Vortex motion, stretching and shedding

Andrew Gilbert, University of Exeter,

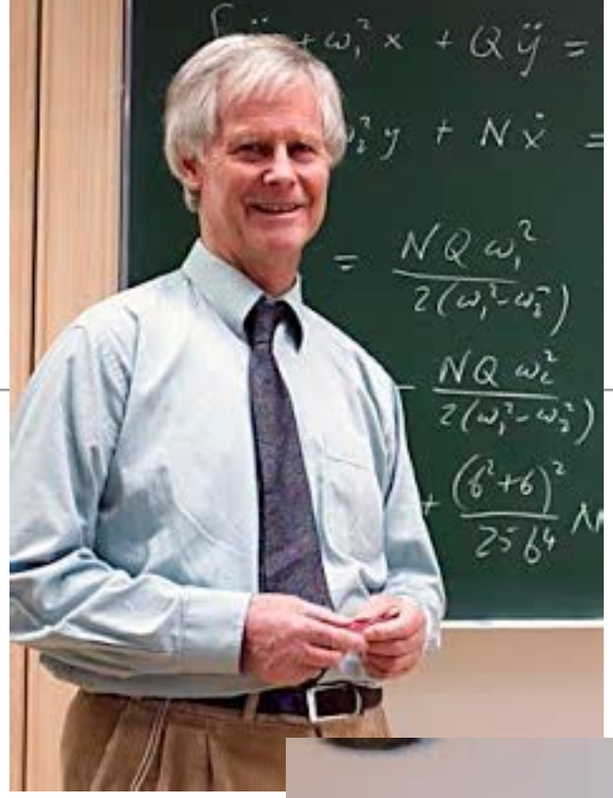
Steve Childress, New York University,

Emmanuel Dormy, ENS, Paris,

Paul Valiant, Brown University.

Happy birthday, Keith!

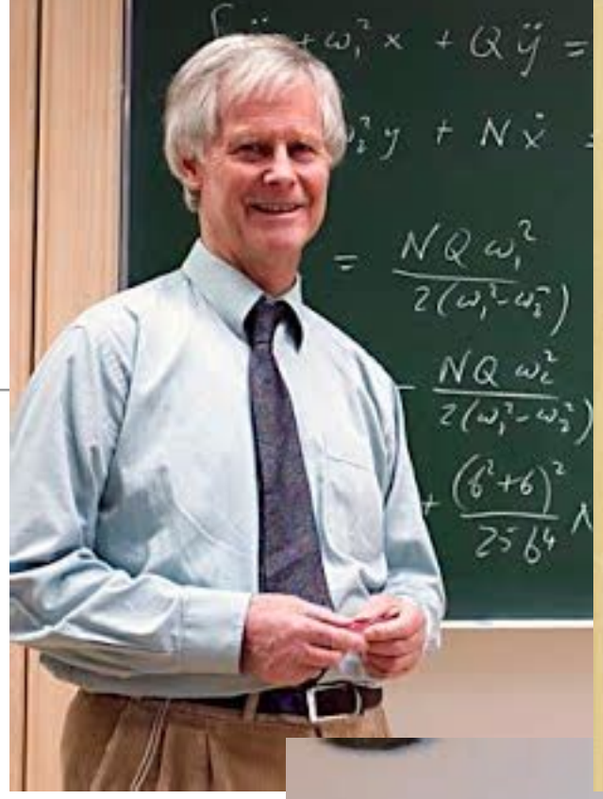
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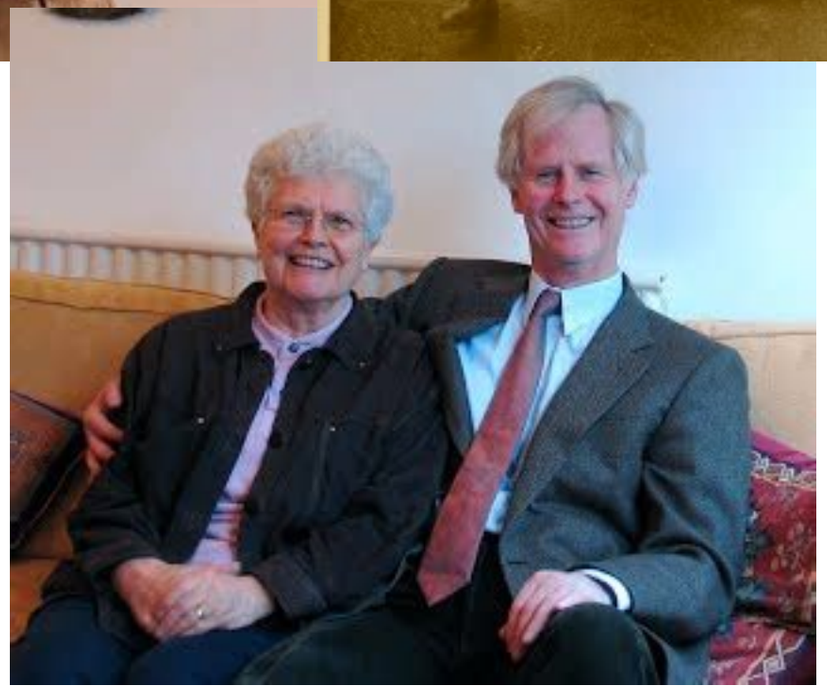


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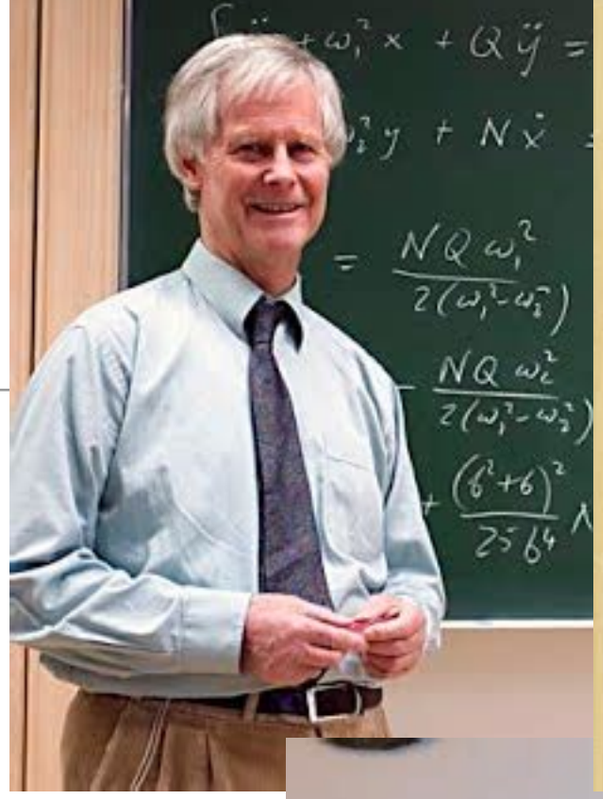








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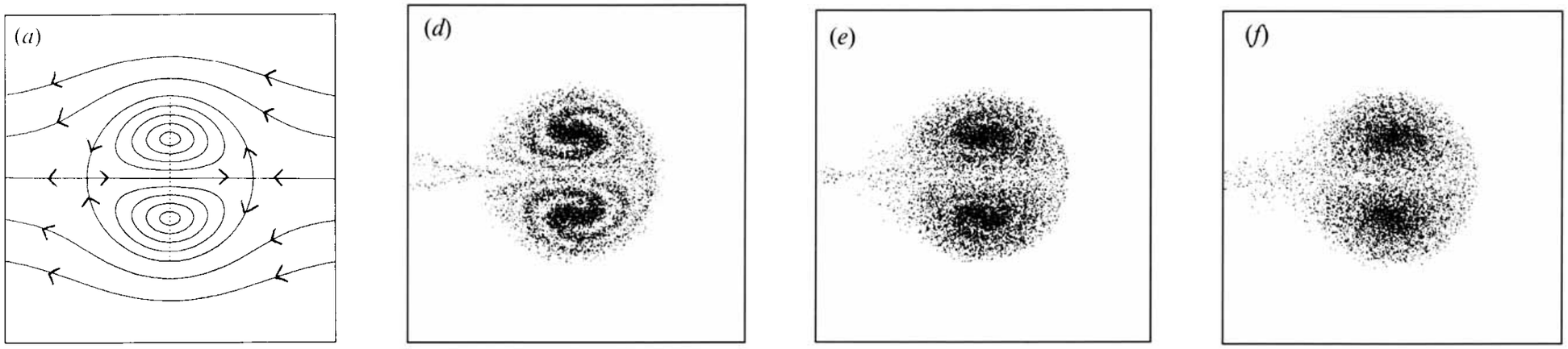
Vortex collisions, stretching, and shedding: axisymmetric geometry:

- T. T. Lim and T. B. Nickels, “Instability and reconnection in the head-on collision of two vortex rings,” *Nature* **357**, 225–227 (1992).

- https://www.youtube.com/watch?v=XJk8ijAUCiI

Vortex propagation and shedding/erosion: planar geometry

- Passive scalar in Lamb dipole, Lingevitch & Bernoff, *J. Fluid Mech.* **270**, 1994.

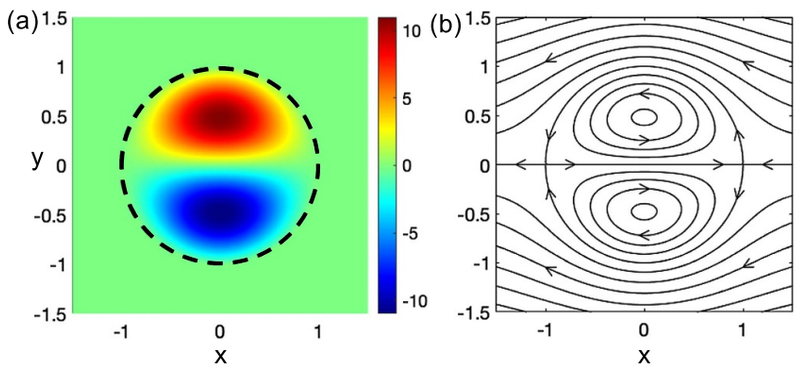


$$\omega(r,\theta) = \begin{cases} \dfrac{2Uk}{J_0(ka)} J_1(kr)\sin\theta, & r \leqslant a, \\ 0, & r > a, \end{cases}$$

Vortex propagation and shedding/erosion: planar geometry

- Vorticity decay of Lamb dipole, Krasny & Xu, *Fluid Dyn. Res.* **53**, 2021.

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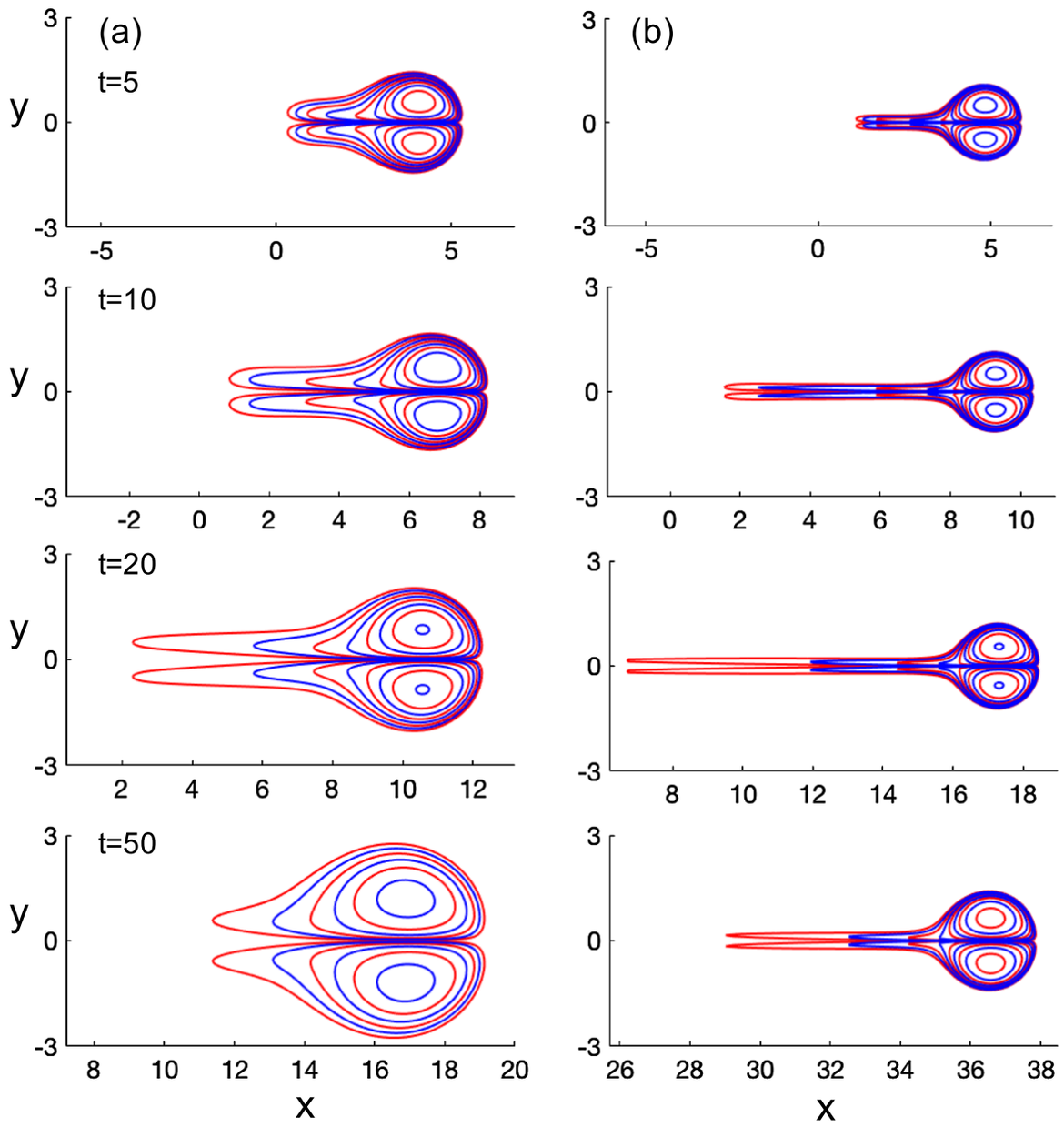
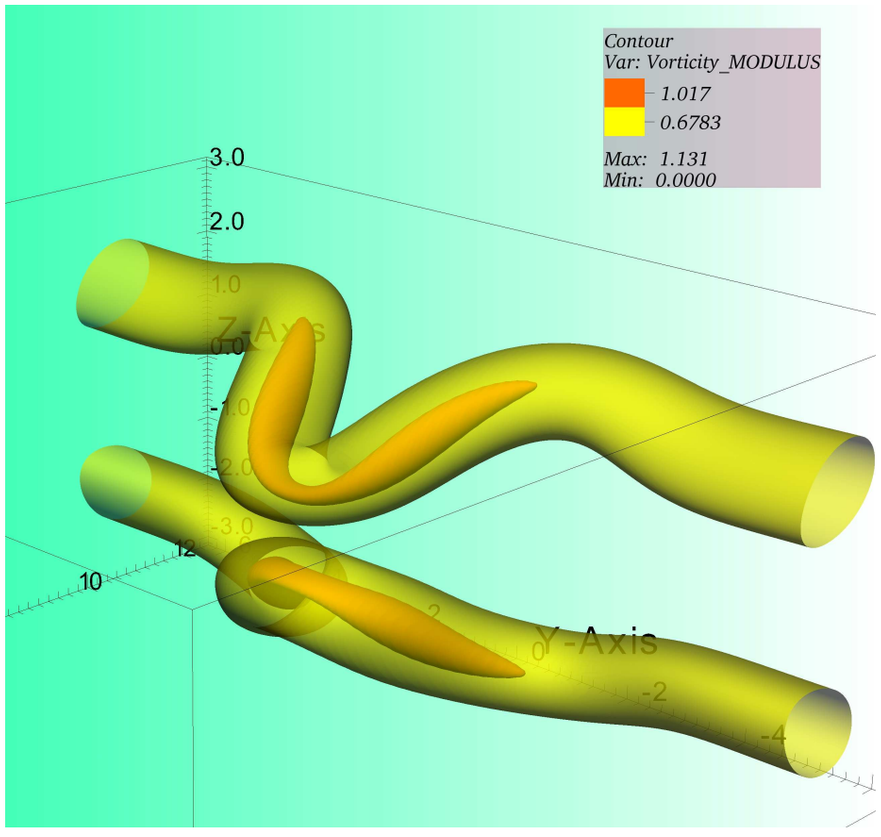


Figure 2. NSE simulations of viscous Lamb dipole, vorticity contours $|\omega| = 2^p$, $p = -6:3$ with alternating color (red/blue), time $t = 5, 10, 20, 50$, (a) $\mathrm{Re} = 125$, (b) $\mathrm{Re} = 1000$, $\omega > 0$ for $y > 0$, $\omega < 0$ for $y < 0$.

Anti-parallel vortices

- Bustamante & Kerr, 3D Euler about a 2D symmetry plane, *Physica D*, **237**, 2008.

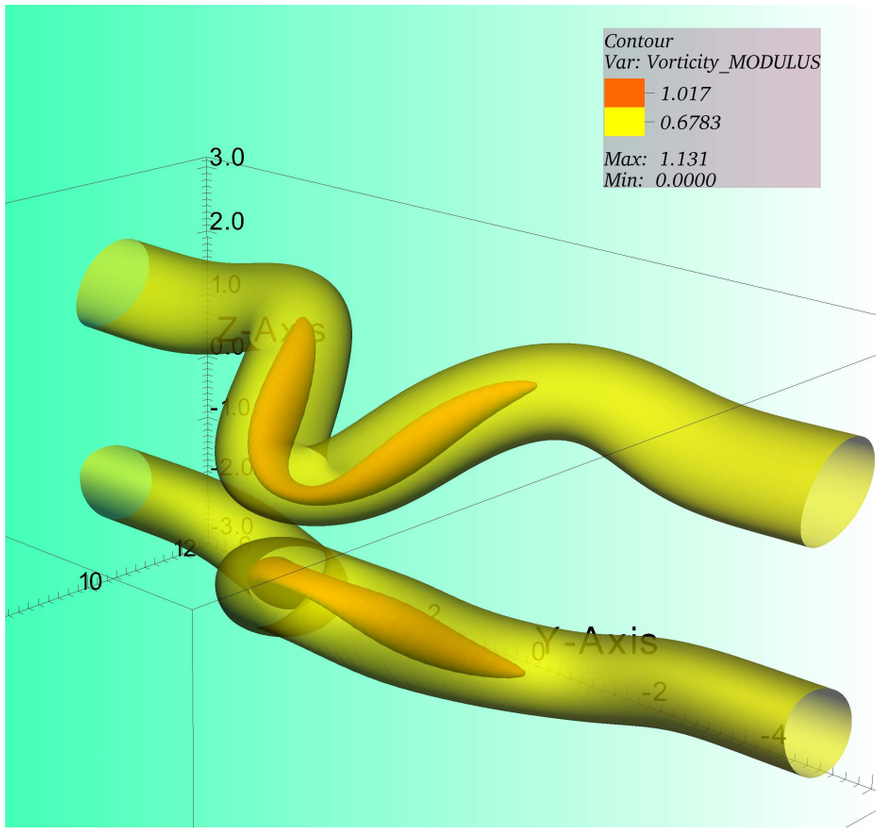
- Euler equations (zero viscosity):

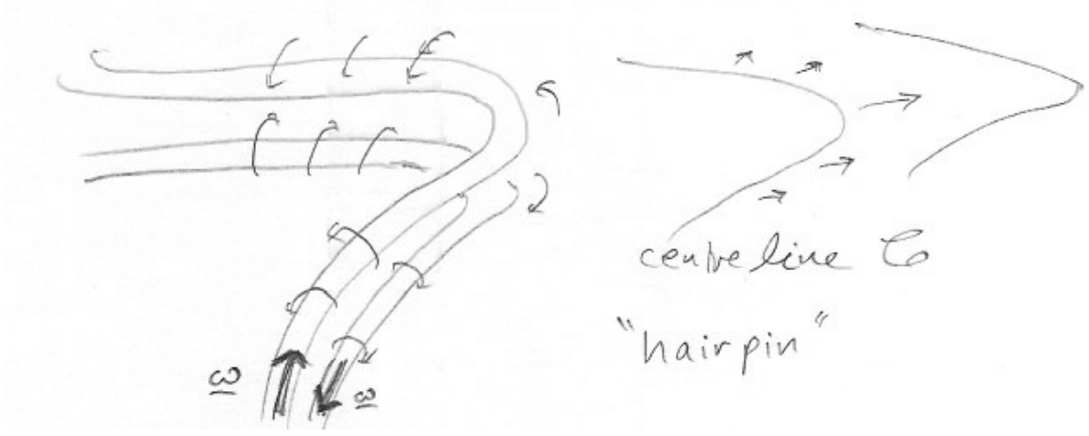


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Vortex stretching

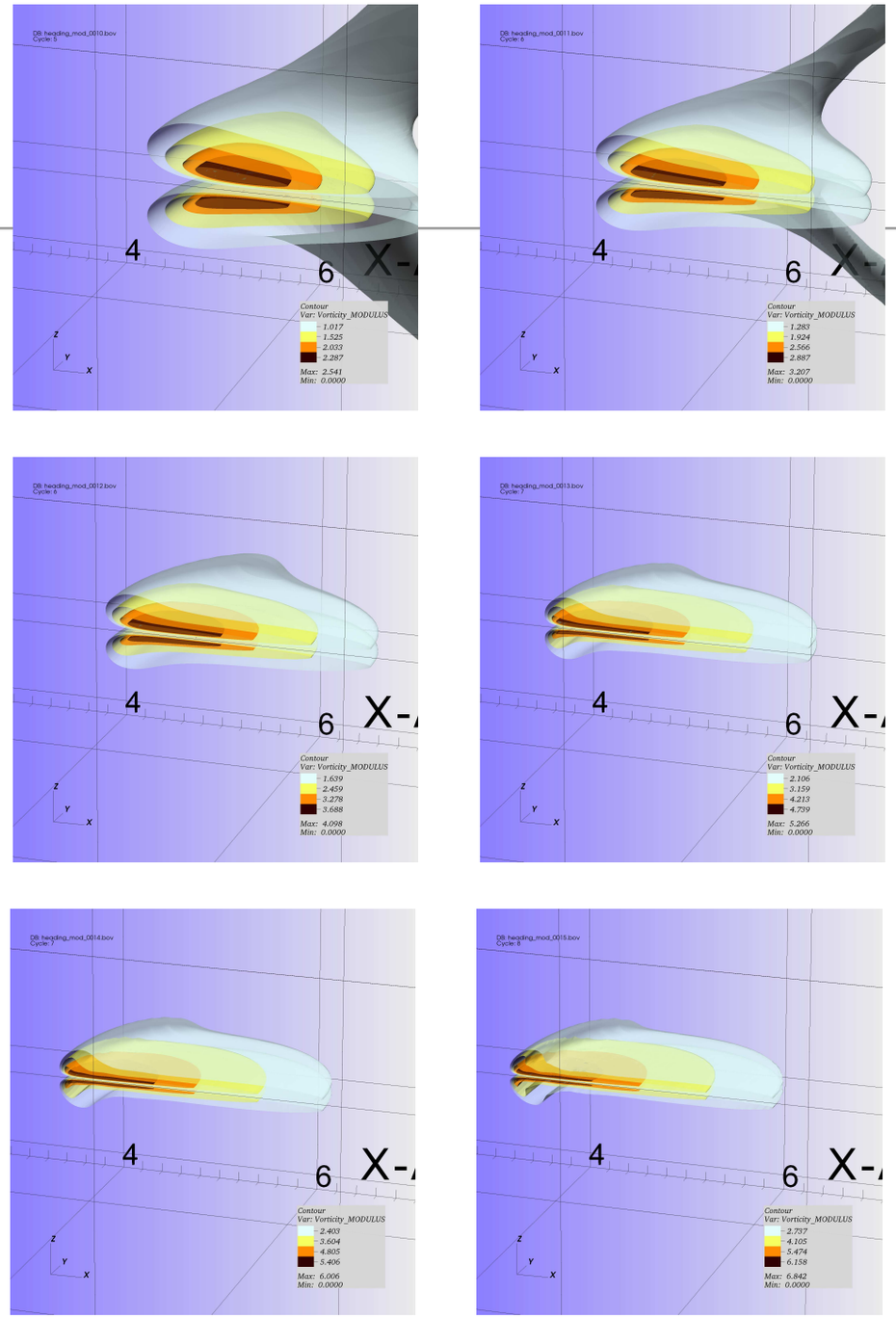


FIG. 6: From left to right, and from top to bottom: six successive, zoomed snapshots of the Euler anti-parallel vortices at times $t = 5.625, 6.25, 6.875, 7.5, 7.8125, 8.125$. The contours are sectioned through the $y = 0$ symmetry plane, to facilitate the view of the structures. The contours are isosurfaces of vorticity modulus corresponding, respectively from outer to inner, to the $40\%, 60\%, 80\%$ and 90% of the value of the instantaneous maximum vorticity modulus.

Vortex stretching

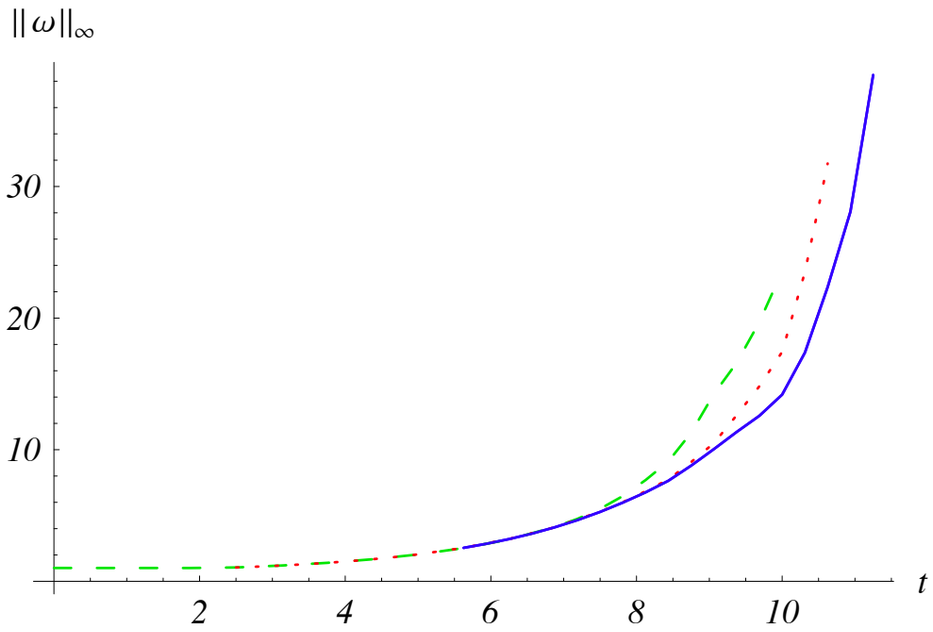


FIG. 3: Top: Resolution study of $||\omega||_\infty$, for resolutions $n_x \times n_y \times n_z$ of: $512 \times 64 \times 2048$ (dashed), $512 \times 128 \times 2048$ (dotted) and $1024 \times 256 \times 2048$ (solid). Bottom: Anisotropic

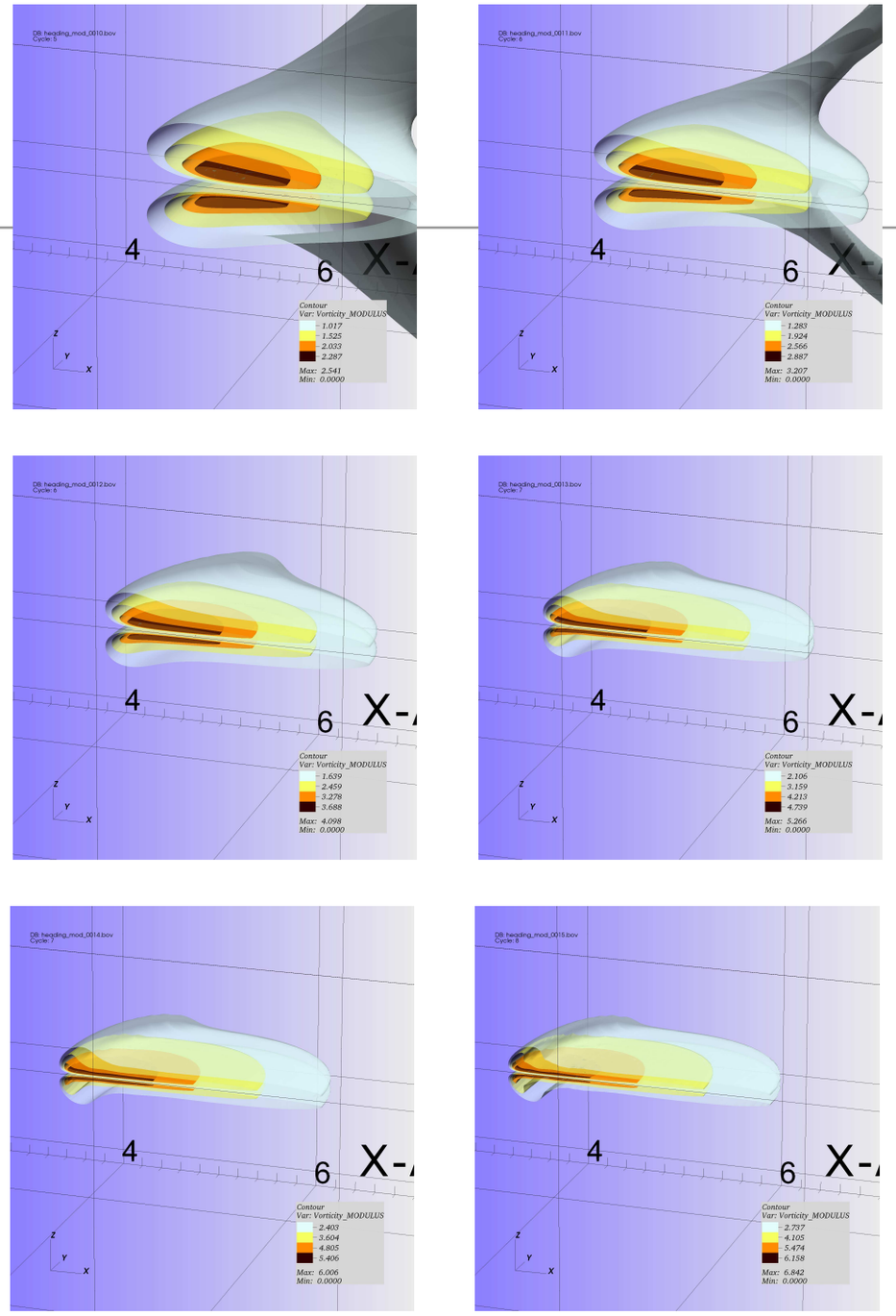


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Beale-Kato-Majda theorem

- Euler equation:

$$\partial_t \boldsymbol{\omega} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u}, \qquad \boldsymbol{\omega} = \nabla \times \boldsymbol{u}$$

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Theorem 1 (Beale *et al* [35]). *There exists a global solution* $\boldsymbol{u} \in C([0, \infty]; H^s) \cap C^1([0, \infty]; H^{s-1})$ *of the Euler equations for* $s \geqslant 3$ *if and only if, for every* $t > 0$,

$$\int_0^t \|\boldsymbol{\omega}(\cdot, \xi)\|_\infty \, \mathrm{d}\xi < \infty. \tag{1.3}$$

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- Clay Millenium Prize: to prove regularity or otherwise of the Navier-Stokes equation (viscosity > 0).

Vortex ring collisions and stretching:

- Moffatt & Kimura, Towards a finite time singularity, *J. Fluid Mech.* **861**, 2019.

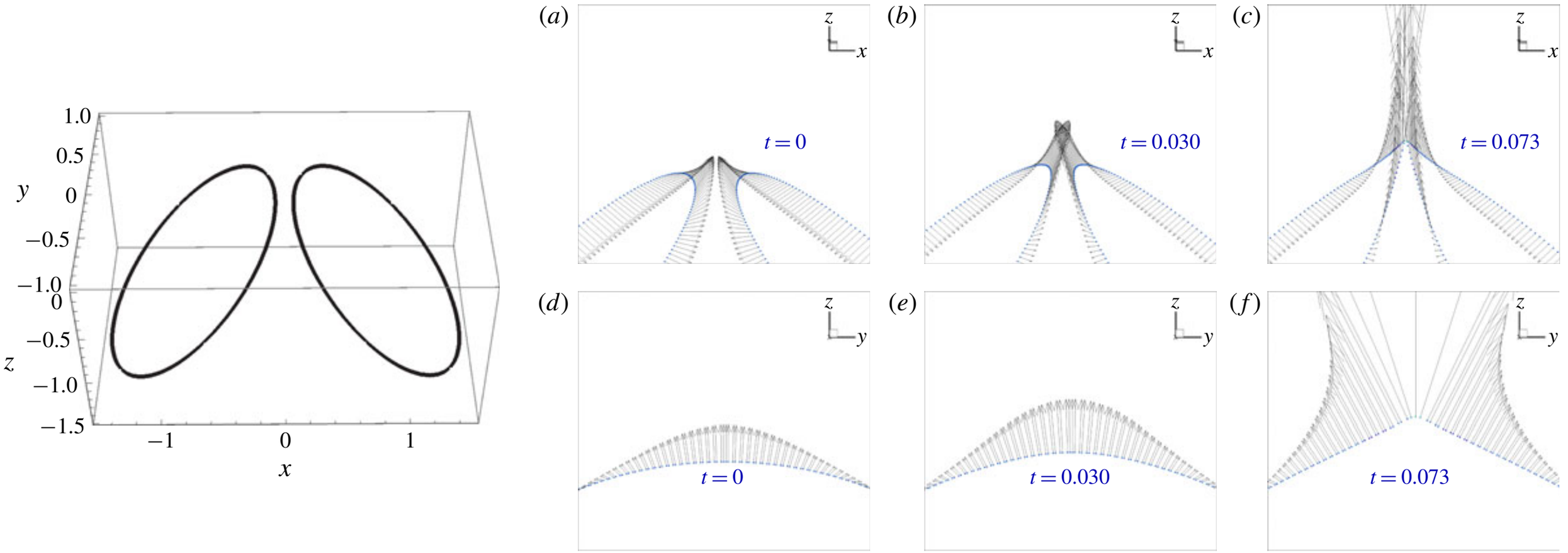
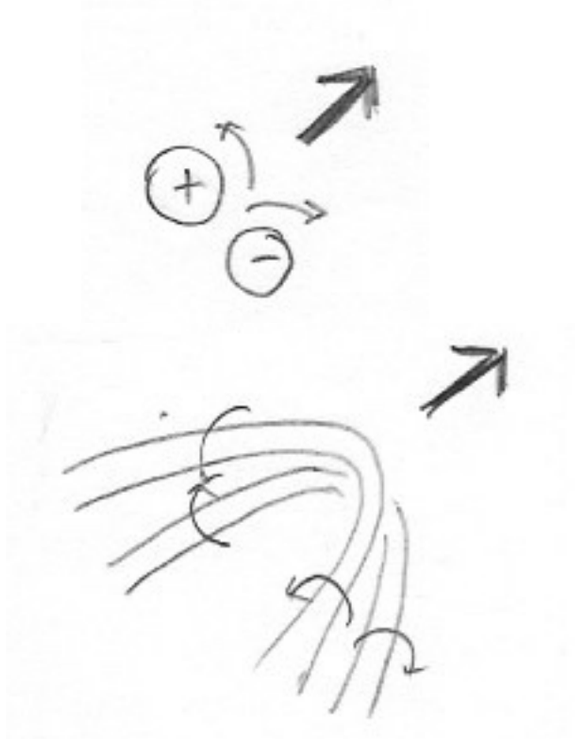


FIGURE 1. (Colour online) Biot–Savart approach to a singularity (using 'cutoff' regularisation) starting with two circular vortices (as in figure 4), radius R and circulations

- https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/towards-a-finitetime-singularity-of-the-navierstokes-equations-part-1-derivation-and-analysis-of-dynamical-system/46FA48CD0063CD0AC00CC608C7AAE311

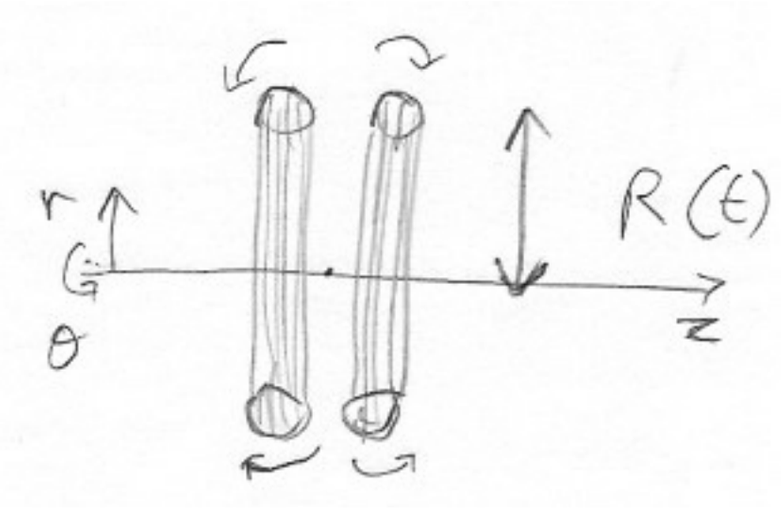
Processes

- vortex dipole pair in 2-d or 3-d
- vortex propagation
- vortex stretching and intensification
- vorticity shedding / vortex erosion
- axial flow….



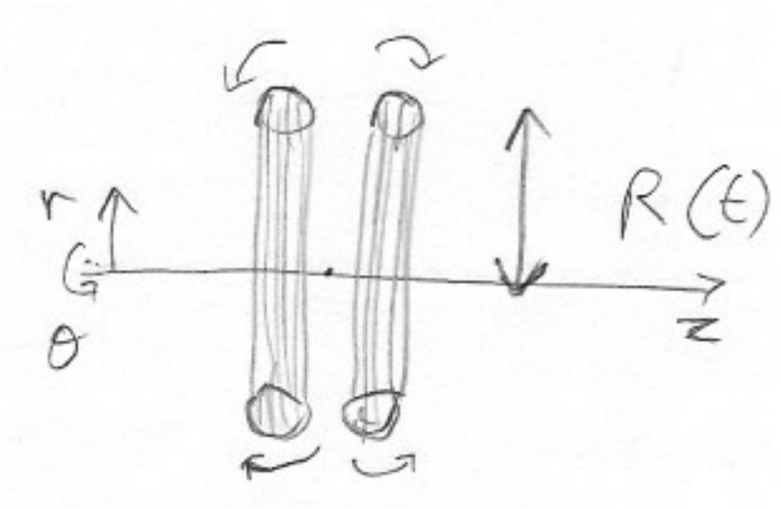
Axisymmetric flow without swirl (Childress 2008)

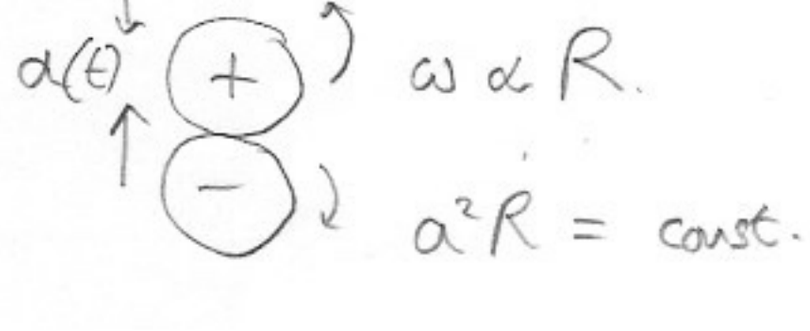
- cylindrical polar coordinates (r, θ, z)
- fluid velocity and vorticity $\boldsymbol{u}(r,z) = u\hat{\boldsymbol{r}} + v\hat{\boldsymbol{z}}$, $\boldsymbol{\omega}(r,z) = \omega\hat{\boldsymbol{\theta}}$
- vortex rings major radius $R(t)$, minor radius $a(t)$



Axisymmetric flow without swirl (Childress 2008)

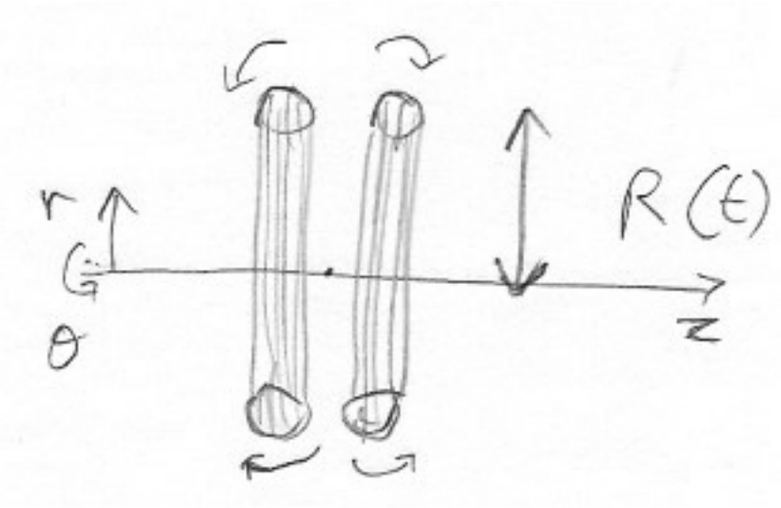
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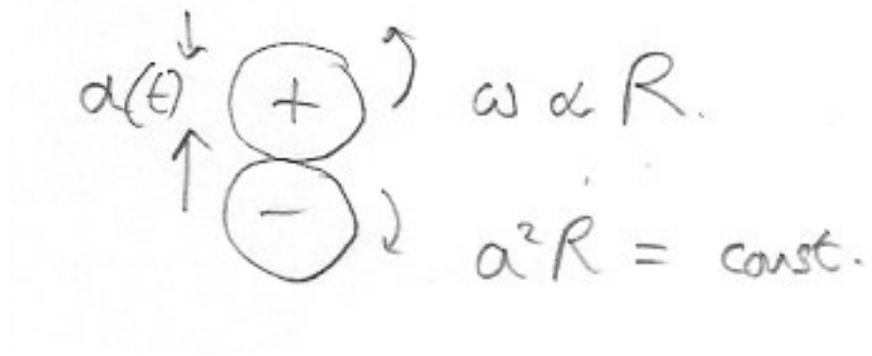




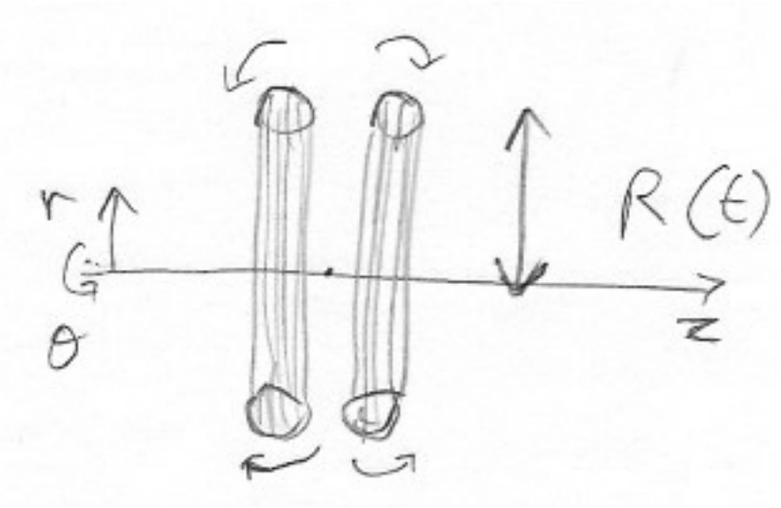
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- conservation of volume $a^2 R \sim 1$
- vortex stretching $\omega \propto R$
- vortex motion $dR/dt \propto a\omega$

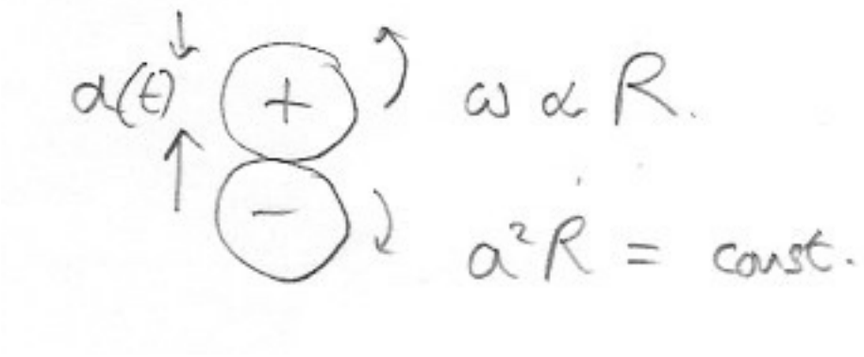




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- conservation of volume $a^2 R \sim 1$
- vortex stretching $\omega \propto R$
- vortex motion $dR/dt \propto a\omega$
- gives $dR/dt \sim \sqrt{R}$ and so $R(t) \sim t^2$

But…

- kinetic energy $E \sim Ra^2(\omega a)^2 \sim R$

- can’t conserve kinetic energy and volume

But…

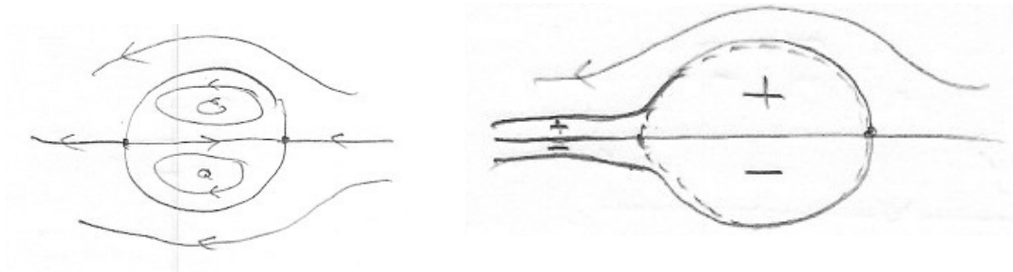
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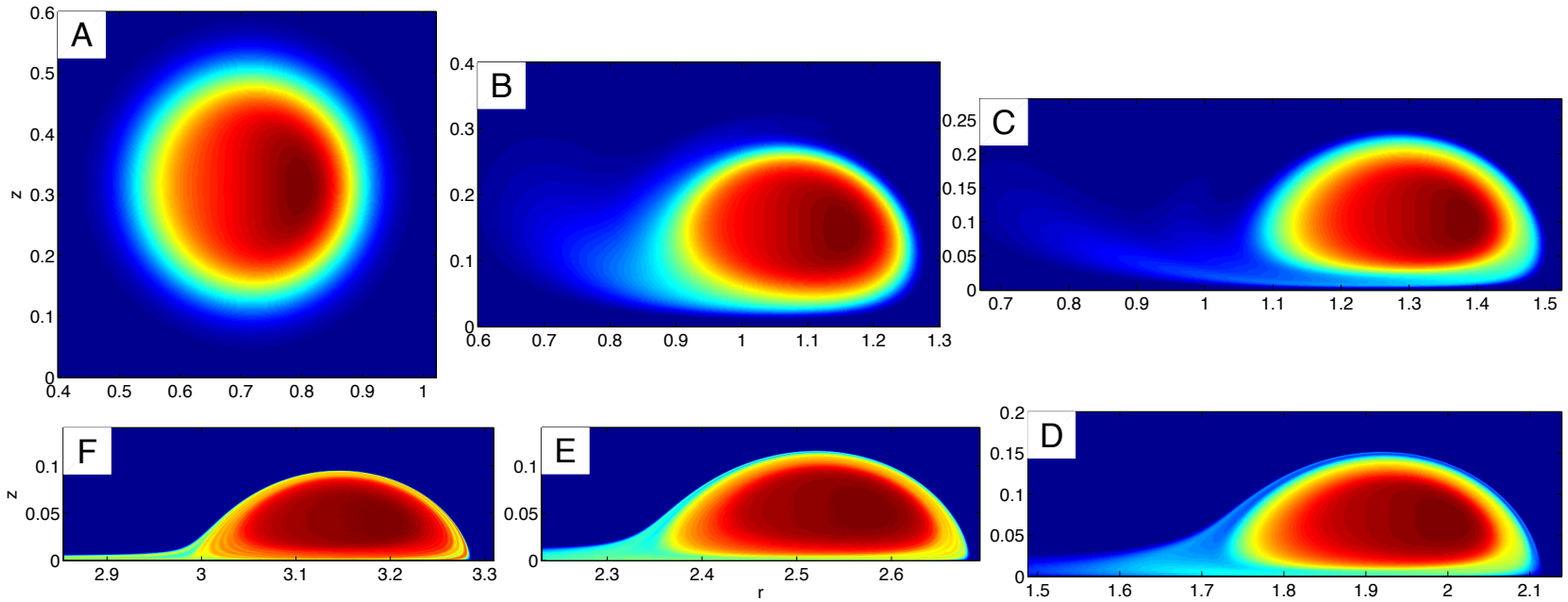
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- to obtain $dR/dt \sim R^{1/4}$ and so $R(t) \sim t^{4/3}$
- conserve energy but lose volume $a^2 R \sim t^{-2/3}$
- vortex stripping, seen in simulations



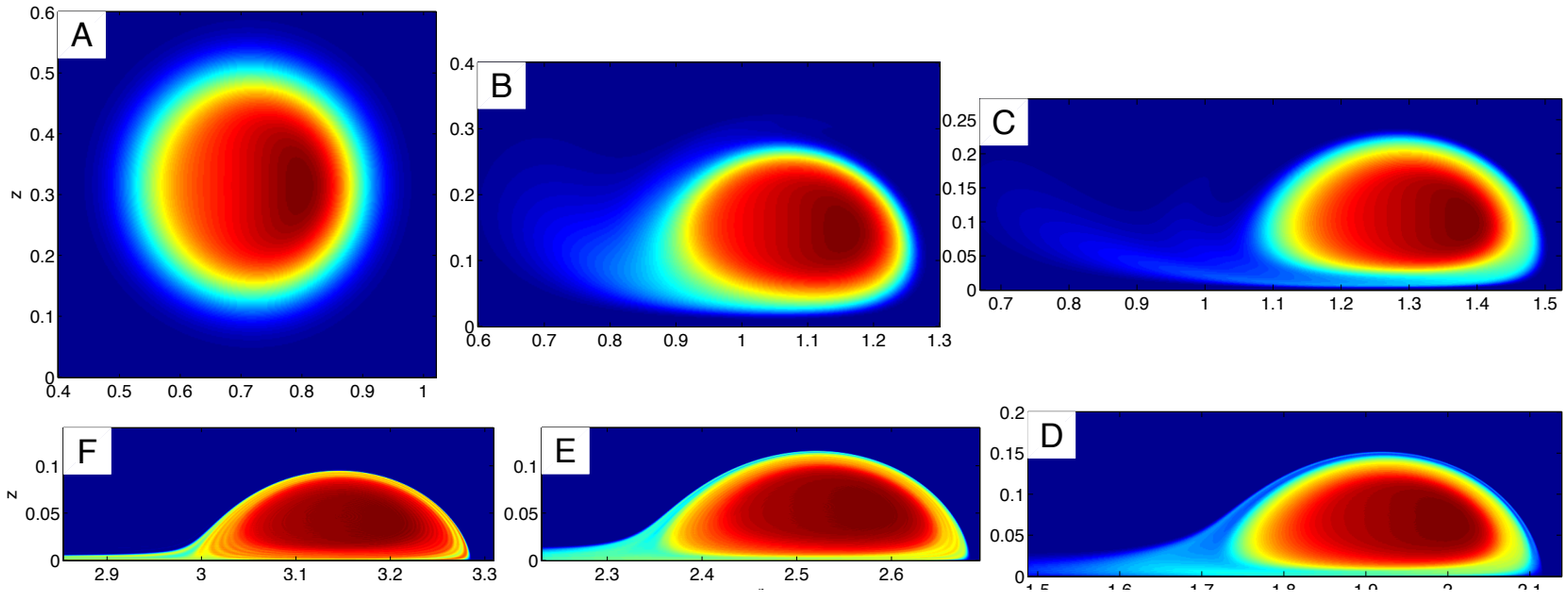
Paul Valiant simulations

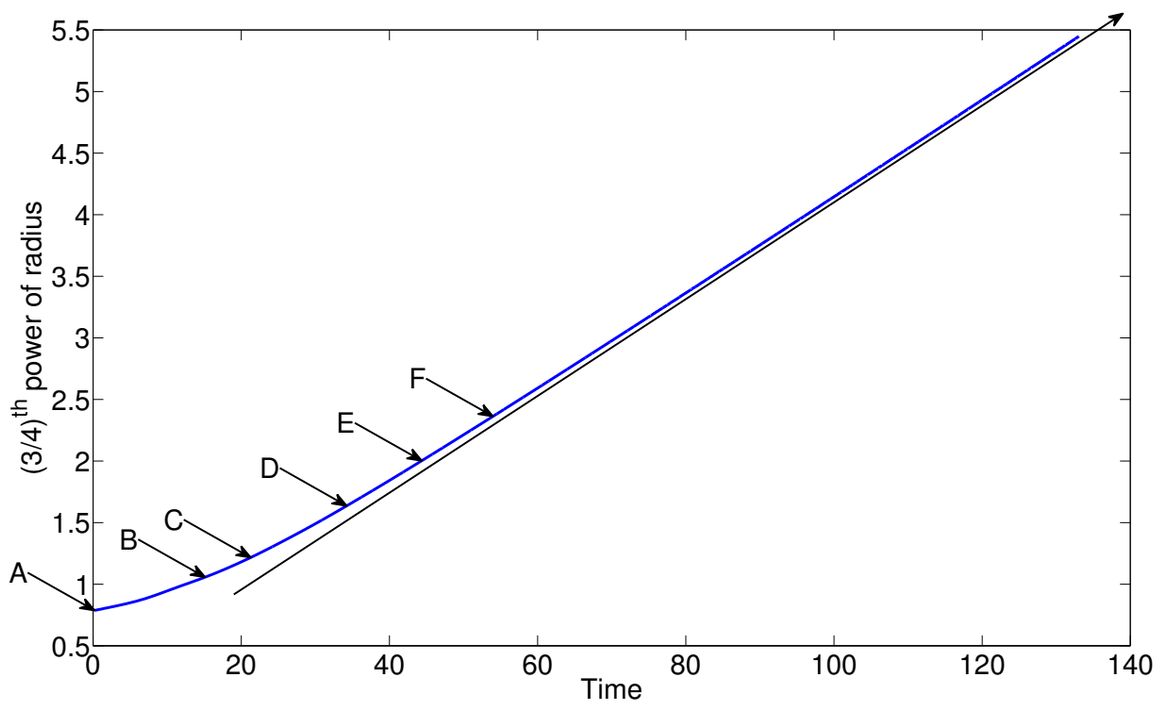
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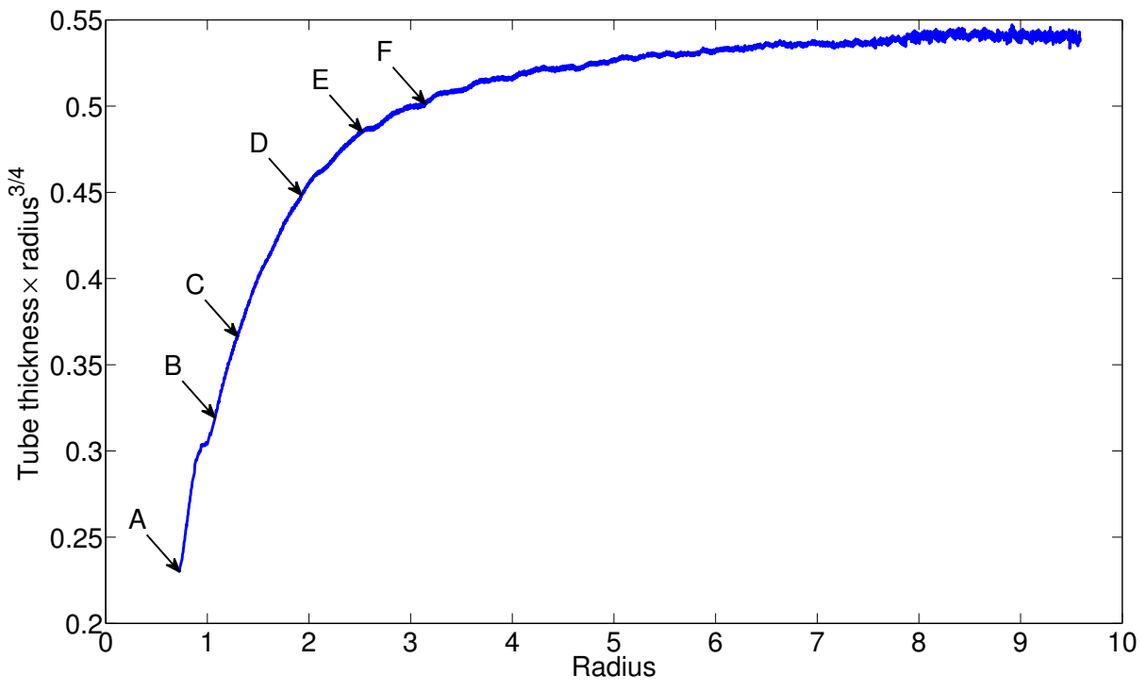


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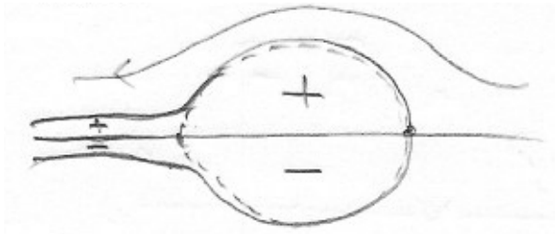
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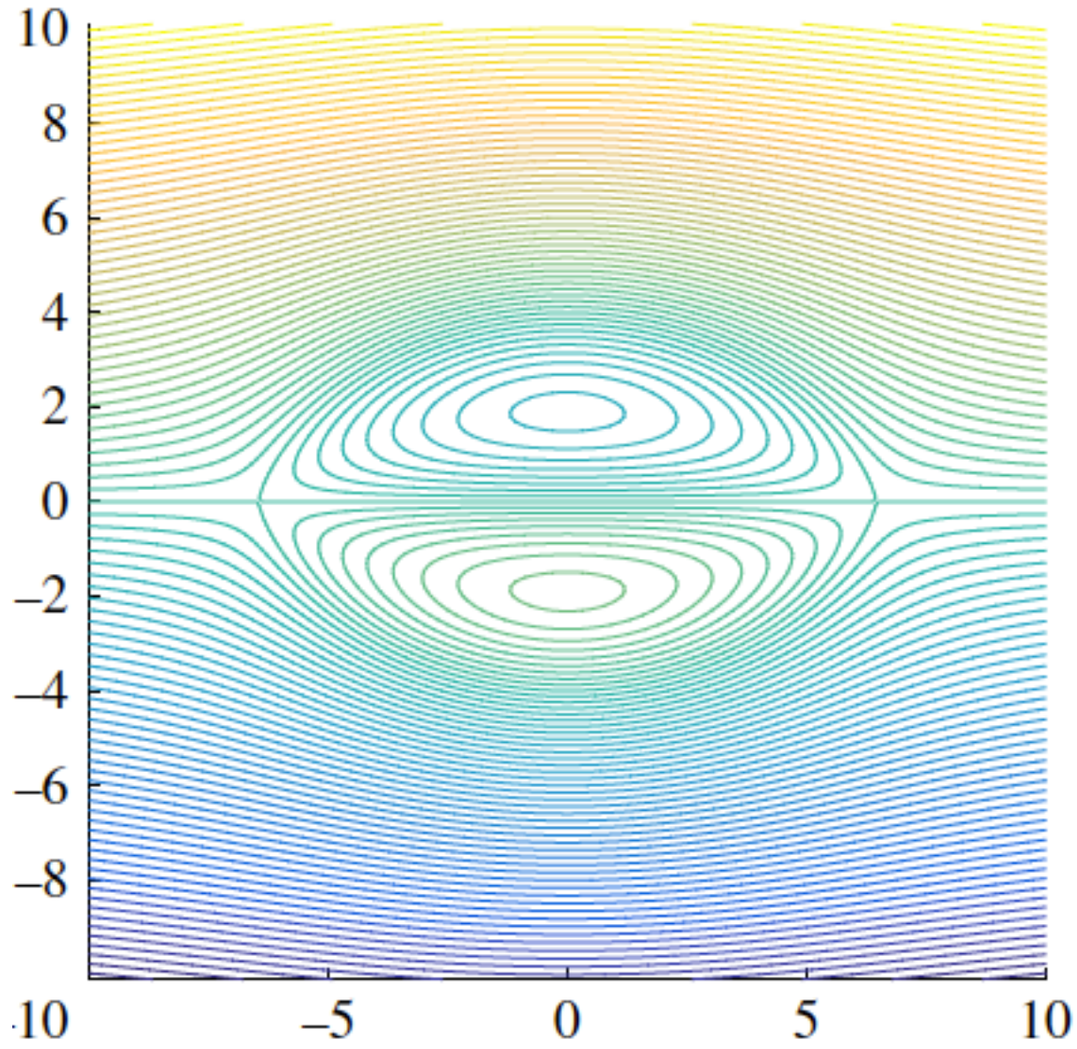




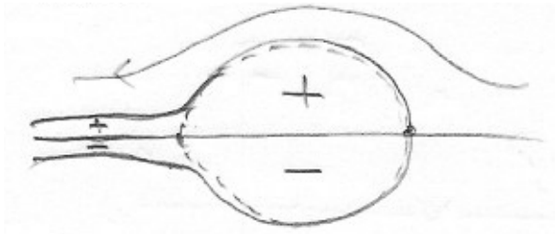
Asymptotic behaviour: Sadovskii vortex



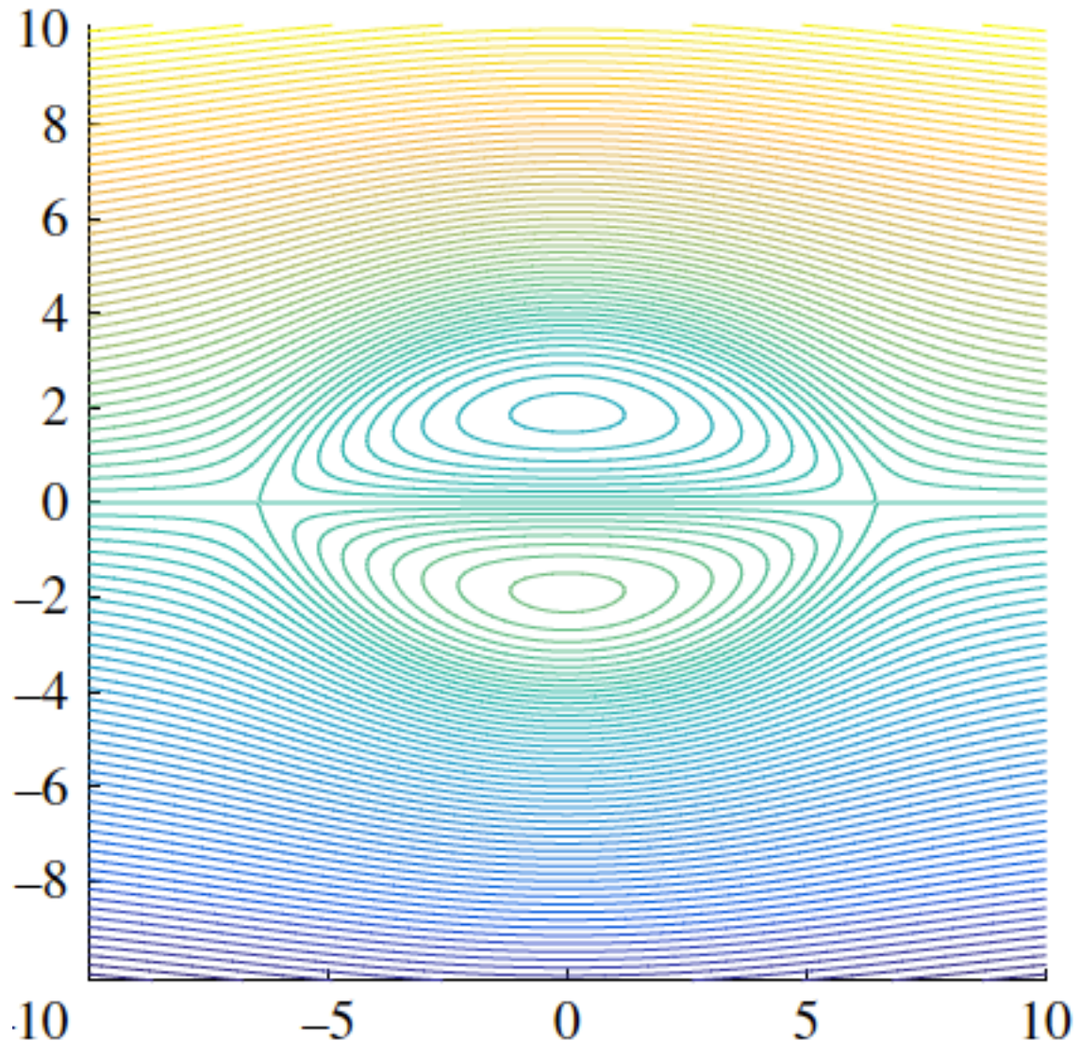
- constant vorticity in two regions
- stream function in co-moving frame



Asymptotic behaviour: Sadovskii vortex



- constant vorticity in two regions
- stream function in co-moving frame



- for speed unity, vorticity +1/-1, area of each lobe is 37.11
- seek long-time solution of approximately this form, to full problem

Asymptotic analysis (sketch)

- Set

$$r\varpi = \omega = \partial u_z/\partial r - \partial u_r/\partial z$$
$$u_r = -r^{-1}\partial\psi/\partial z,\ u_z = r^{-1}\partial\psi/\partial r$$

Asymptotic analysis (sketch)

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- and solve

$$\frac{\partial\varpi}{\partial t} + \frac{1}{r}\frac{\partial(\psi,\varpi)}{\partial(r,z)} = 0, \quad \varpi = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2}$$

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$$\varpi(r,z,t) = \frac{\omega_0}{R_0}\,\tilde{\varpi}(\xi,\eta,\tau), \quad \psi(r,z,t) = \omega_0 R_0^3 f^2 g^2\,\tilde{\psi}(\xi,\eta,\tau)$$

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$$\varpi(r,z,t) = \frac{\omega_0}{R_0}\,\widetilde{\varpi}(\xi,\eta,\tau), \quad \psi(r,z,t) = \omega_0 R_0^3 f^2 g^2\,\widetilde{\psi}(\xi,\eta,\tau)$$

- to obtain, exactly

$$\frac{\partial\varpi}{\partial\tau} - \frac{\dot g}{f}\frac{\partial\varpi}{\partial\xi} - \frac{\dot g}{g}\xi\frac{\partial\varpi}{\partial\xi} - \frac{\dot f}{f}\left(\xi\frac{\partial\varpi}{\partial\xi} + \eta\frac{\partial\varpi}{\partial\eta}\right) + \mathcal{J}(\psi,\varpi) = 0$$

$$\varpi = \frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{1+2fg^{-1}\xi}\frac{\partial^2\psi}{\partial\eta^2},$$

continued…

- we have in mind

$$\frac{\partial \varpi}{\partial \tau} - \frac{\dot{g}}{f}\frac{\partial \varpi}{\partial \xi} - \frac{\dot{g}}{g}\xi\frac{\partial \varpi}{\partial \xi} - \frac{\dot{f}}{f}\left(\xi\frac{\partial \varpi}{\partial \xi} + \eta\frac{\partial \varpi}{\partial \eta}\right) + \mathcal{J}(\psi, \varpi) = 0$$

$$\varpi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{1 + 2fg^{-1}\xi}\frac{\partial^2 \psi}{\partial \eta^2},$$

$$g = R/R_0 \gg 1 \qquad f = a/R_0 \ll 1 \qquad \dot{f}/f \ll 1, \quad \dot{g}/g \ll 1$$

continued…

$$\frac{\partial \varpi}{\partial \tau} - \frac{\dot{g}}{f}\frac{\partial \varpi}{\partial \xi} - \frac{\dot{g}}{g}\xi\frac{\partial \varpi}{\partial \xi} - \frac{\dot{f}}{f}\left(\xi\frac{\partial \varpi}{\partial \xi} + \eta\frac{\partial \varpi}{\partial \eta}\right) + \mathcal{J}(\psi, \varpi) = 0$$

$$\varpi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{1 + 2fg^{-1}\xi}\frac{\partial^2 \psi}{\partial \eta^2},$$

- we have in mind $g = R/R_0 \gg 1 \qquad f = a/R_0 \ll 1 \qquad \dot{f}/f \ll 1, \quad \dot{g}/g \ll 1$
- and set $\varpi = \varpi_0 + \frac{f_0}{g_0}\varpi_1 + \cdots, \quad \psi = \psi_0 + \frac{f_0}{g_0}\psi_1 + \cdots, \quad g = g_0 + g_1 + \cdots, \quad f = f_0 + f_1 +$

continued…

$$\frac{\partial \varpi}{\partial \tau} - \frac{\dot g}{f}\frac{\partial \varpi}{\partial \xi} - \frac{\dot g}{g}\xi\frac{\partial \varpi}{\partial \xi} - \frac{\dot f}{f}\left(\xi\frac{\partial \varpi}{\partial \xi} + \eta\frac{\partial \varpi}{\partial \eta}\right) + \mathcal{J}(\psi, \varpi) = 0$$

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- and set $\varpi = \varpi_0 + \frac{f_0}{g_0}\varpi_1 + \cdots, \quad \psi = \psi_0 + \frac{f_0}{g_0}\psi_1 + \cdots, \quad g = g_0 + g_1 + \cdots, \quad f = f_0 + f_1 +$

- at leading order, have pure 2-d vortex dynamics

$$\frac{\partial \varpi_0}{\partial \tau} - c_0\frac{\partial \varpi_0}{\partial \xi} + \mathcal{J}(\psi_0, \varpi_0) = 0, \qquad \varpi_0 = \frac{\partial^2 \psi_0}{\partial \xi^2} + \frac{\partial^2 \psi_0}{\partial \eta^2} \qquad c_0 = \dot g_0/f_0$$

continued…

$$\frac{\partial \varpi}{\partial \tau} - \frac{\dot{g}}{f}\frac{\partial \varpi}{\partial \xi} - \frac{\dot{g}}{g}\xi\frac{\partial \varpi}{\partial \xi} - \frac{\dot{f}}{f}\left(\xi\frac{\partial \varpi}{\partial \xi} + \eta\frac{\partial \varpi}{\partial \eta}\right) + \mathcal{J}(\psi, \varpi) = 0$$

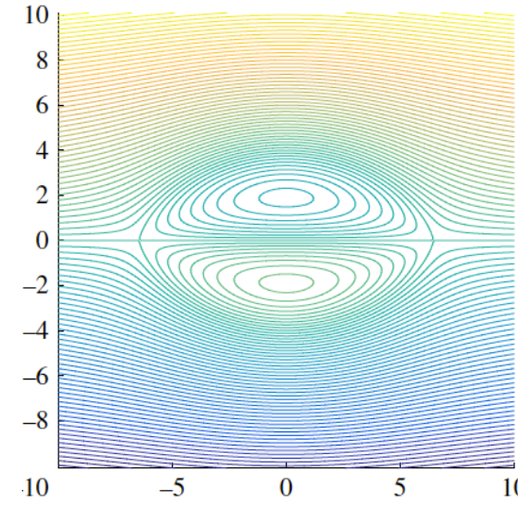
$$\varpi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{1}{1 + 2fg^{-1}\xi}\frac{\partial^2 \psi}{\partial \eta^2},$$

- we have in mind $g = R/R_0 \gg 1 \quad f = a/R_0 \ll 1 \quad \dot{f}/f \ll 1, \quad \dot{g}/g \ll 1$
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- at leading order, have pure 2-d vortex dynamics

$$\frac{\partial \varpi_0}{\partial \tau} - c_0\frac{\partial \varpi_0}{\partial \xi} + \mathcal{J}(\psi_0, \varpi_0) = 0, \qquad \varpi_0 = \frac{\partial^2 \psi_0}{\partial \xi^2} + \frac{\partial^2 \psi_0}{\partial \eta^2}$$

$c_0 = \dot{g}_0/f_0$

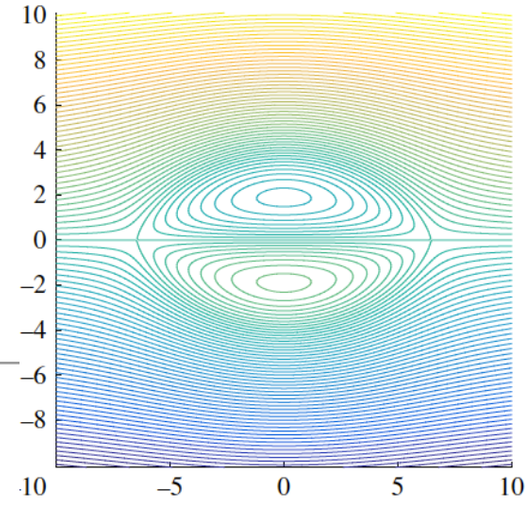
- choose solution as steady Sadovskii vortex $\varpi_0 = \pm 1 \quad c_0 = 1$
- obtain one relationship $c_0 = \dot{g}_0/f_0$ but need another one



continued…

$$c_0 = 1$$

$$\varpi_0 = \pm 1$$



- leading order Sadovskii vortex with $c_0 = \dot{g}_0/f_0$

- next order

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right)$$
$$+ \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

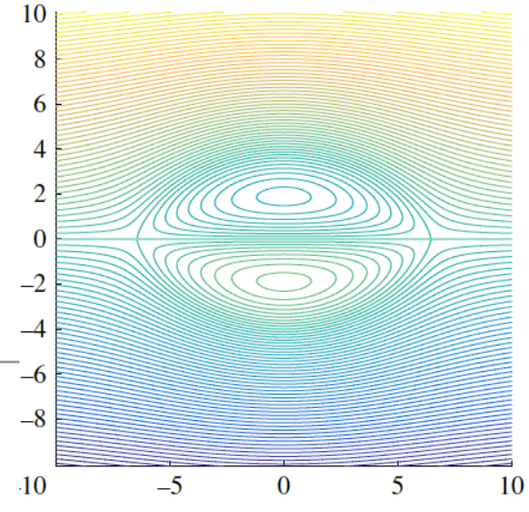
$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

-

continued…

$$c_0 = 1$$

$$\varpi_0 = \pm 1$$



- leading order Sadovskii vortex with $c_0 = \dot{g}_0 / f_0$

- next order

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right)$$
$$+ \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

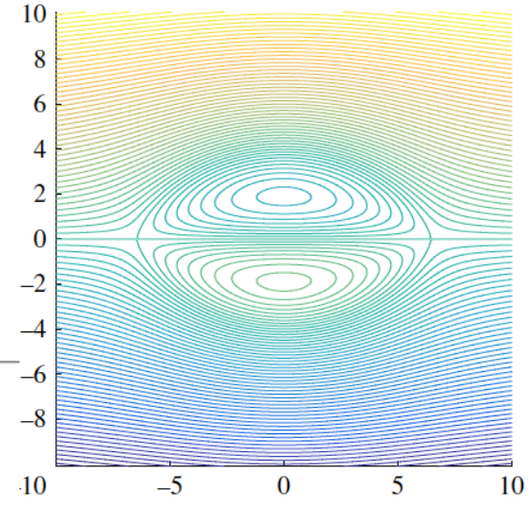
- with

$$c_1 = \frac{g_0}{f_0^2} (\dot{g}_1 - c_0 f_1), \quad p_1 = -\frac{\dot{f}_0 g_0}{c_0 f_0^2}$$

continued…

$$c_0 = 1$$

$$\varpi_0 = \pm 1$$



- leading order Sadovskii vortex with $c_0 = \dot{g}_0 / f_0$

- next order

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right)$$
$$+ \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

- with

$$c_1 = \frac{g_0}{f_0^2} (\dot{g}_1 - c_0 f_1), \quad p_1 = -\frac{\dot{f}_0 g_0}{c_0 f_0^2}$$

- need to fix p_1 to gain a steady propagating state at this order and then

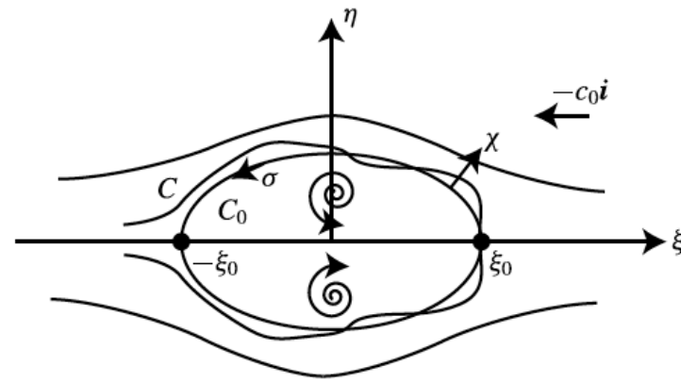
$$f_0 \propto \tau^{p_1/(1+p_1)} \propto (\omega_0 t)^{-1}, \quad g_0 \propto \tau^{1/(1+p_1)} \propto (\omega_0 t)^{1/p_1}, \quad \tau \propto (\omega_0 t)^{1+1/p_1}$$

continued…

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right) + \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

- equation for ϖ_1 in the flow field of ψ_0 with `source terms'



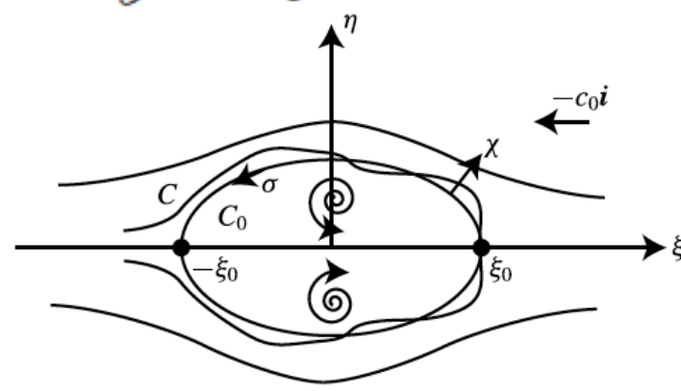
continued…

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right) + \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

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- equation for ϖ_1 in the flow field of ψ_0 with `source terms'

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} + \mathcal{J}(\psi_0, \varpi_1) = c_0 \xi \frac{\partial \varpi_0}{\partial \xi} - \mathcal{J}(\mathcal{G}\varpi_1, \varpi_0) - \mathcal{J}(\psi_1^{sq}, \varpi_0) + c_1 \varpi_0^{trans} - c_0 p_1 \varpi_0^{scale}$$



continued…

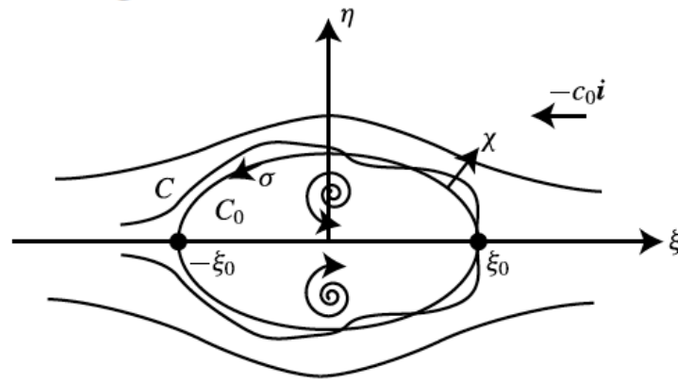
$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right) + \mathscr{J}(\psi_0, \varpi_1) + \mathscr{J}(\psi_1, \varpi_0) = 0,$$

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$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} + \mathscr{J}(\psi_0, \varpi_1) = c_0 \xi \frac{\partial \varpi_0}{\partial \xi} - \mathscr{J}(\mathscr{G}\varpi_1, \varpi_0) - \mathscr{J}(\psi_1^{sq}, \varpi_0) + c_1 \varpi_0^{trans} - c_0 p_1 \varpi_0^{scale}$$

$$\psi_1 = \mathscr{G}\varpi_1 + \psi_1^{sq}, \qquad \psi_1^{sq} \equiv 2\mathscr{G}\left(\xi \frac{\partial^2 \psi_0}{\partial \eta^2} \right)$$



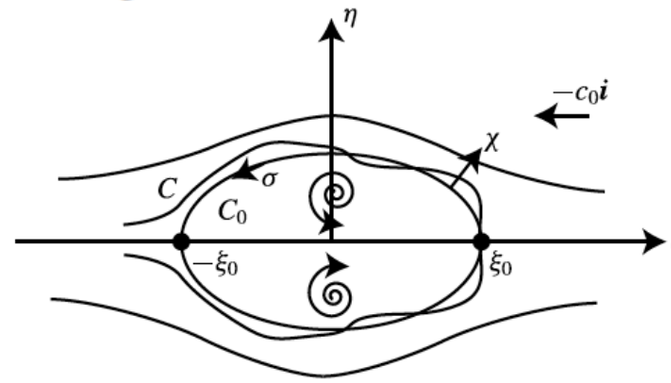
continued…

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right) + \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

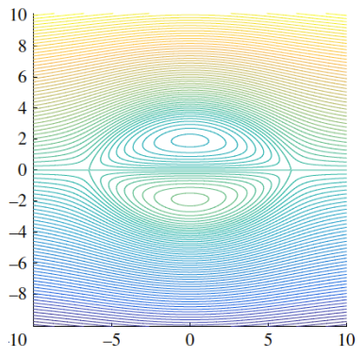
$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

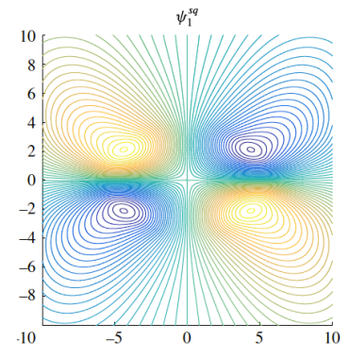
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$$\psi_1 = \mathcal{G}\varpi_1 + \psi_1^{sq}, \qquad \psi_1^{sq} \equiv 2\mathcal{G}\left(\xi \frac{\partial^2 \psi_0}{\partial \eta^2} \right)$$





- note `squeezing' mode ψ^{sq} from residual curvature
- term from loss of volume $c_0 \xi \frac{\partial \varpi_0}{\partial \xi}$ (flow has divergence!)

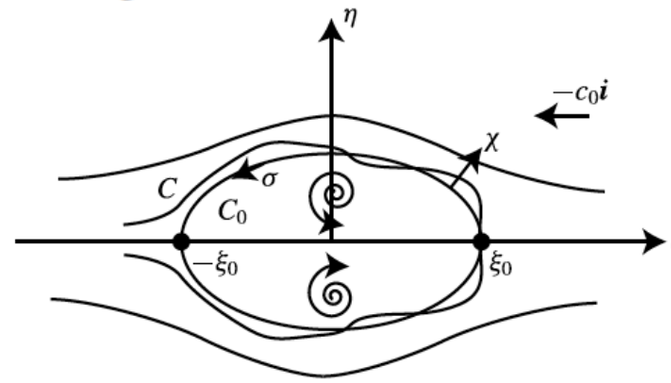
continued…

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} - c_1 \frac{\partial \varpi_0}{\partial \xi} - c_0 \xi \frac{\partial \varpi_0}{\partial \xi} + c_0 p_1 \left(\xi \frac{\partial \varpi_0}{\partial \xi} + \eta \frac{\partial \varpi_0}{\partial \eta} \right) + \mathcal{J}(\psi_0, \varpi_1) + \mathcal{J}(\psi_1, \varpi_0) = 0,$$

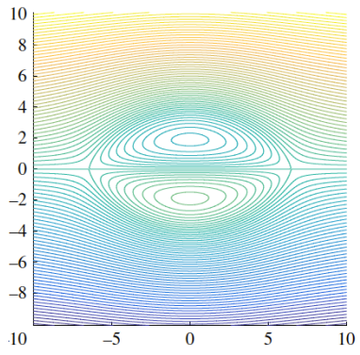
$$\varpi_1 = \frac{\partial^2 \psi_1}{\partial \xi^2} + \frac{\partial^2 \psi_1}{\partial \eta^2} - 2\xi \frac{\partial^2 \psi_0}{\partial \eta^2}.$$

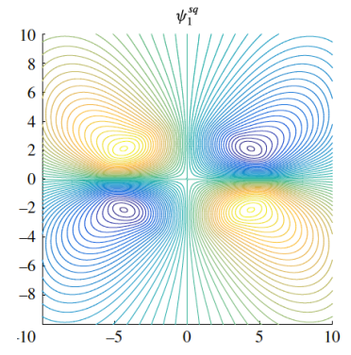
- equation for ϖ_1 in the flow field of ψ_0 with `source terms'

$$\frac{\partial \varpi_1}{\partial \tau} - c_0 \frac{\partial \varpi_1}{\partial \xi} + \mathcal{J}(\psi_0, \varpi_1) = c_0 \xi \frac{\partial \varpi_0}{\partial \xi} - \mathcal{J}(\mathcal{G}\varpi_1, \varpi_0) - \mathcal{J}(\psi_1^{sq}, \varpi_0) + c_1 \varpi_0^{trans} - c_0 p_1 \varpi_0^{scale}$$



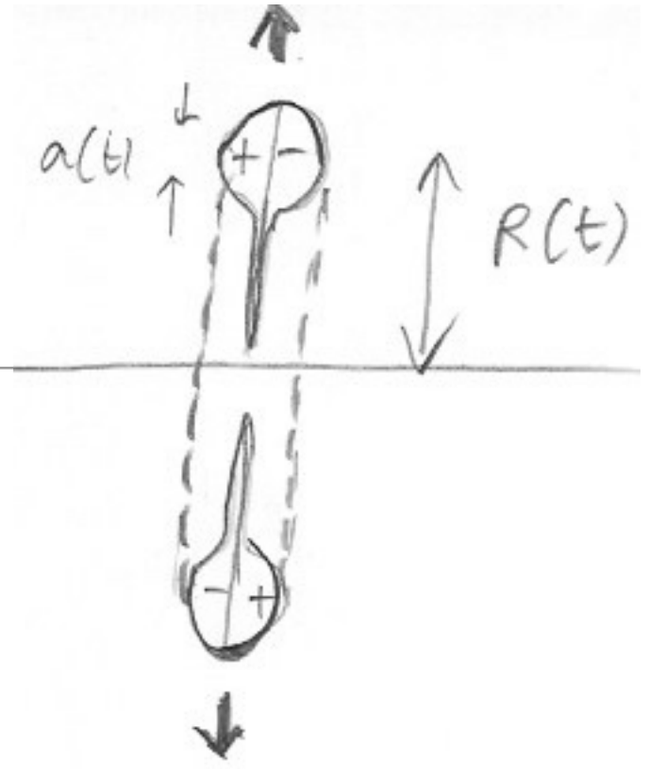
$$\psi_1 = \mathcal{G}\varpi_1 + \psi_1^{sq}, \qquad \psi_1^{sq} \equiv 2\mathcal{G}\left(\xi \frac{\partial^2 \psi_0}{\partial \eta^2} \right)$$



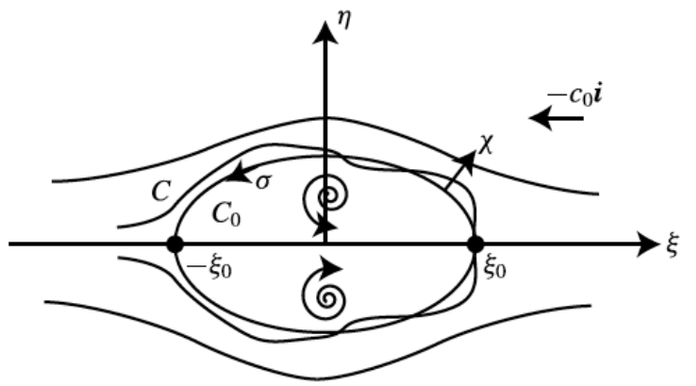


- note `squeezing' mode ψ^{sq} from residual curvature
- term from loss of volume $c_0 \xi \frac{\partial \varpi_0}{\partial \xi}$ (flow has divergence!)
- translation mode, scaling mode

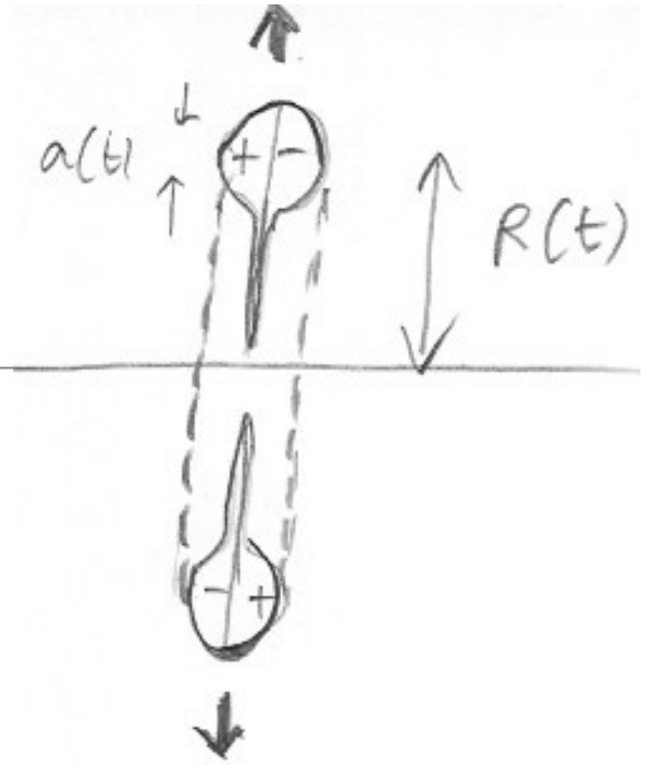
more analysis + numerical solution



- value $p_1 \simeq 0.74$
- energy conservation (previous analysis) gives $p_1 = 3/4$
- recall $a(t) = R_0 f(\tau)$ $R(t) = R_0 g(\tau)$



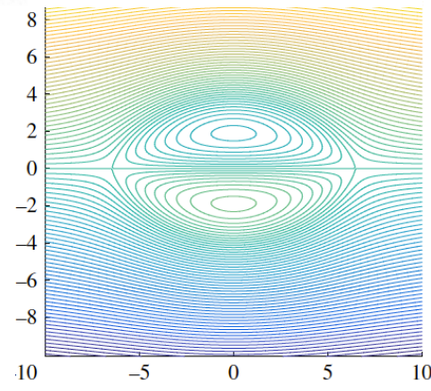
more analysis + numerical solution



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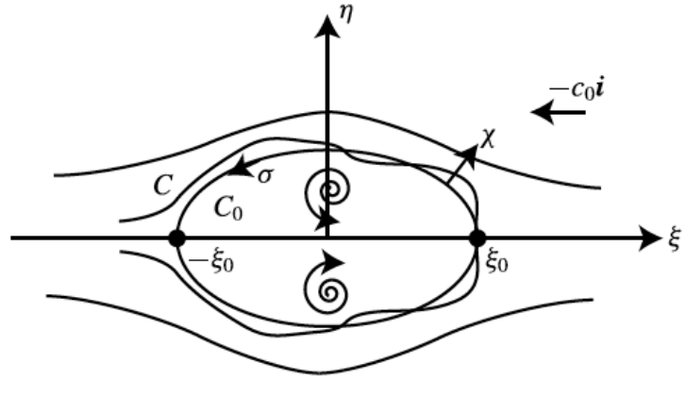
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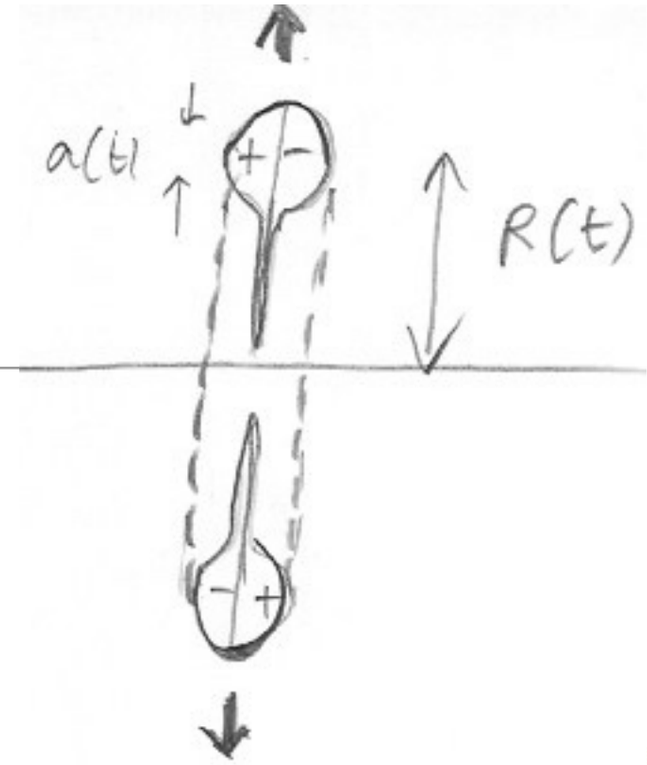


- at leading order $c_0 = \dot{g}_0 / f_0$ and fix Sadovskii vortex with $c_0 = 1$

- at next order we needed a steady propagating vortex with $p_1 = -\dfrac{\dot{f}_0 g_0}{c_0 f_0^2}$



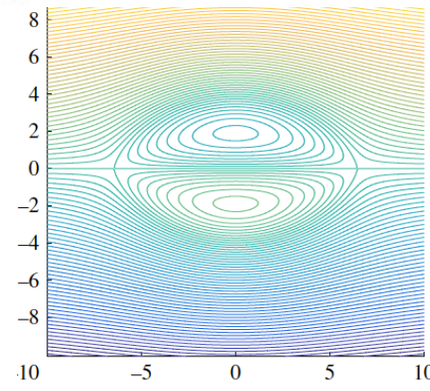
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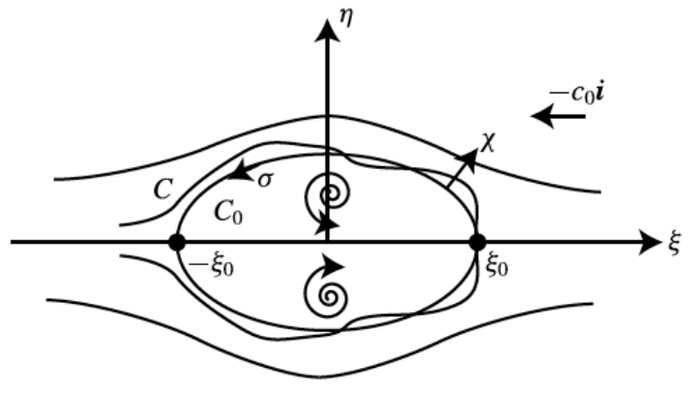


- at leading order $c_0 = \dot{g}_0/f_0$ and fix Sadovskii vortex with $c_0 = 1$

- at next order we needed a steady propagating vortex with $p_1 = -\dfrac{\dot{f}_0 g_0}{c_0 f_0^2}$

- fix $p_1 = 3/4$ and so complete problem

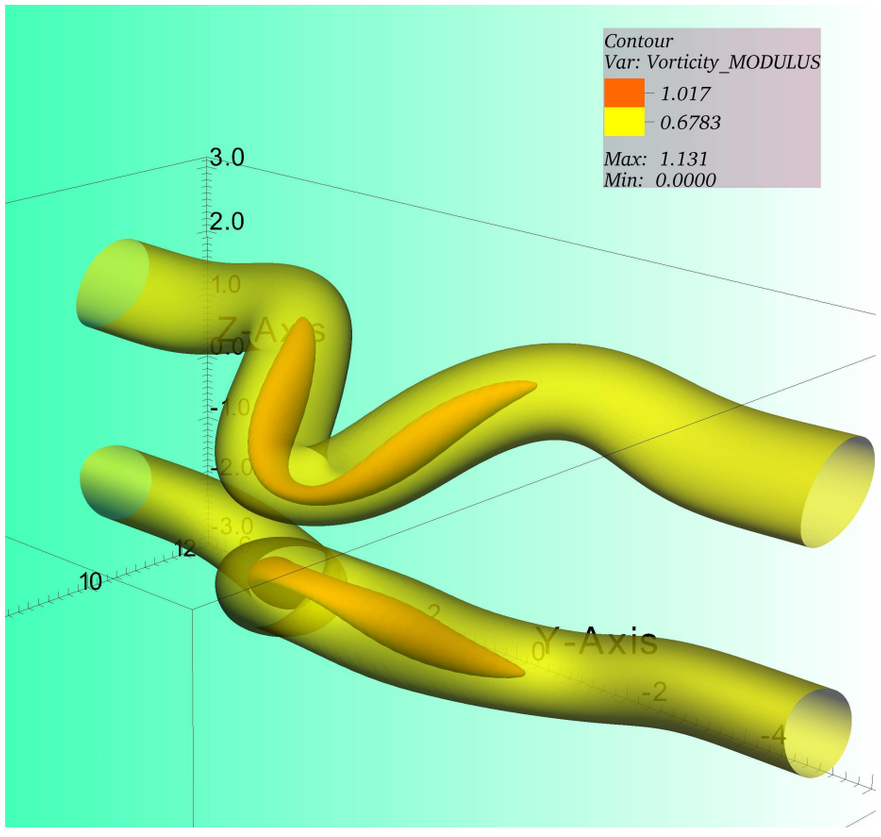
$$f_0 \propto \tau^{p_1/(1+p_1)} \propto (\omega_0 t)^{-1}, \qquad g_0 \propto \tau^{1/(1+p_1)} \propto (\omega_0 t)^{1/p_1}, \qquad \tau \propto (\omega_0 t)^{1+1/p_1}$$

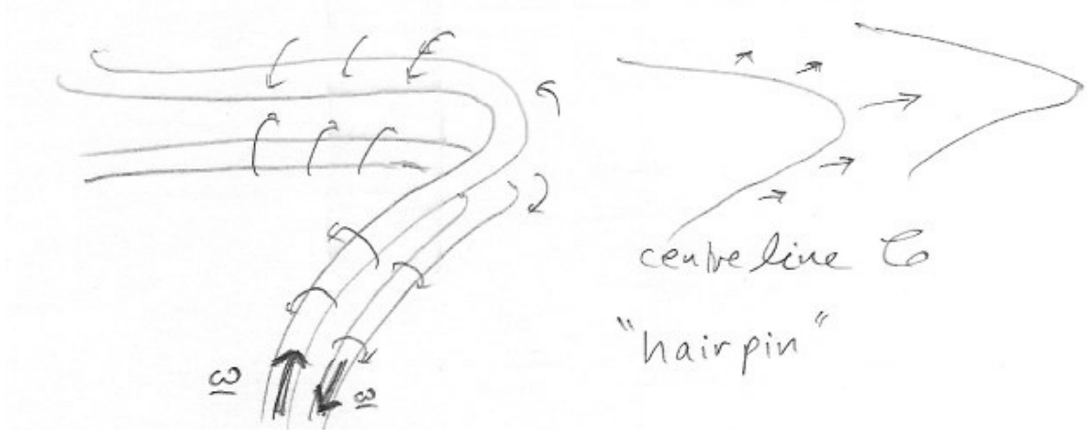


Anti-parallel vortices

- Bustamante & Kerr, 3D Euler about a 2D symmetry plane, *Physica D*, **237**, 2008.

- Euler equations (zero viscosity):

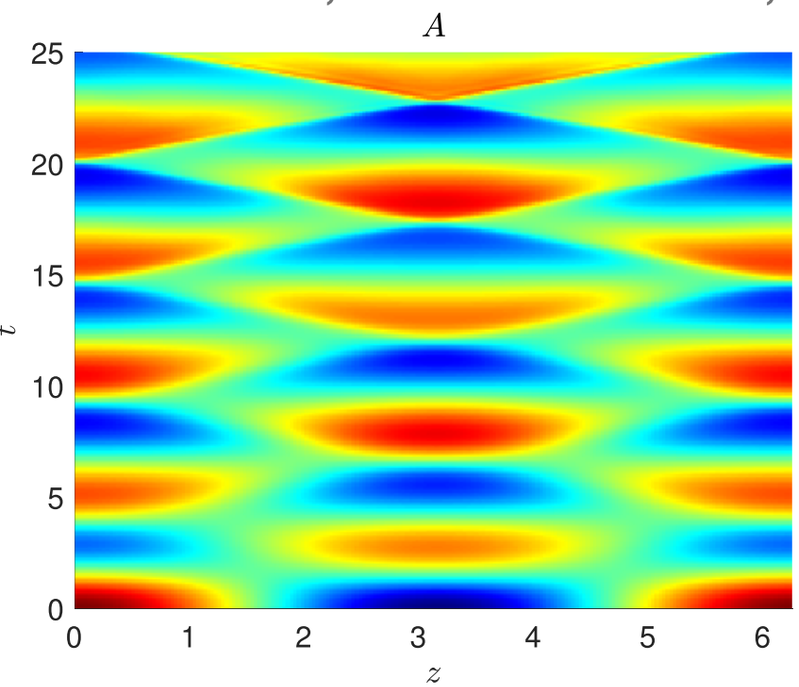




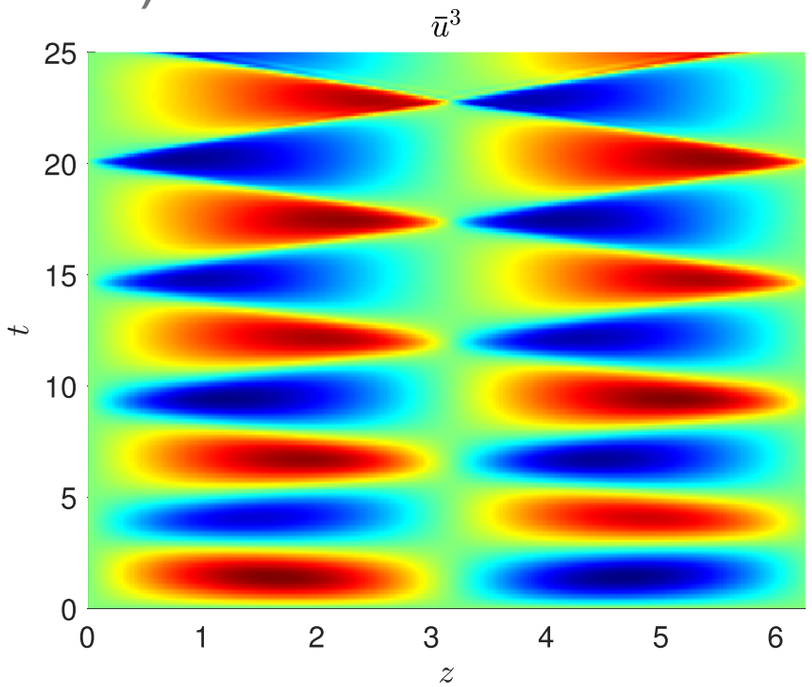
Axial flow and area waves

- Axisymmetric geometry without swirl avoids any axial flow

- can have *area waves* on vortices (Lundgren & Ashurst 1989, Leonard 1994, C & G 2021)



(b)



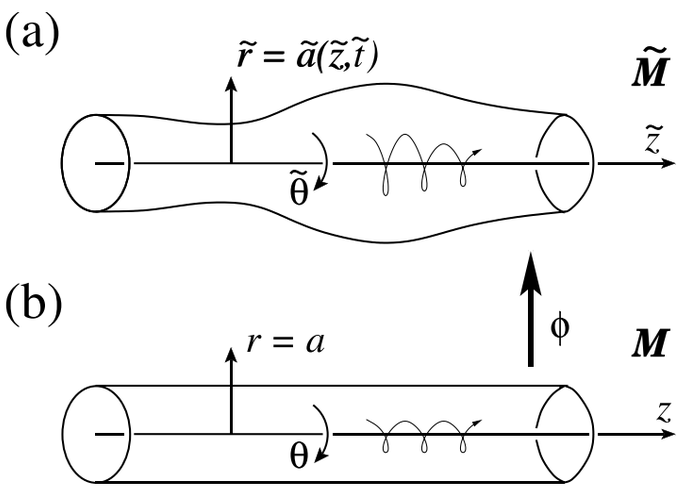
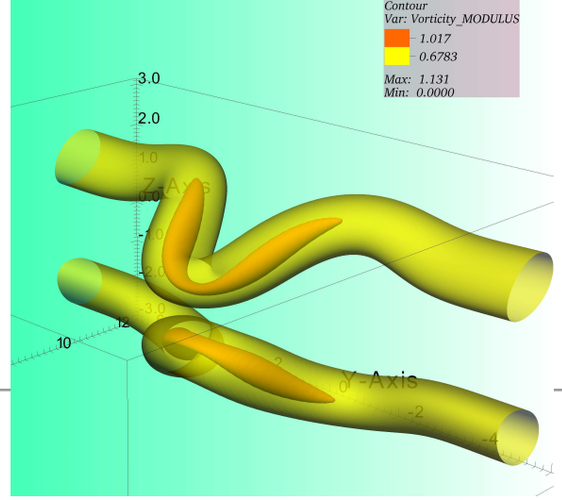
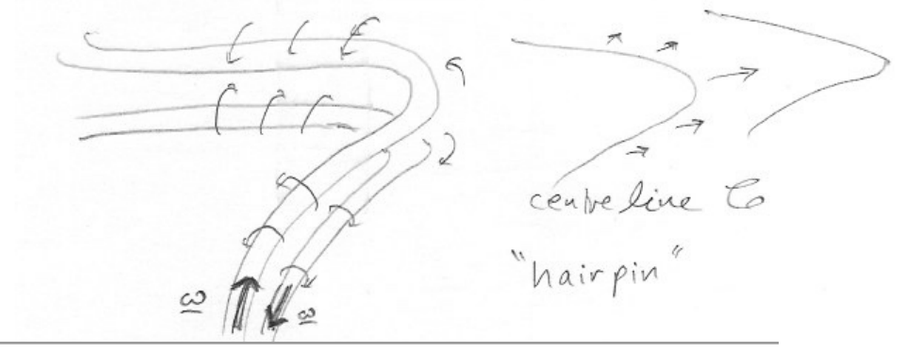


Figure 2. Slender vortex simulations for the initial $A(z, 0)$ in (5.16) and $\bar{u}^3(z, 0) = W_0 = 0$: (a) A (from 0.9π to 1.1π) and (b) $\bar{u}^3$ (from -0.87 to 0.87) are depicted on a colour

- effect suppresses vortex thinning and singularity formation

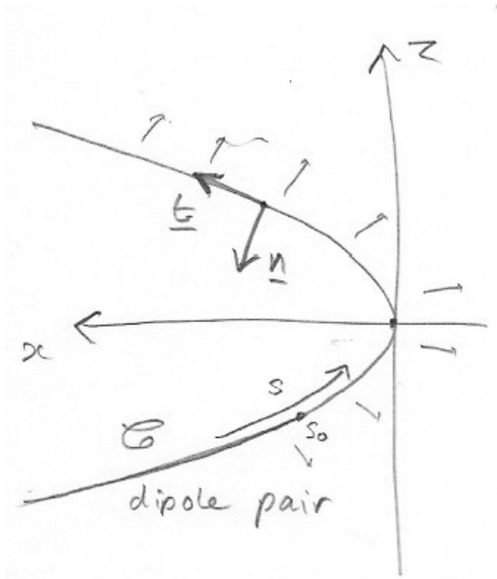
Hairpin geometry





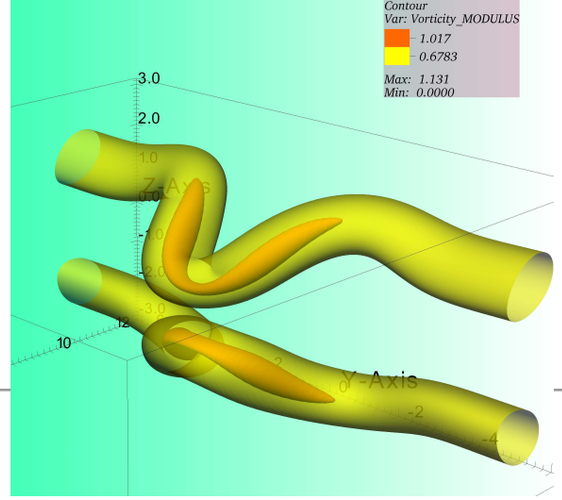
- equation of motion for centreline curve $\mathscr{C}(t)$

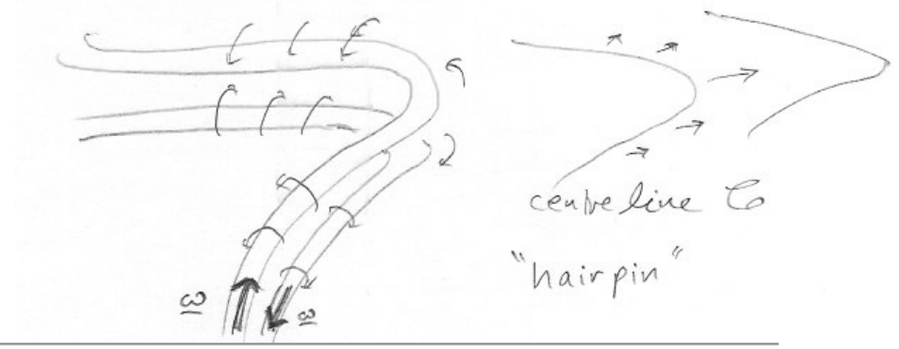
$$\left.\frac{\partial \mathbf{x}}{\partial t}\right|_{s_0} = U\mathbf{n} + W\mathbf{t}, \quad U < 0$$



-

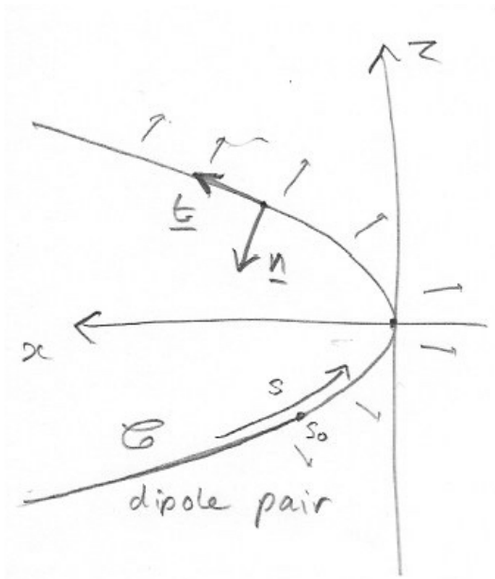
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- Lagrangian point s_0 , Jacobian $J = \partial s / \partial s_0$

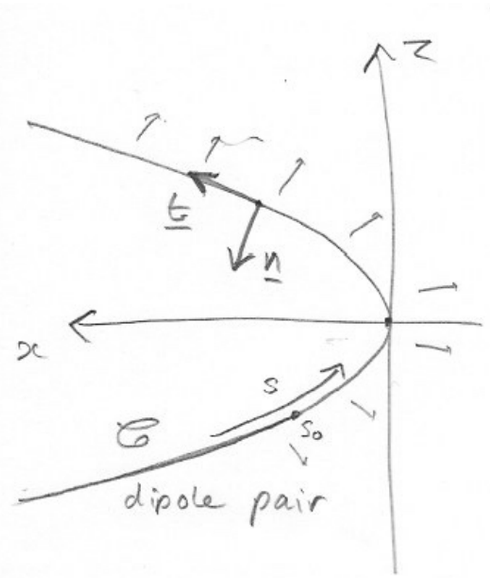
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- equation for evolving curve given U, W

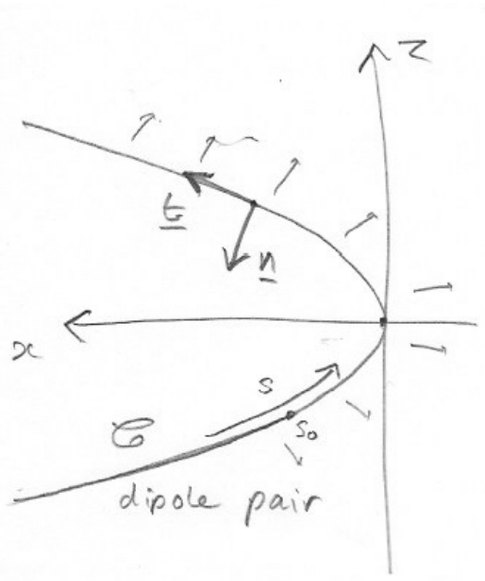
Evolution without axial flow

- set zero axial flow $W = 0$
- need to define U

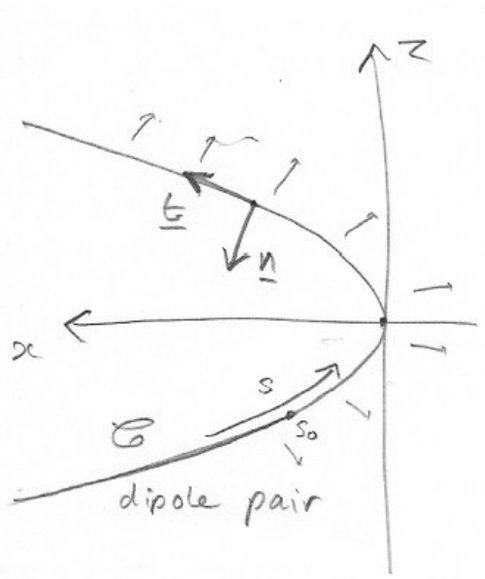


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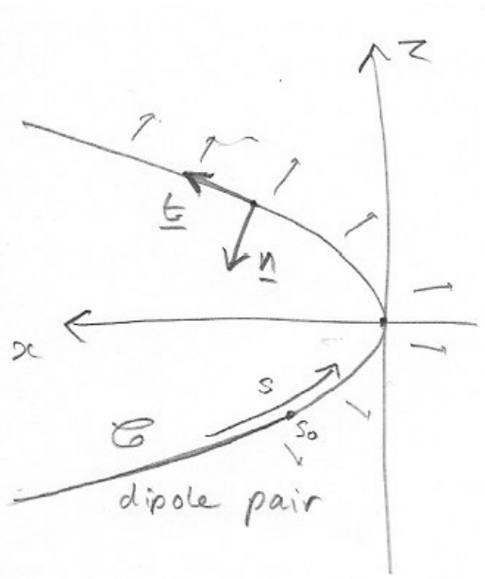


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$$J \sim U^4$$

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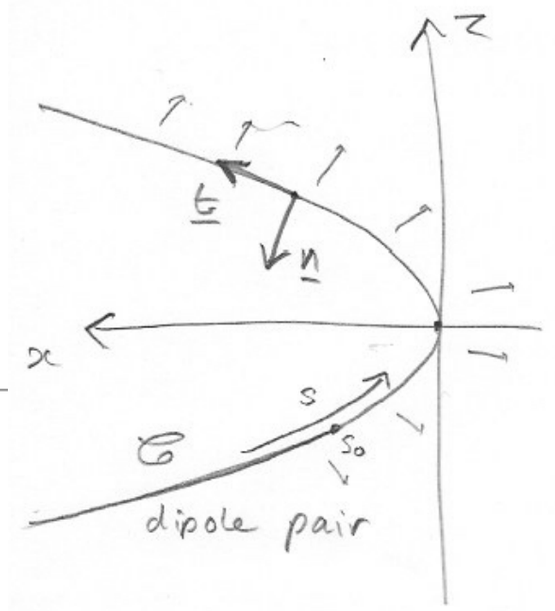
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- and so set $J = (U/U_0)^4$
- reference velocity U_0

Similarity solutions (Childress)



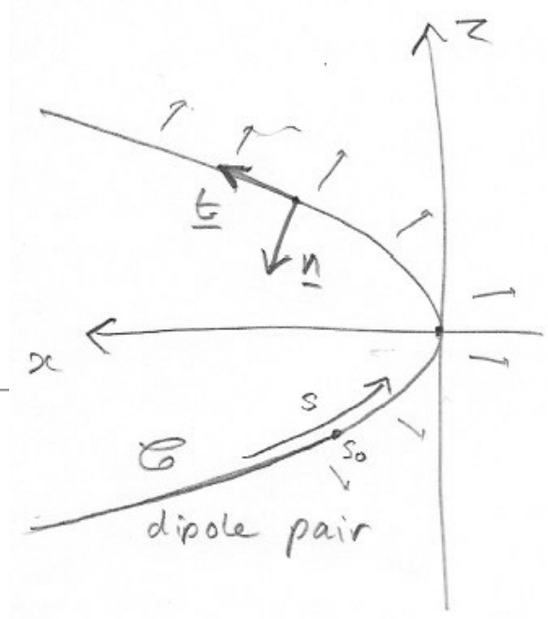
- use s_0 as Lagrangian parameter
- and set $J = (\tau_0/\tau)^{4\gamma}[g(\sigma)]^4 \qquad U = (\tau_0/\tau)^{\gamma} g(\sigma)$

$$\tau = -t$$

$$\sigma = \frac{s_0 \tau_0^{3\gamma}}{U_0 \tau^{3\gamma+1}}$$

- solutions singular as $t \to 0^- \qquad \tau \to 0^+$

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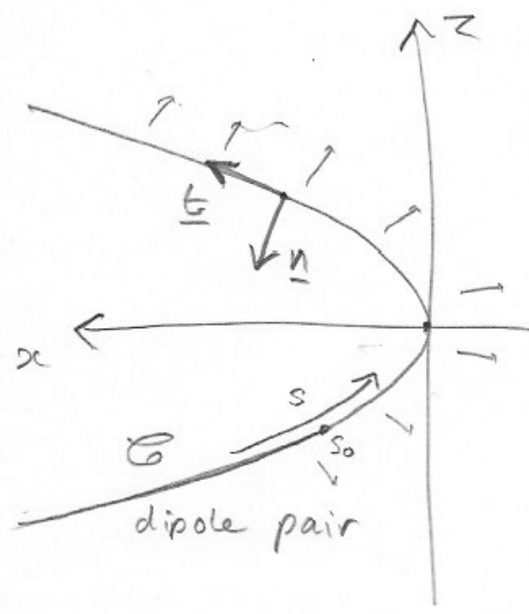
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- obtain similarity ODE $\gamma(3\gamma + 1)\sigma g^3 + (3\gamma + 1)^2 \sigma^2 g^2 g_\sigma + \dfrac{g_\sigma}{4g^4} = 0.$

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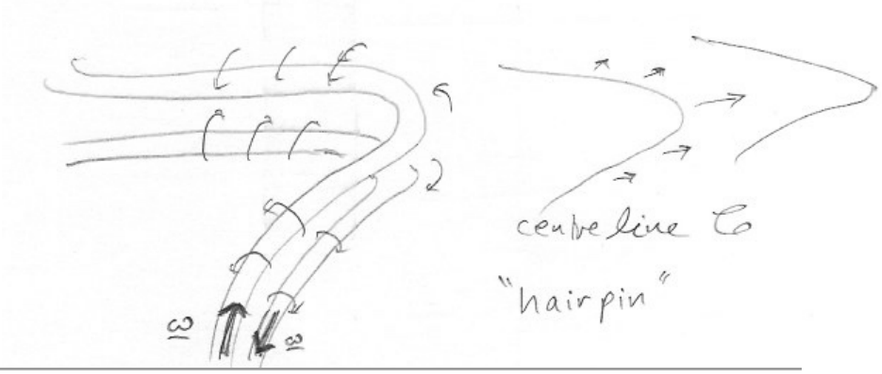
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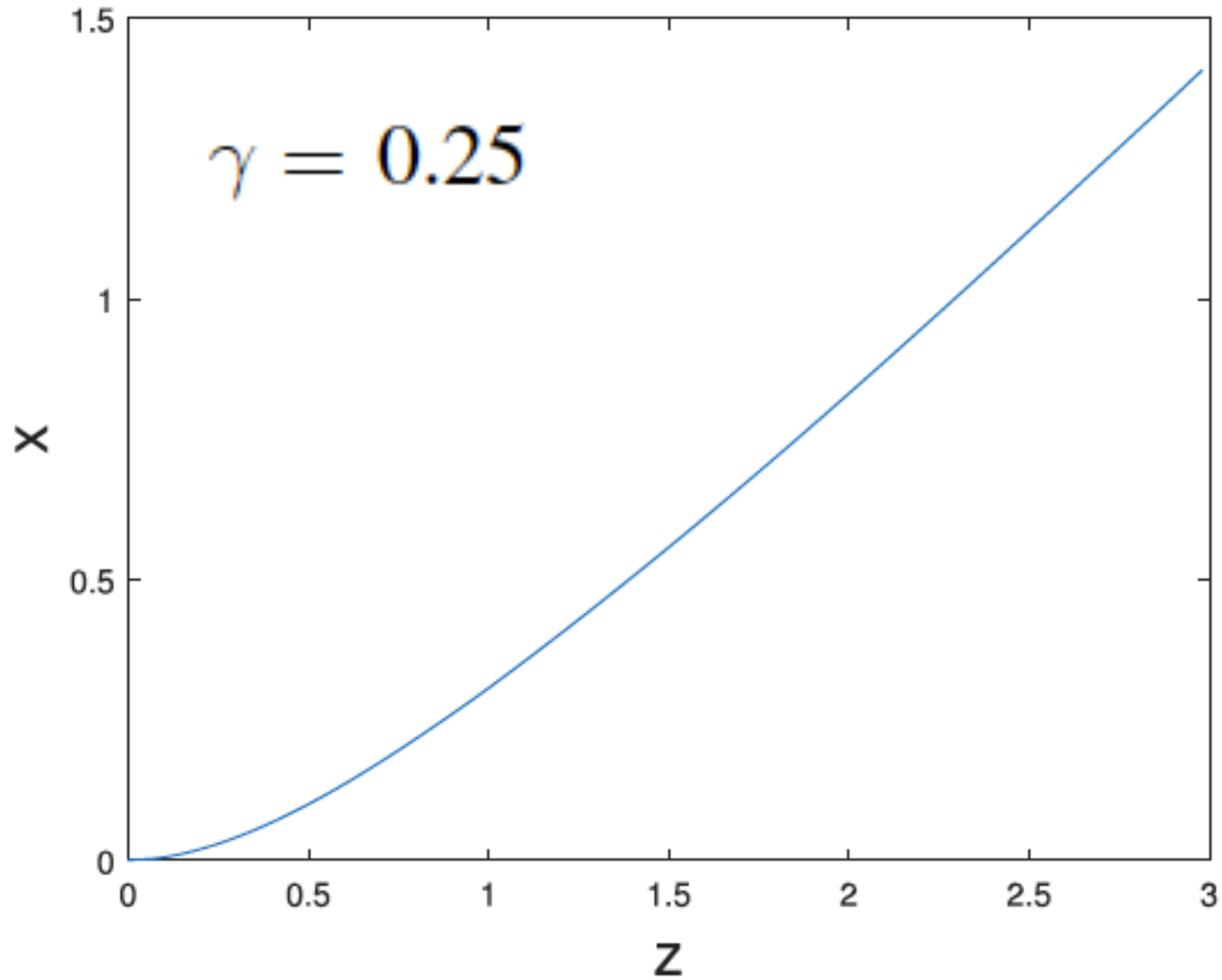
- can solve and reconstruct curve via $\kappa = -\dfrac{J_t}{JU}$

$$\kappa = \frac{\tau^{\gamma-1}}{\tau_0^{\gamma} U_0} \frac{4\gamma g + 4(3\gamma + 1)\sigma g_\sigma}{g^2}$$

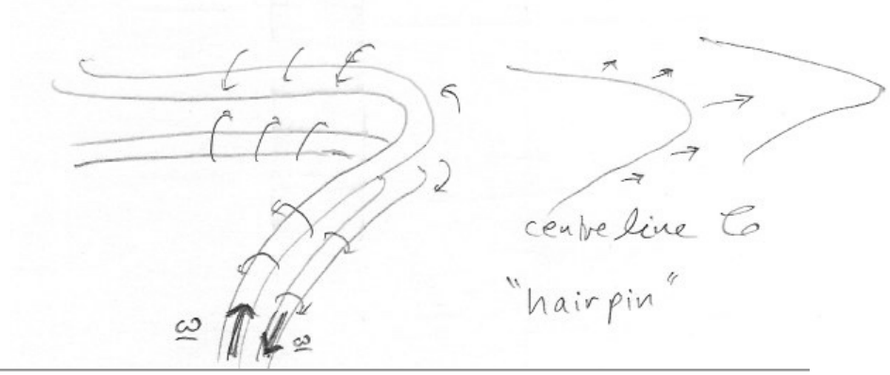
Hairpin similarity solution



- solving gives the shape



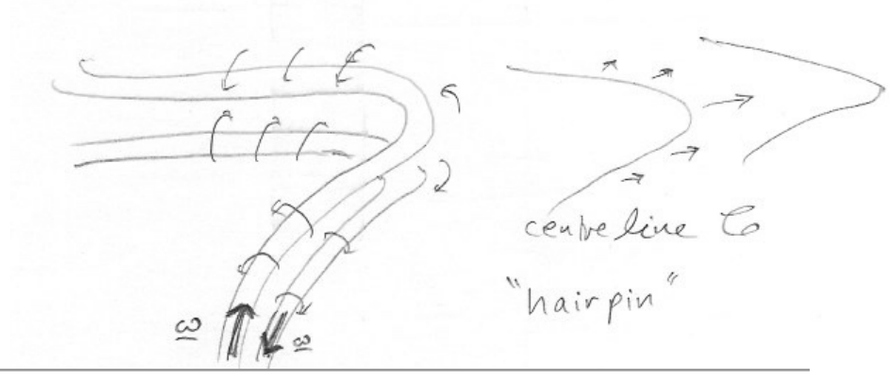
Behaviour



- as singularity time is approached $t \to 0^-$ $\tau \to 0^+$

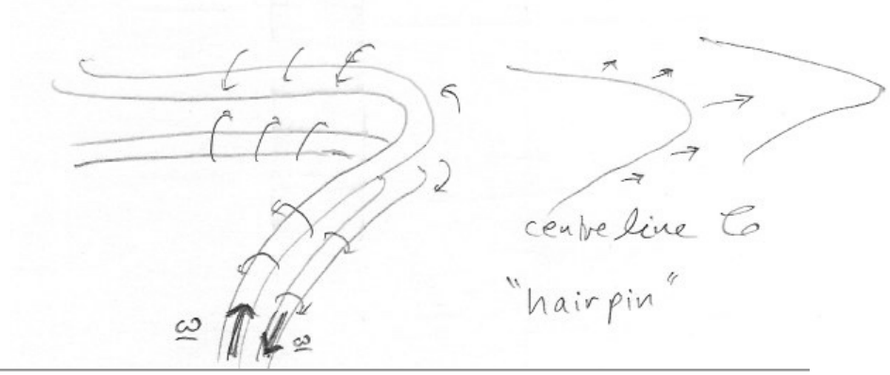
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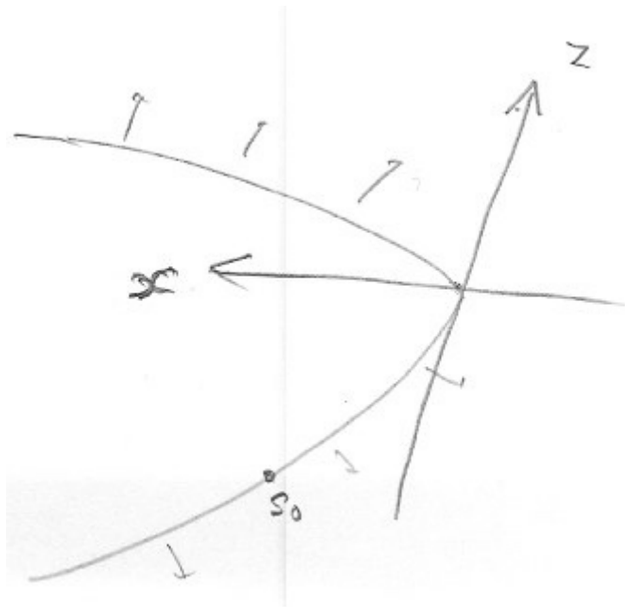


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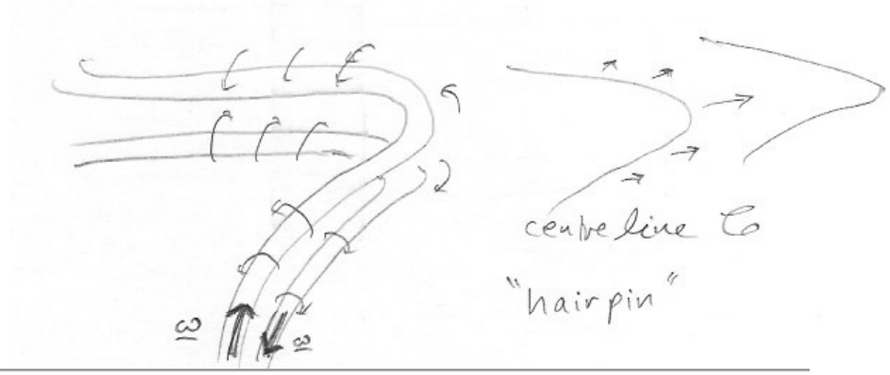


- as singularity time is approached $t \to 0^-$ $\tau \to 0^+$
- take $1/4 \leqslant \gamma < 1$, then each segment of hairpin has finite stretching
- infinite stretching $J = \infty$ occurs at a single point
- cross section area goes to zero there



$$\int_0^{s_0} J \, \mathrm{d}s_0 = U_0 \tau_0^{\gamma} \tau^{1-\gamma} \int_0^{\sigma} g^4 \mathrm{d}\sigma$$

Fitting the vortex to the hairpin

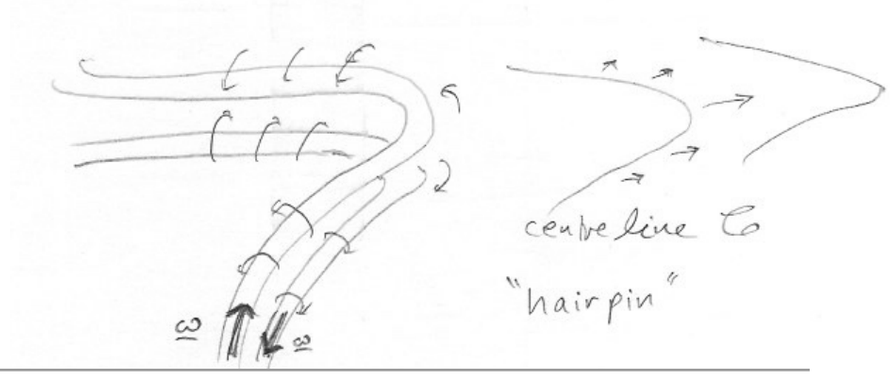


- need to now incorporate the vortex dipole along the curve $\mathscr{C}(t)$

- and estimate velocities U , W in $\partial \mathbf{x}/\partial t = U\mathbf{n} + W\mathbf{t}$

-

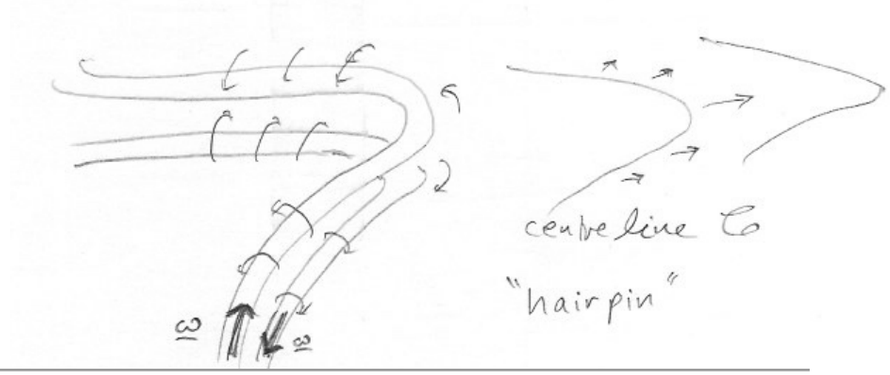
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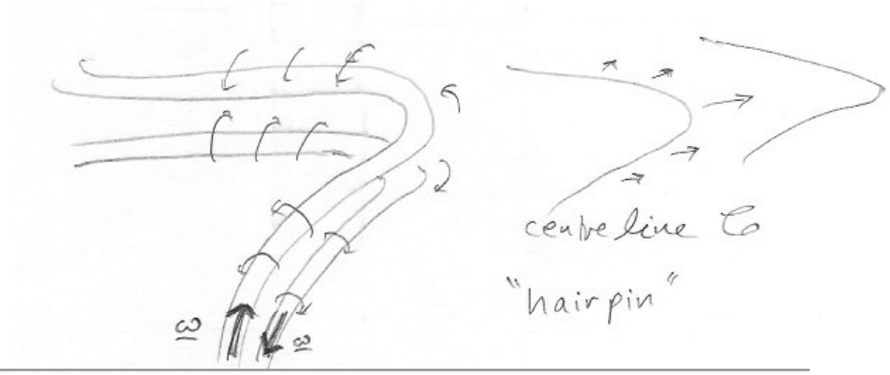
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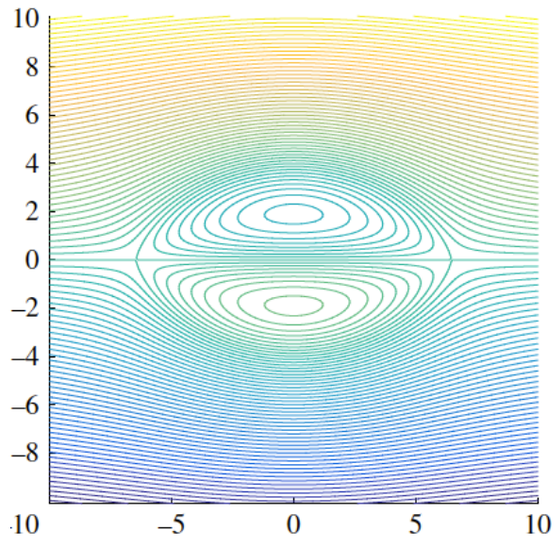
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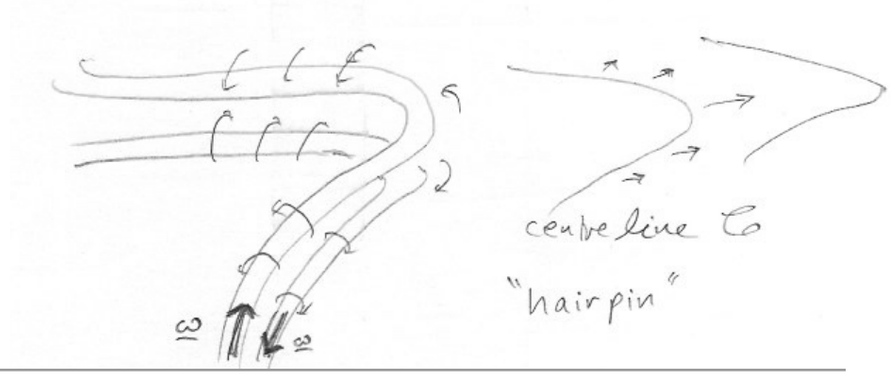


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- this is asymptotic in time provided provided $1/4 < \gamma \leqslant 1/2$
- and remains so if initially it is for $\gamma = 1/4$
- picture holds for $W = 0$ based on Sadovskii vortex



Axial flow…. non-zero W



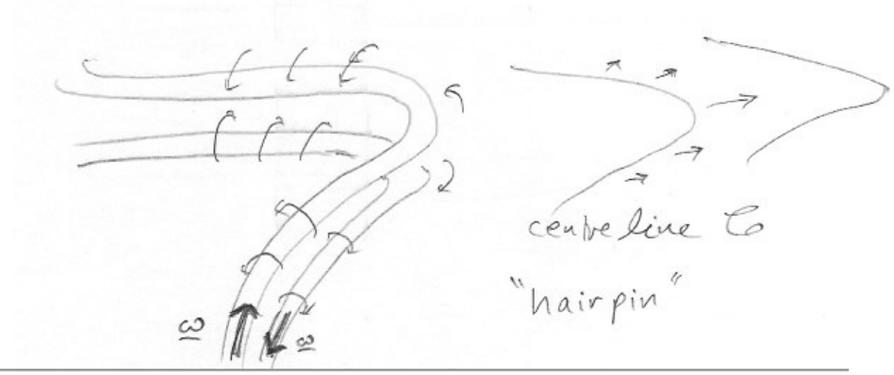
$$\partial \mathbf{x}/\partial t = U\mathbf{n} + W\mathbf{t}$$

- immediately gets very complicated…
- equations for centre line

$$\left.\frac{\partial J}{\partial t}\right|_{s_0} = \left(\frac{\partial W}{\partial s} - U\kappa\right)J,$$

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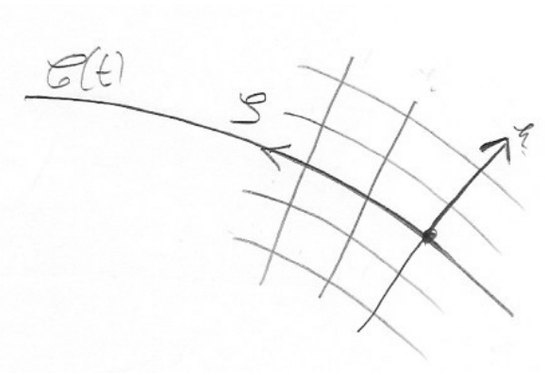
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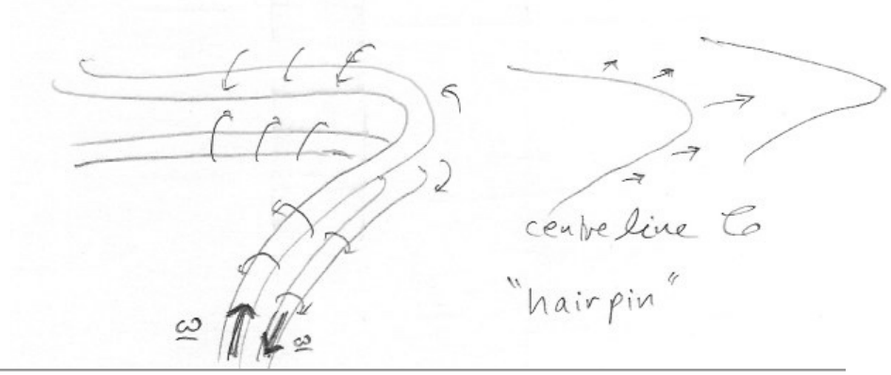
$$\left.\frac{\partial \kappa}{\partial t}\right|_{s_0} - W\frac{\partial \kappa}{\partial s} - U\kappa^2 - \frac{\partial^2 U}{\partial s^2} = 0.$$

- triad $(\mathbf{n}, \mathbf{b}, \mathbf{t})$,, coordinates (ξ, η, ζ), metric

$$ds^2 = d\xi^2 + d\eta^2 + h^2 d\zeta^2$$
$$h = 1 - \xi\kappa$$



Axial flow…. non-zero W



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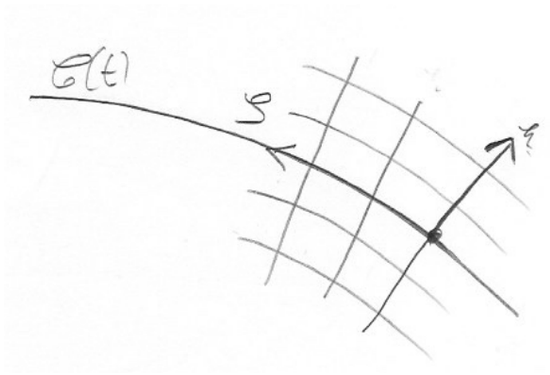
$$\mathrm{d}s^2 = \mathrm{d}\xi^2 + \mathrm{d}\eta^2 + h^2\mathrm{d}\zeta^2$$
$$h = 1 - \xi\kappa$$

- Euler and continuity equations

$$\mathscr{D}u + 2h^{-1}(W + w)\frac{\partial U}{\partial \zeta} + h^{-1}(W + w)^2\kappa + \frac{\partial p}{\partial \xi} + \frac{\partial U}{\partial t} = 0,$$

$$\mathscr{D}v + \frac{\partial p}{\partial \eta} = 0,$$

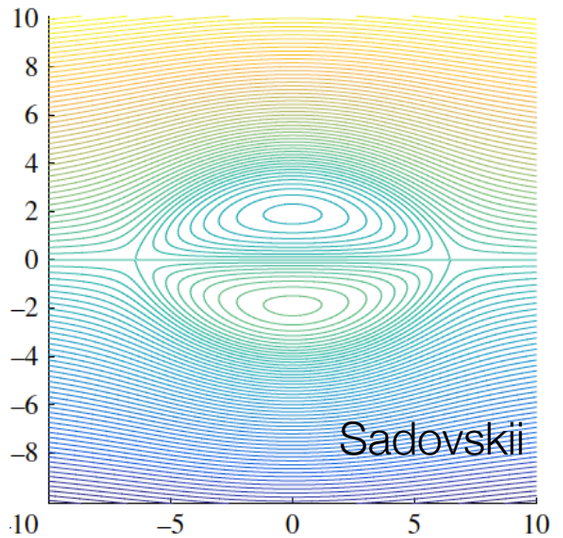
$$\mathscr{D}w + h^{-1}w\frac{\partial W}{\partial \zeta} - h^{-1}u\frac{\partial U}{\partial \zeta} - h^{-1}(W + w)(U + u)\kappa + h^{-1}\frac{\partial (p - \frac{1}{2}U^2)}{\partial \zeta} + \frac{\partial W}{\partial t} + h^{-1}W\frac{\partial W}{\partial \zeta} = 0.$$



$$\frac{\partial h(U + u)}{\partial \xi} + \frac{\partial hv}{\partial \eta} + \frac{\partial (W + w)}{\partial \zeta} = 0$$

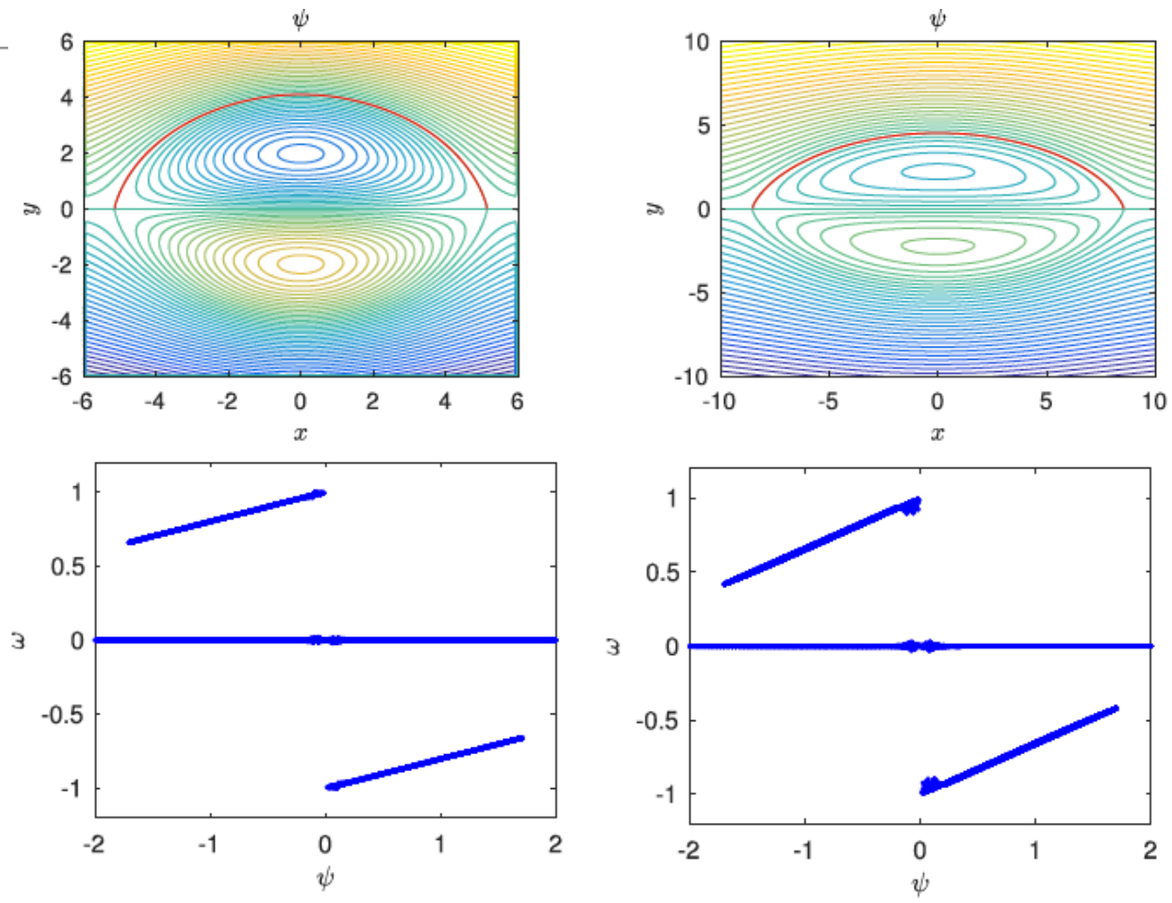
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- picture more complicated, much more
- lose Sadovskii vortex,
- need a different vortex cross-section



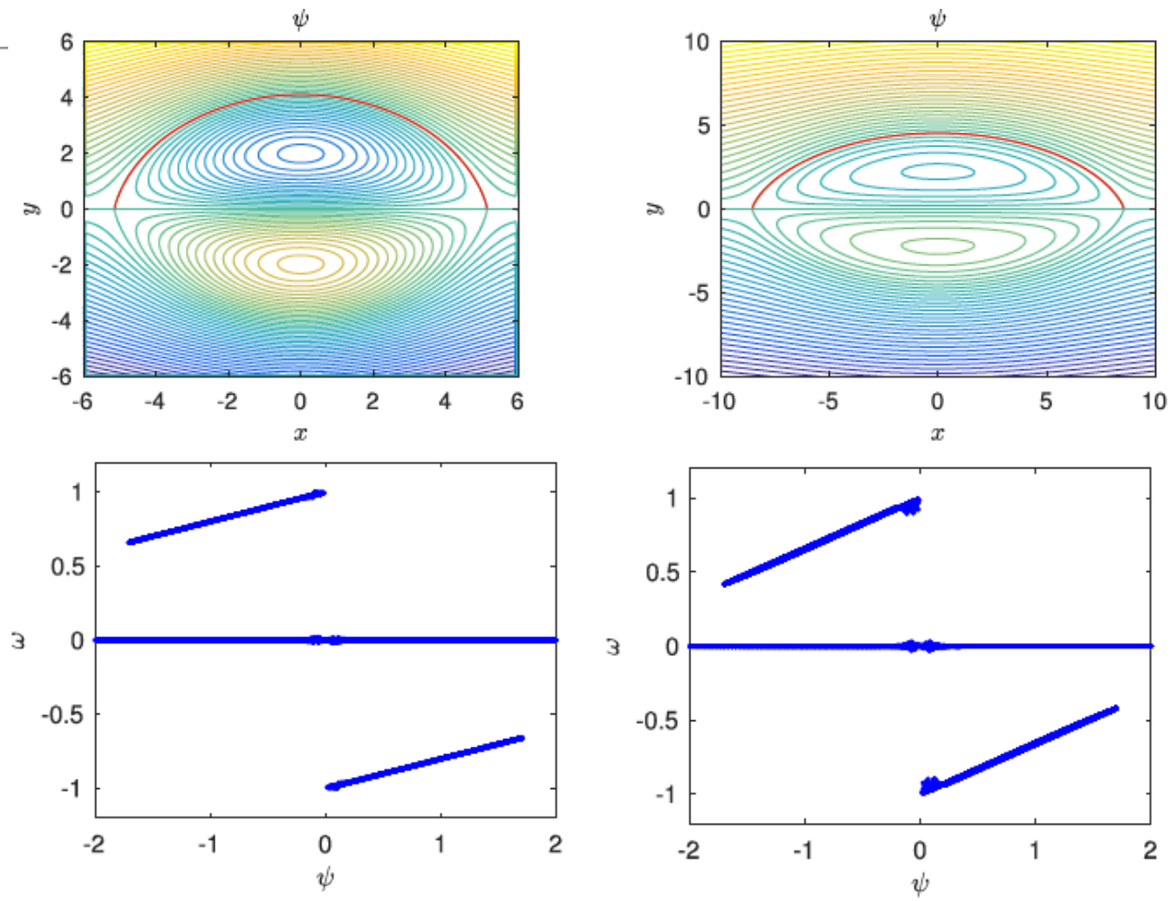
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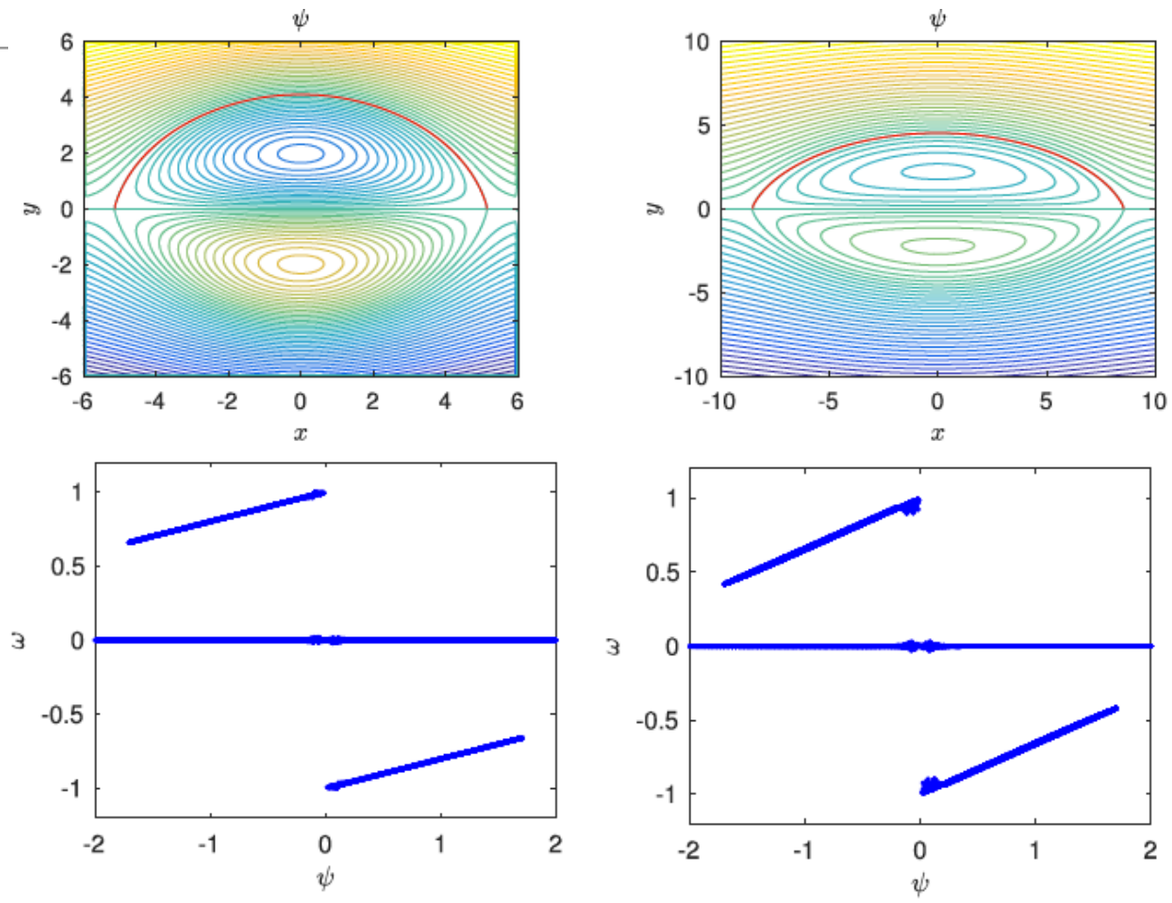
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- axial flow may act to suppress singular behaviour
- by driving fluid to the nose of the hairpin



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- need a different vortex cross-section
- modifies the velocity U
- axial flow may act to suppress singular behaviour
- by driving fluid to the nose of the hairpin
- delicate issues - unresolved - even in *asymptopia*



Conclusions

- intensification of vorticity important process in high Reynolds number flows
- vortices easily form dipole pairs which lead to stretching
- in axisymmetric geometry, vortex ring collisions give $R(t) \sim t^{4/3}$
- (local) conservation of energy important, vorticity stripping occurs
- vortex cross section is Sadovskii (piecewise constant vorticity)
- in general geometry, with **zero axial flow** W, similarity solutions show finite time blow-up: *hairpin vortices*
- **with axial flow**, need a new vortex cross section; blow-up a delicate issue

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- S. Childress, A.D.Gilbert & P. Valiant 2016 Eroding dipole and vorticity growth for Euler flows in **R**^3: axisymmetric flow without swirl. *J. Fluid Mech.*, **805,** 1-30.

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