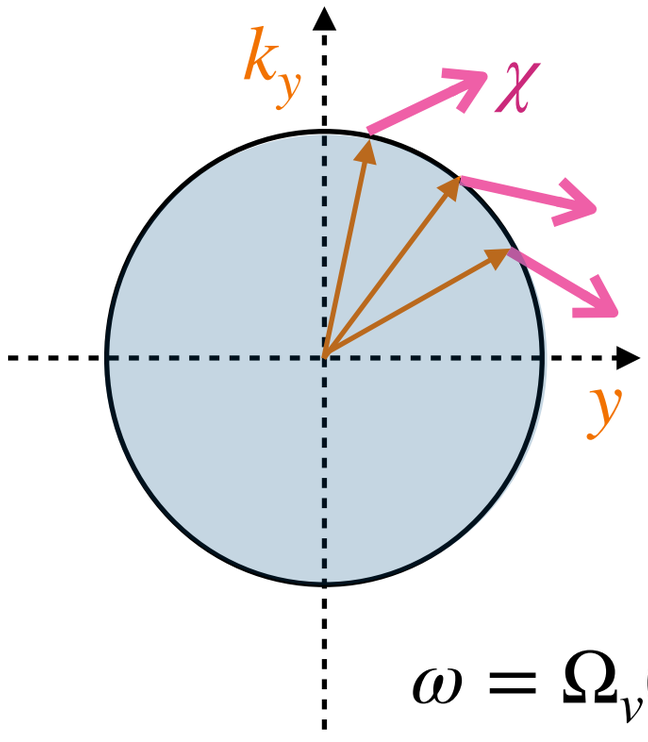
$$\omega = \Omega_v(y, k_y)$$

Step 3: Bohr-Sommerfed condition

Dynamical phase **Geometric phase** **Maslov** **Number of modes below ω**

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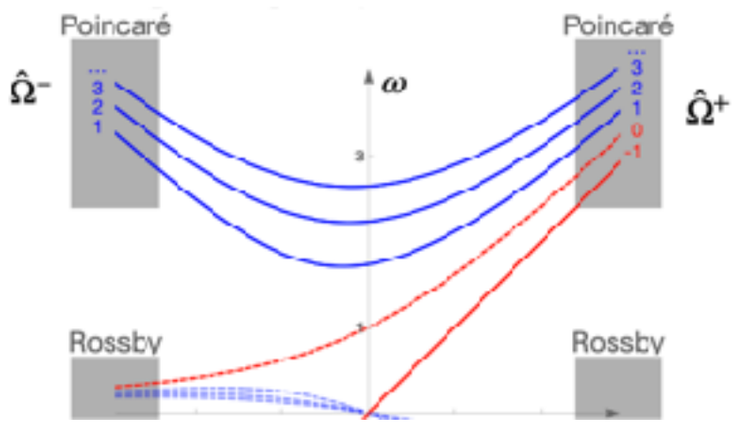
$$= \frac{1}{2\pi}\int\int F_{y,k_y} dy dk_y = \frac{\Gamma}{2\pi}$$

$\omega = \Omega_v(y, k_y)$

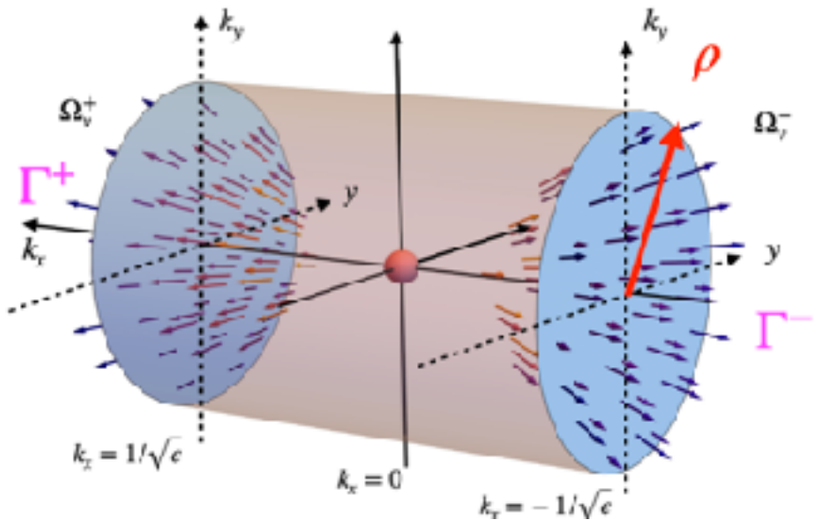
Γ is the Berry flux through the area $\Omega_v(y, k_y) < \omega$

Topological origin of the spectral flow

Dispersion relation



Parameter space / phase space



Number of modes between $\omega = 0$ **and** $\omega(\rho)$

$$n^+(\rho) - n^-(\rho) = \frac{\Gamma^+ - \Gamma^-}{2\pi} = \underset{\rho \to +\infty}{\mathscr{C}}$$

Chern-Gauss-Bonnet

Bohr-Sommerfeld

Summary

Latitudinal variations of the Coriolis parameter, aka beta effect is key to

- **The western intensification of ocean currents**
- Stommel 1948, Munk 1952 with an incompressible, **linearized** 2D flow model
- **The equatorial wave guide**
- Matsuno 1966, with a **linearized** shallow water model

Low energy western intensified quasilinear gyres persist in the turbulent regime

- **Instantaneous field dominated by a strongly out-of-equilibrium vortex gas**
- Miller Deremble Venaille 2024

Topology yields useful information on astrophysical and geophysical waves

- **Bulk-interface correspondence (Kelvin - Matsuno correspondence in this talk)**
- Delplace Marston Venaille 2017

Manifestation of Berry curvature in geophysical ray tracing

- **Semiclassical interpretation of bulk-interface correspondence**
- Venaille Onuki Perez Leclerc 2023