Multi-Particle Dispersion in Rotating-Stratified Turbulent Flows

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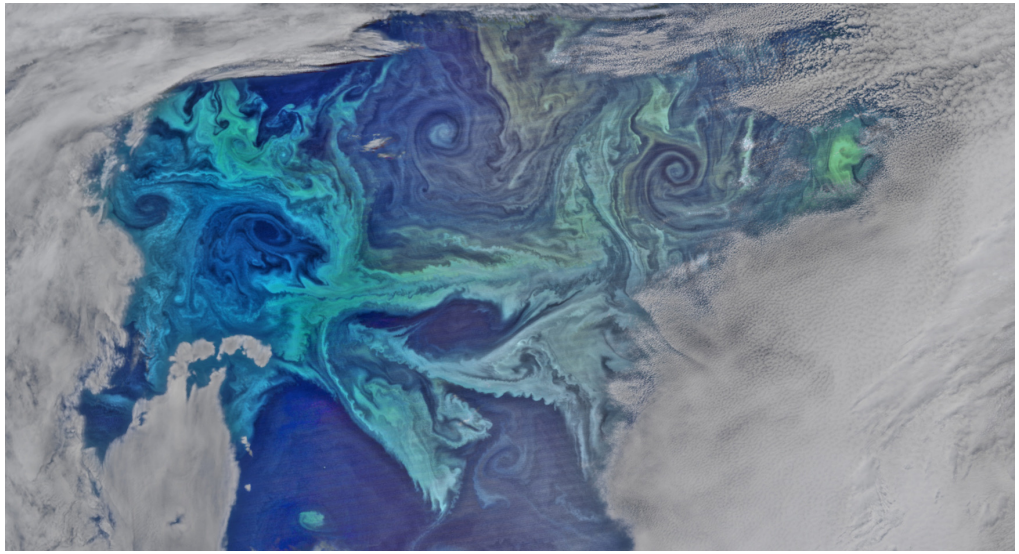
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Advanced summer school on mathematical fluid dynamics, Cargèse, April 2025

Dispersion in stratified an Turbulent Flows

- Spread of material substances in turbulent flows is important for many natural systems
  - Volcanic ashes in the atmosphere
  - Air quality in urban systems
  - Mixing in the ocean
- Idealized framework to study dispersion:

Relative movement of tracer particle pairs



Phytoplankton bloom in the southern ocean
NASA Earth Observatory / VIIRIS



An example of multiparticle dispersion in the atmosphere: the “Impact” campaign; northern Finland; June 2024. Experimental group at MPI-DS (E. Bodenschatz; G. Bagheri, F. Falkinhoff):

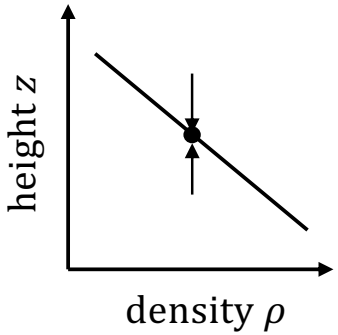
Simultaneous release of several small balloons;
study their mean separation rate, and how it is affected by different atmospheric conditions

Rotating and stratified turbulence

Stratification and rotation



- Stratification:



➢ Brunt-Väisälä frequency N

- Rotation:
  - ➢ Coriolis parameter f

  (f = two times the rotation frequency)

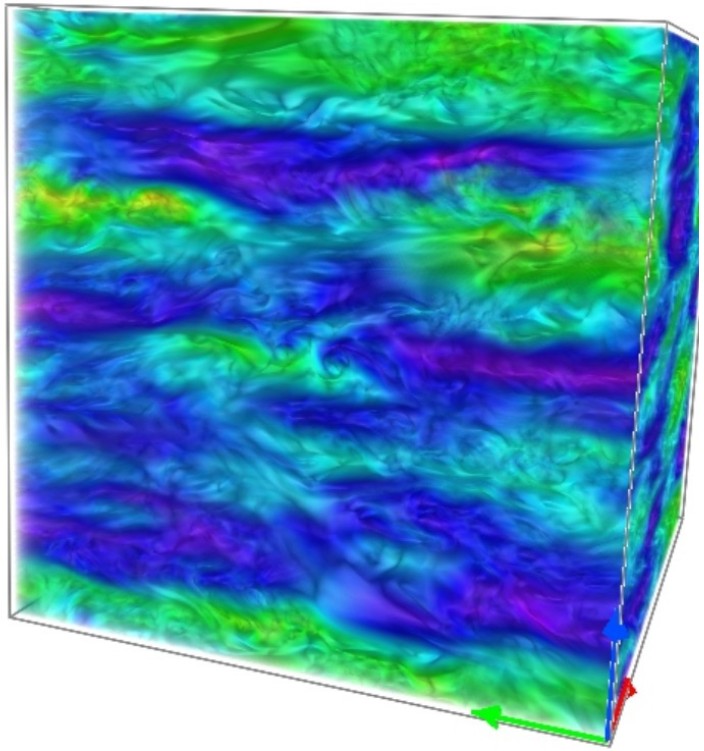
- $\frac{N}{f} = 5$

- [relevant in the Southern ocean, Garabato et al, Science 2004]

ROTSTRAT Turbulence?

- DNS of the Boussinesq equation with a **linear density profile**, $\mathrm{Pr} = \frac{\nu}{\kappa} = 1$ using pseudospectral method (GHOST solver):

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p - f\,\mathbf{e}_z \times \mathbf{u} - N\theta \mathbf{e}_z + \nu\,\nabla^2 \mathbf{u} + \mathbf{\Phi}$$
$$\nabla \cdot \mathbf{u} = 0$$
$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = Nw + \kappa \nabla^2 \theta$$



- Series of 8 runs at resolutions 512^3 and 1024^3

- increasing N and f, keeping: $\frac{N}{f} = 5$:

Buoyancy Reynolds Number

$$R_{IB} = \frac{\langle \omega^2 \rangle}{N^2} = \left(\frac{\ell_O}{\eta}\right)^{\frac{4}{3}}$$

From Homogeneous-Isotropic Turbulence-like: $R_{IB} \gg 1$

to stratification dominated $R_{IB} < 1$

Small time scales?

Small time scales in the system:

- Waves due to rotation/stratification; here frequency $\sim N \sim (U/L)/Fr$
- Turbulent "frequency" $\sim \langle\omega^2\rangle^{1/2} \sim (U/L)\, Re^{1/2}$

Comparison between the two:

$$R_{IB} = \langle\omega^2\rangle/N^2 = \varepsilon/\nu N^2 \sim Re\, Fr^2$$

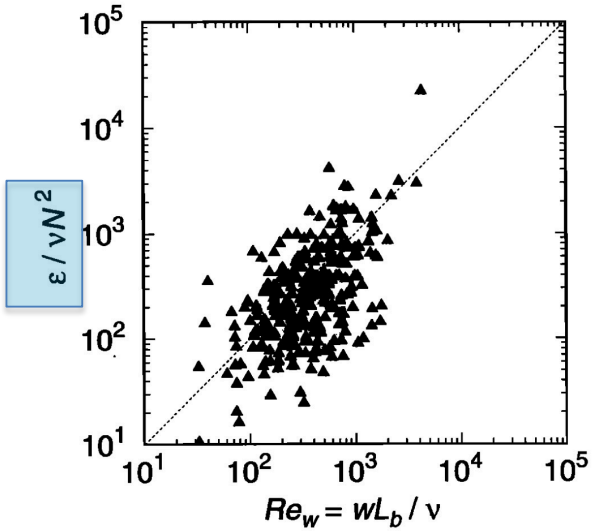


Figure 6. Vertical Reynolds number, $Re_w = w\ell/\nu$ (where ℓ has been replaced by L_b) versus $\epsilon/\nu N^2$. The parameter $\epsilon/\nu N^2$ is referred to as a buoyancy Reynolds number or turbulence activity.

In the oceans, times scales due to turbulence are faster than due to stratification…

…but not so much faster.

Relevance: mixing in the oceans (Ivey et al. Ann. Rev. Fluid Mech, 2008, Gregg et al. Ann. Rev Marine Science, 2018)…

Moum, JGR Oceans, 1996

Irreversibility in turbulence/ particle dispersion?

- A fundamental aspect of turbulent flows:

  the $t \rightarrow -t$ symmetry is broken.

[energy cascade: in 3d, energy flows from large to small scales; not the other way around]

=> manifestations of irreversibility in particle dispersion?

Two-particle dispersion

Pair Dispersion in Turbulent Flows

- Consider the relative pair distance

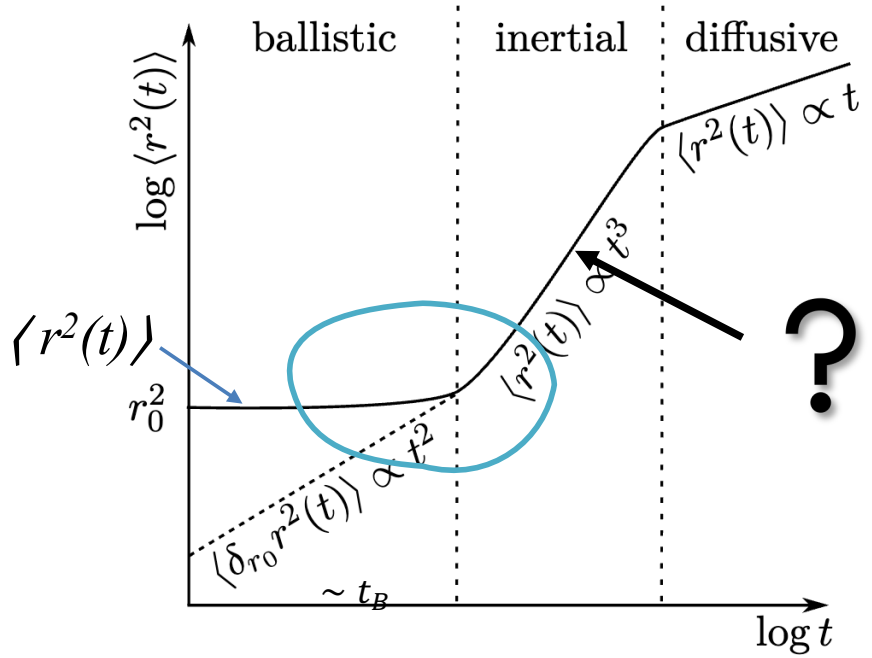
$$\mathbf{r}(t) = \mathbf{x}^{(2)}(t) - \mathbf{x}^{(1)}(t)$$

$$\mathbf{r}_0 = \mathbf{r}(0)$$

- Quantifying dispersion with:

$$\delta r^2(t) = |\mathbf{r}(t) - \mathbf{r}(0)|^2$$

$$r^2(t) = |\mathbf{r}(t)|^2$$



$$t_B = \frac{\langle \Delta_{\mathbf{r}} \mathbf{u}^2 \rangle}{|\langle \Delta_{\mathbf{r}} \mathbf{a} \cdot \Delta_{\mathbf{r}} \mathbf{u} \rangle|}$$

Richardson dispersion

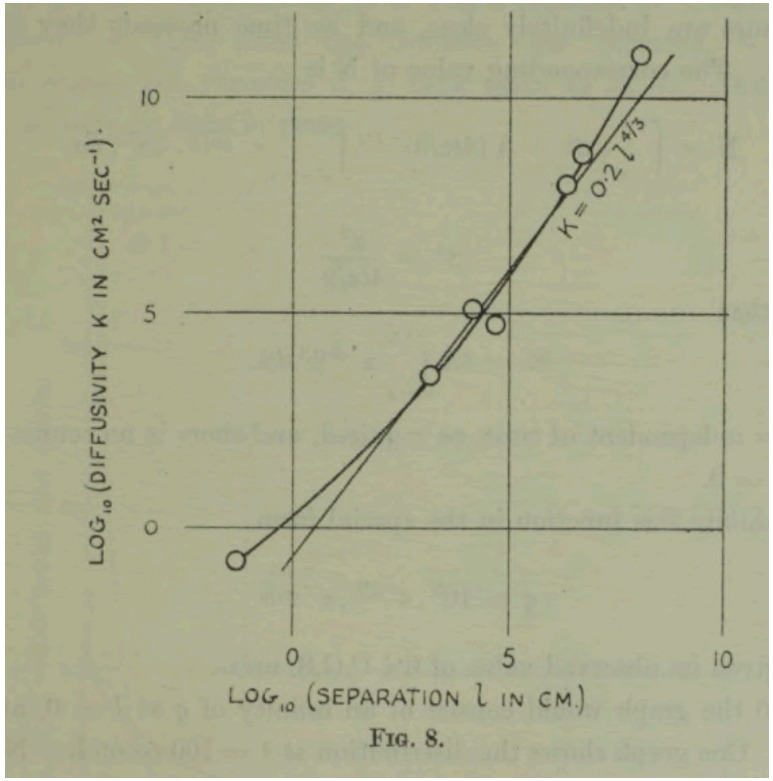


FIG. 8.

From limited data (...), Richardson (1926) inferred that

$$\langle \delta r(t)^2 \rangle \sim \varepsilon t^3.$$

- Looks very sensible from a theoretical point of view.
- But experimental/numerical support has remained very weak.

Richardson, Proc Roy Soc 1926.

Fundamentals of turbulence from the relative motion of two particles.

2-particle energy considerations

"Theorem": (Ott & Mann, 2000, Pumir et al, 2001, Falkovich et al, 2001).

$$\frac{1}{2}\frac{d}{dt}\left\langle (\mathbf{u}_2 - \mathbf{u}_1)^2 \right\rangle = -2\epsilon$$

for $\eta \ll |\mathbf{r}_2 - \mathbf{r}_1| \ll L$

Significance: this equation rests on an energy budget, directly from the Navier-Stokes equations. It can be viewed as the Lagrangian version of the (exact) von Karman-Howarth-Kolmogorov relation:

$$\left\langle (\delta_r u_L)^3 \right\rangle = -\frac{4}{5}\epsilon r + 6\nu \frac{d}{dr}\left\langle (\delta_r u_L)^2 \right\rangle$$

Lagrangian energy budget (1)

Estimate the rate of change of the energy of the particles:

$$\begin{aligned} &\tfrac{1}{2}\, d/dt \,\langle (\boldsymbol{u}_2 - \boldsymbol{u}_1)^2 \rangle \\ &= \langle (\boldsymbol{u}_2 - \boldsymbol{u}_1)\, (d\boldsymbol{u}_2/dt - d\boldsymbol{u}_1/dt) \rangle \\ &= \langle \boldsymbol{u}_2\, d\boldsymbol{u}_2/dt \rangle + \langle \boldsymbol{u}_1\, d\boldsymbol{u}_1/dt \rangle - \langle \boldsymbol{u}_2\, d\boldsymbol{u}_1/dt \rangle - \langle \boldsymbol{u}_1\, d\boldsymbol{u}_2/dt \rangle \end{aligned}$$

Use the Navier-Stokes equations:

$$d\boldsymbol{u}_1/dt = -\nabla p_1 + \boldsymbol{f}_1 + \nu \nabla^2 \boldsymbol{u}_1$$
$$d\boldsymbol{u}_2/dt = -\nabla p_2 + \boldsymbol{f}_2 + \nu \nabla^2 \boldsymbol{u}_2$$

Lagrangian energy budget (2)

Substitute:

$$\langle \boldsymbol{u}_1 \, d\boldsymbol{u}_1/dt \rangle = \langle \boldsymbol{u}_1 . (- \nabla p_1 + f_1 + \nu \nabla^2 \boldsymbol{u}_1) \rangle$$
$$= -\langle u_1 . \nabla p_1 \rangle + \langle \boldsymbol{u}_1 . \boldsymbol{f}_1 \rangle + \nu \langle \boldsymbol{u}_1 . \nabla^2 \boldsymbol{u}_1 \rangle$$
$$= \quad 0 \quad + \quad \varepsilon_\nu \quad - \quad \varepsilon_\nu$$

Incompressibility => the correlation between $\boldsymbol{u}$ *and pressure gradient = 0.*

and:

$$\langle \boldsymbol{u}_1 \, d\boldsymbol{u}_2/dt \rangle = \langle \boldsymbol{u}_1 . (- \nabla p_2 + \boldsymbol{f}_2 + \nu \nabla^2 \boldsymbol{u}_2) \rangle$$
$$= -\langle \boldsymbol{u}_1 . \nabla p_2 \rangle + \langle \boldsymbol{u}_1 . \boldsymbol{f}_2 \rangle + \nu \langle \boldsymbol{u}_1 . \nabla^2 \boldsymbol{u}_2 \rangle$$

- Use the fact that the force has a large scale correlation, *L*, and if $|\boldsymbol{r}_2 - \boldsymbol{r}_1| \ll L$

$$\langle \boldsymbol{u}_1 . \boldsymbol{f}_2 \rangle \simeq \langle \boldsymbol{u}_1 . \boldsymbol{f}_1 \rangle = \varepsilon_\nu \simeq \langle \boldsymbol{u}_1 . \boldsymbol{f}_2 \rangle$$

- Velocity and the its Laplacian have short correlations: if $|\boldsymbol{r}_2 - \boldsymbol{r}_1| \gg \eta$

$$\nu \langle \boldsymbol{u}_1 . \nabla^2 \boldsymbol{u}_2 \rangle = \nu \langle \boldsymbol{u}_1 . \nabla^2 \boldsymbol{u}_2 \rangle \simeq 0$$

… the rest follows from elementary algebra…

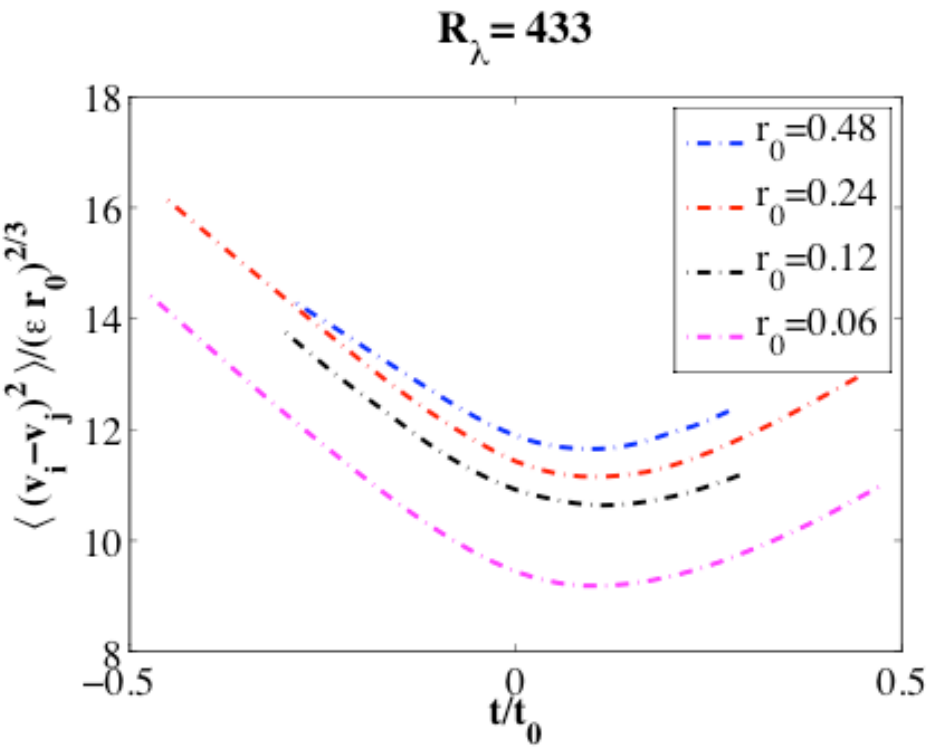
Role of the spatial dimension

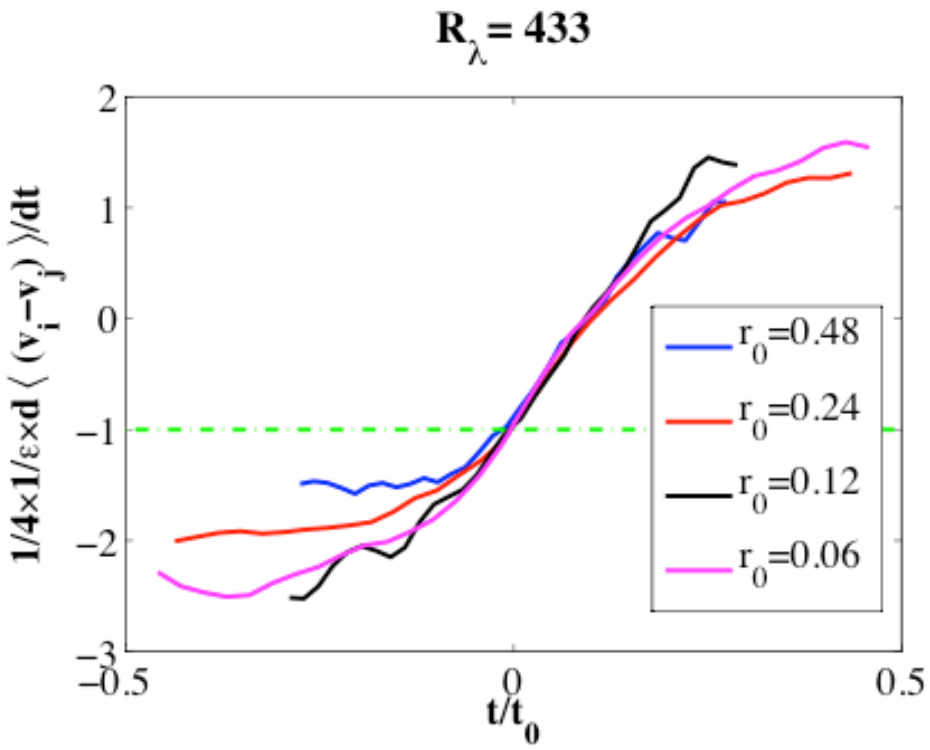
Important remark: in 2-dimensional turbulence: inverse cascade;

- the forcing and u are only correlated at "small scales".
- energy dissipation has to come from large scales

=> the sign of ε differs in 2 and 3d.

Numerical confirmation





Homogeneous turbulent flow from the Johns Hopkins Turbulence Database (η=*0.003; L=1.38*).

All pairs have the exactly same length at t = 0, well within the inertial range.

Two particle dispersion and irreversibility (1)

Study the relative dispersion between *two* particles:

$$\boldsymbol{R}(t) = ([\boldsymbol{r}_2(t) - \boldsymbol{r}_1(t)])$$

$$\delta\mathbf{R}(t) = \mathbf{R}(t) - \mathbf{R}(0)$$

*+ do Taylor-expansion (**R**(0) = **0**):*

$$\frac{\langle \delta\mathbf{R}(t)^2 \rangle}{R_0^2} = \frac{\langle \mathbf{u}(0)^2 \rangle}{R_0^2} t^2 + \frac{\langle \mathbf{u}(0) \cdot \mathbf{a}(0) \rangle}{R_0^2} t^3 + \mathcal{O}(t^4),$$

notation: $\boldsymbol{u} = \boldsymbol{u}_2 - \boldsymbol{u}_1$ $\boldsymbol{a} = \boldsymbol{a}_2 - \boldsymbol{a}_1$

Two particle dispersion and irreversibility (2)

- Use the (Kolmogorov) time characterizing the eddy at scale R_0:

$$t_0 = (R_0^2/\varepsilon)^{1/3}$$

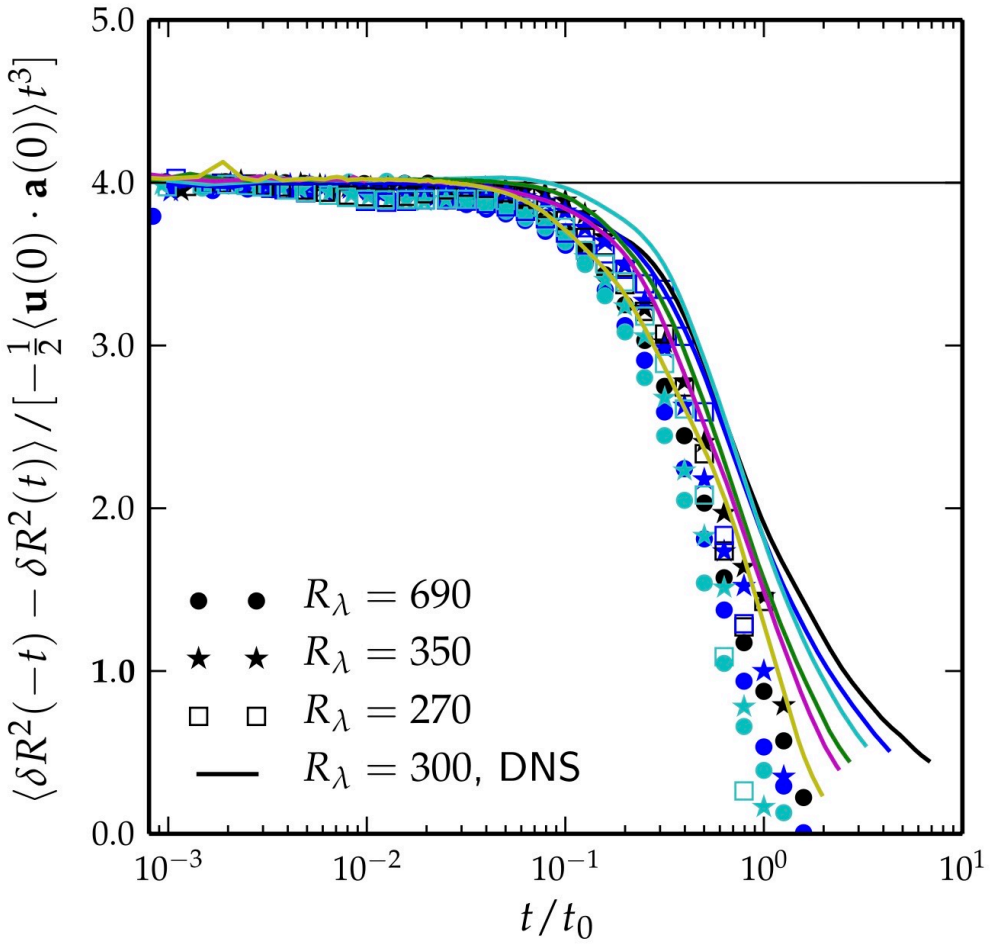
[…dimensional considerations, in the spirit of Kolmogorov similarity hypothesis…]

+ re-express the identity :

$$\frac{\langle \delta\mathbf{R}(-t)^2 - \delta\mathbf{R}(t)^2 \rangle}{R_0^2} = -2\frac{\langle \mathbf{u}(0)\cdot\mathbf{a}(0)\rangle}{R_0^2} t^3 + \mathcal{O}(t^5)$$

$$= 4(t/t_0)^3 + \mathcal{O}(t^5),$$

Irreversibility for the relative motion of 2 particles



The t^3-behavior is very well satisfied up to $\sim t_0/10$.

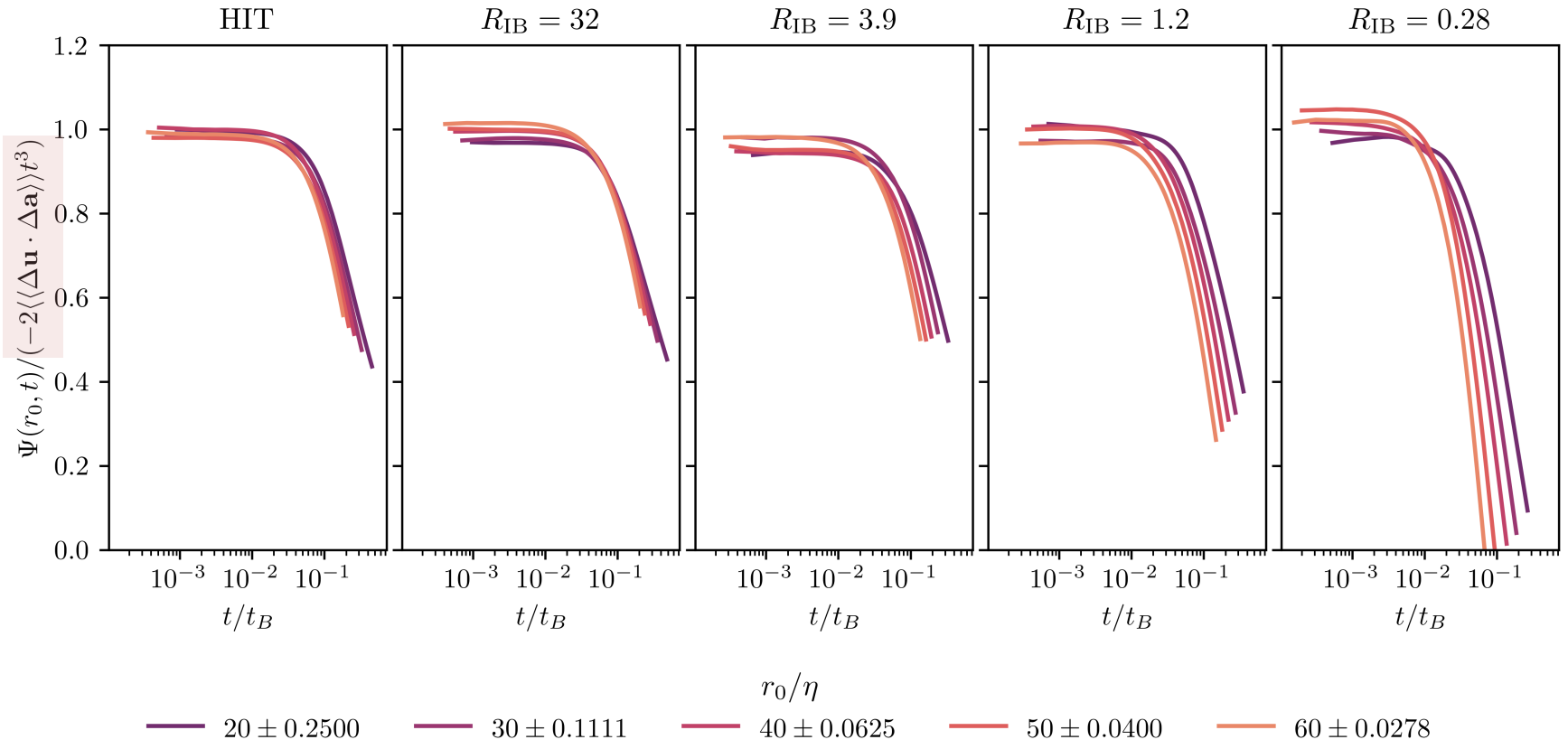
=> *works for Navier-Stokes turbulence:*

Clear manifestation of irreversibility

Jucha et al., PRL 2014

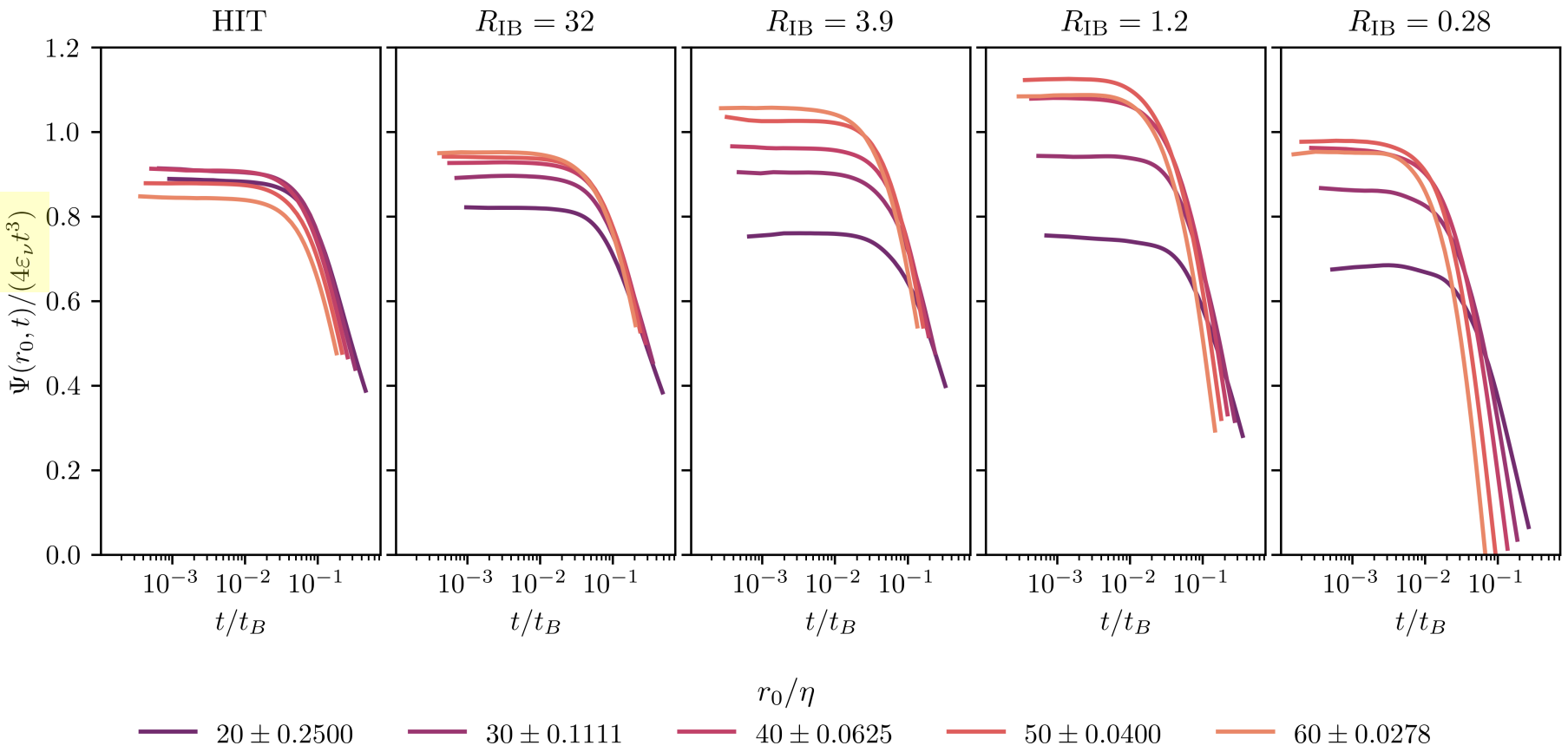
Forward – Backward Pair Dispersion III

stratification and rotation increases



Forward – Backward Pair Dispersion IV

stratification and rotation increases



For ***HIT*** : Lagrangian 4/5-law: $\langle \Delta\mathbf{u} \cdot \Delta\mathbf{a} \rangle = -2\ \varepsilon_\nu$

For *ROTSTRAT*: No collapse ⇒ $\langle \Delta\mathbf{u} \cdot \Delta\mathbf{a} \rangle \neq -2\ \varepsilon_\nu$ & dependence on scale

The Energy Budget for Stratified Flows I

- Averaging the equations of motion (assuming homogenity and stationarity) yields (Sozza et al, 2015):

$$\langle \Delta\mathbf{u} \cdot \Delta\mathbf{a} \rangle = -2\,\varepsilon_\nu - N\,\langle \Delta w\,\Delta\theta \rangle$$



  - Brunt-Väisälä frequency N
  - relative vertical velocity Δw
  - relative density fluctuation $\Delta\theta$
- Note that

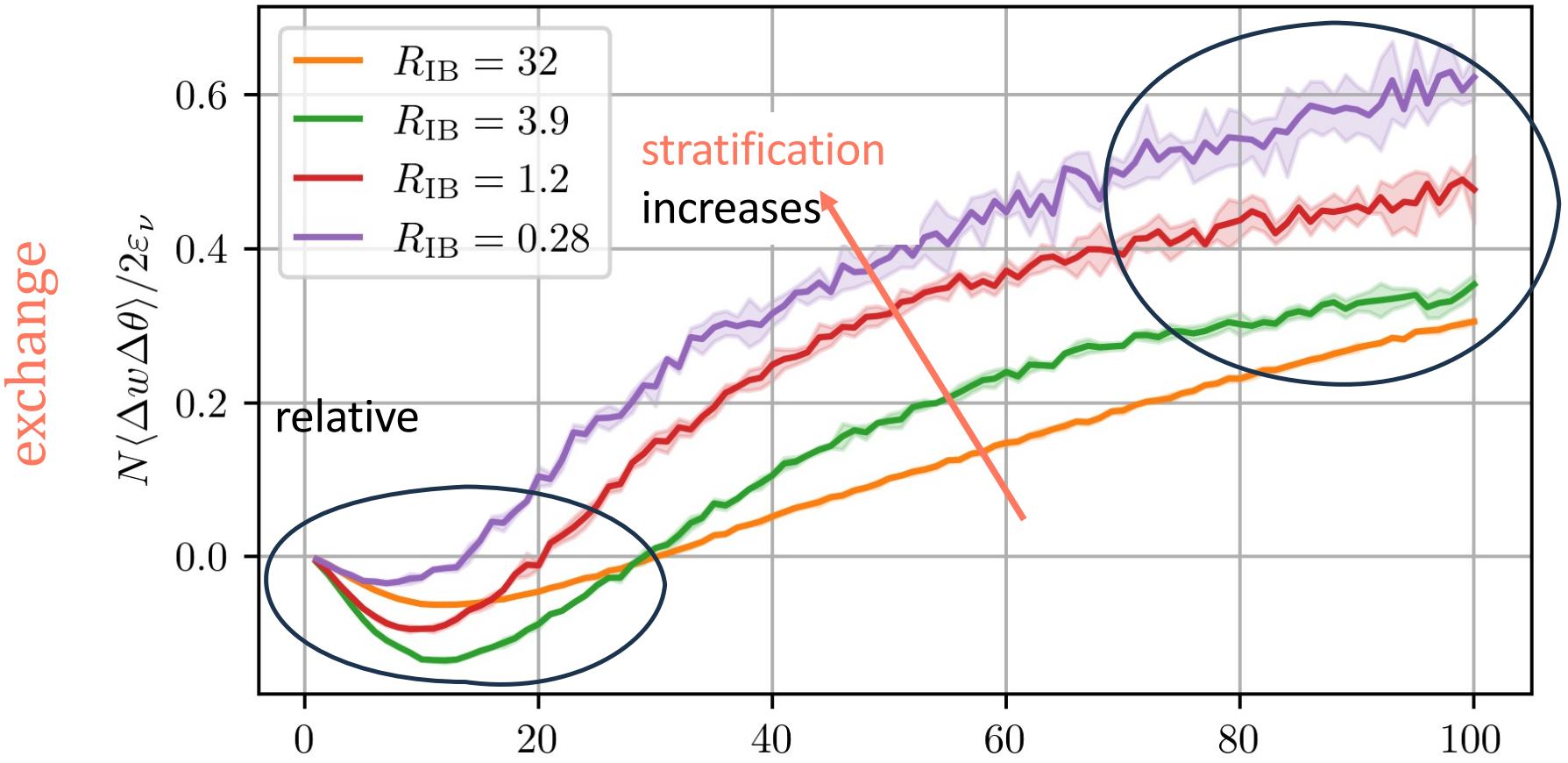
$$\langle \Delta\mathbf{u} \cdot \Delta\mathbf{a} \rangle = \frac{\mathrm{D}}{\mathrm{D}\,t} E_{\mathrm{kin}}^{\mathrm{pair}} \text{ , with } \quad E_{\mathrm{kin}}^{\mathrm{pair}} = \frac{1}{2}\Delta\mathbf{u}^2$$

Energy exchange between $E_{\mathrm{kin}}^{\mathrm{pair}}$ and $E_{\mathrm{pot}}^{\mathrm{pair}}$

- $\Delta\theta$ introduces potential energy $E_{\mathrm{pot}}^{\mathrm{pair}} = \frac{1}{2}\Delta\theta^2$, therefore (Sozza et al., 2015):

$$\frac{\mathrm{D}}{\mathrm{D}\,t} E_{\mathrm{pot}}^{\mathrm{pair}} = \langle \Delta\theta\Delta\dot{\theta} \rangle = -2\varepsilon_\kappa + N\,\langle \Delta w\,\Delta\theta \rangle$$

The Energy Budget for Stratified Flows II



negative exchange
$E_{\mathrm{pot}}^{\mathrm{pair}} \rightarrow E_{\mathrm{kin}}^{\mathrm{pair}}$
decreased time irreversibility

positive exchange
$E_{\mathrm{kin}}^{\mathrm{pair}} \rightarrow E_{\mathrm{pot}}^{\mathrm{pair}}$
Increased time irreversibility

The change of sign

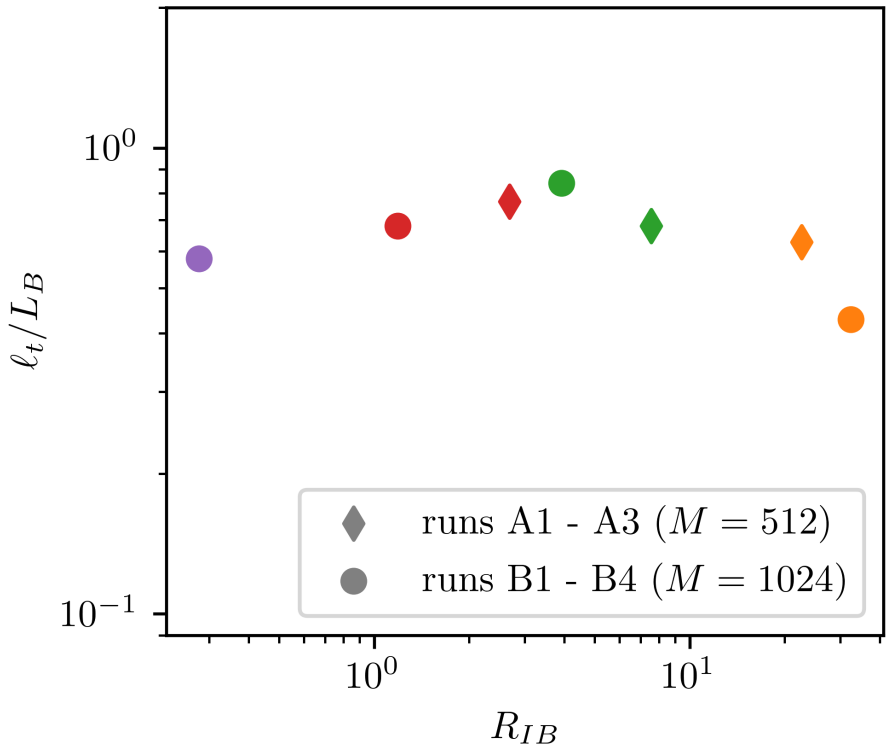
- Consider the scale ℓ_t where the exchange term vanishes

$$N\langle \Delta w \, \Delta\theta \rangle = 0$$

- Numerical observation:

$$\frac{\ell_t}{L_B} = \frac{\ell_t}{U/N} = \text{const} = 0.8$$

in the range of parameters studied here

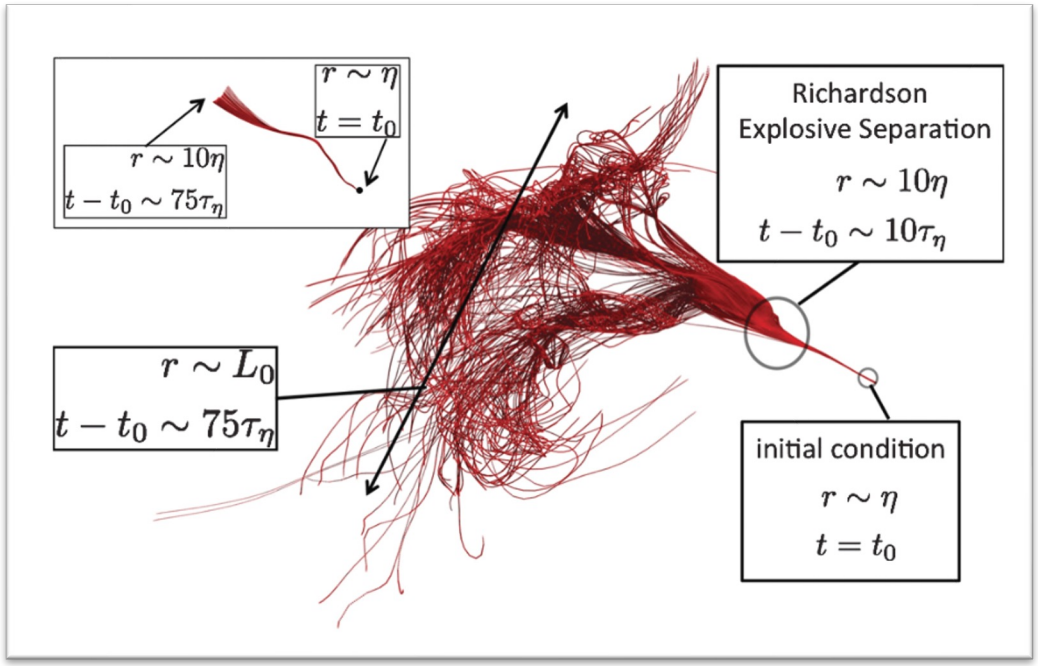


Intermittency in particle pair dispersion

Pair Dispersion in Turbulent Flows: intermittency

- Intermittency: Some pairs separate (much) quicker than others.

In Homogeneous Isotropic (HIT) flows:



Scatamacchia et al. (2012)

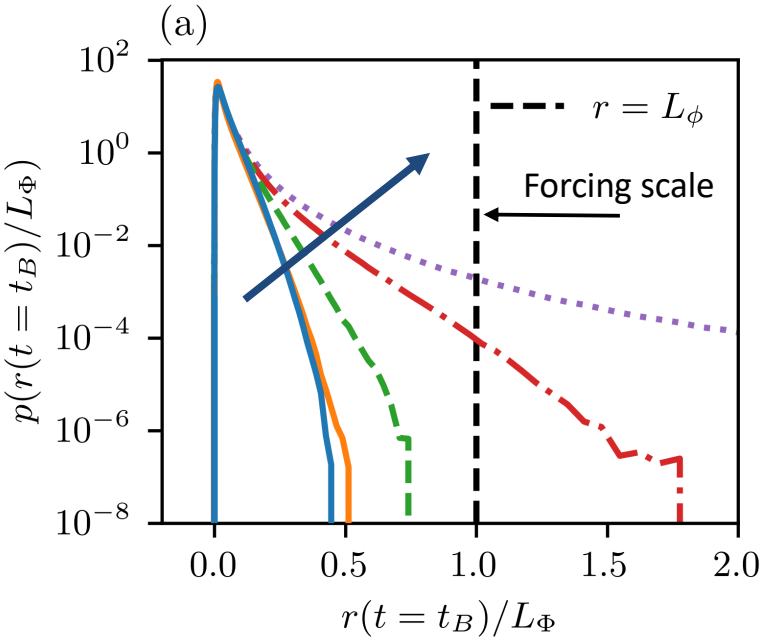
Here: Study particle pairs in Rotating-Stratified Turbulence.

Dispersion Statistics

Initial separation:

$r_0/\eta = 4 \pm 0.4000$

$$R_{\mathrm{IB}} = \frac{\langle \omega^2 \rangle}{N^2} = \left(\frac{\ell_o}{\eta}\right)^{\frac{4}{3}}$$



run

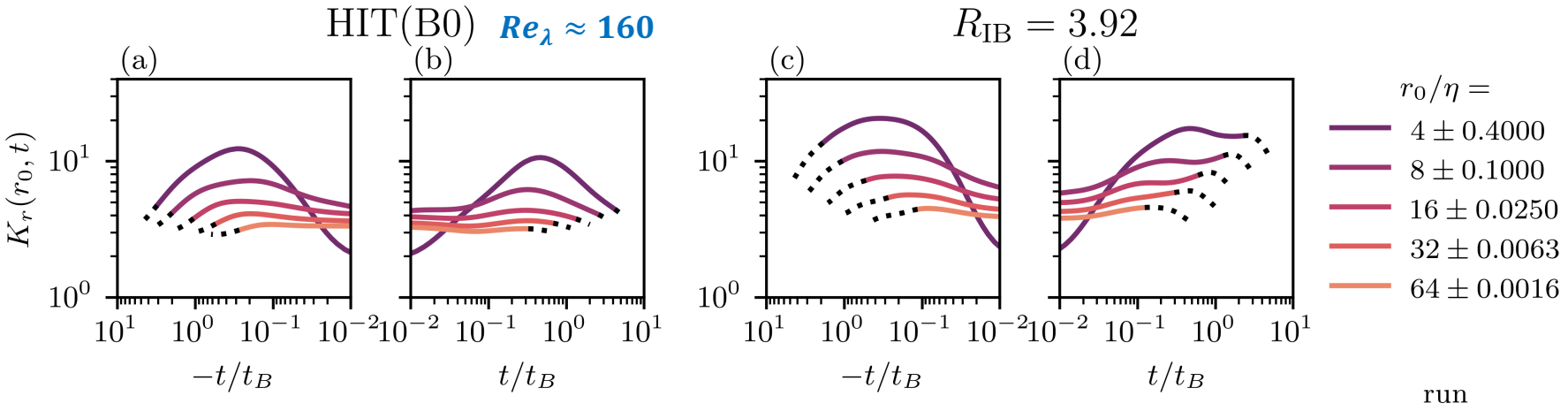
- HIT(B0)
- $R_{\mathrm{IB}} = 31.9$
- $R_{\mathrm{IB}} = 3.92$
- $R_{\mathrm{IB}} = 1.19$
- $R_{\mathrm{IB}} = 0.755$

Stratification and rotation increase

Pair Dispersion Intermittency

- Kurtosis of pair separation
  - $K_r(r_0,t) = \frac{\langle [r(t)-\langle r(t)\rangle]^4 \rangle}{\langle [r(t)-\langle r(t)\rangle]^2 \rangle^2}$ (Gaussian reference : $K = 3$)
- Pair dispersion kurtosis increases with:

Decreasing initial separation



Pair Dispersion Intermittency

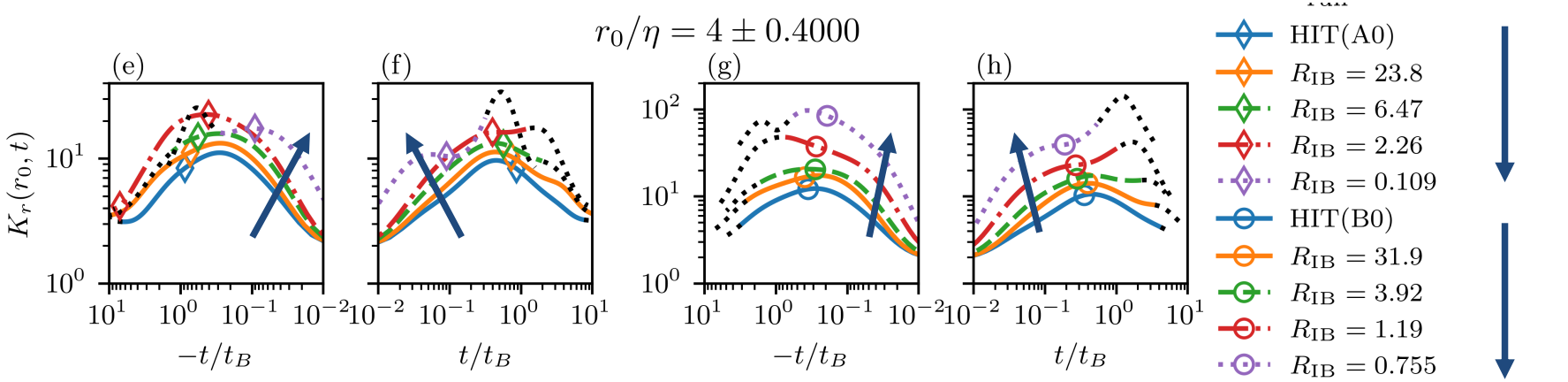
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- Pair dispersion kurtosis increases with:

$$R_{\mathrm{IB}} = \frac{\langle \omega^2 \rangle}{N^2} = \left(\frac{\ell_O}{\eta}\right)^{\frac{4}{3}}$$

Decreasing initial separation

Increasing strength of rotation and stratification

Stratification and rotation increase



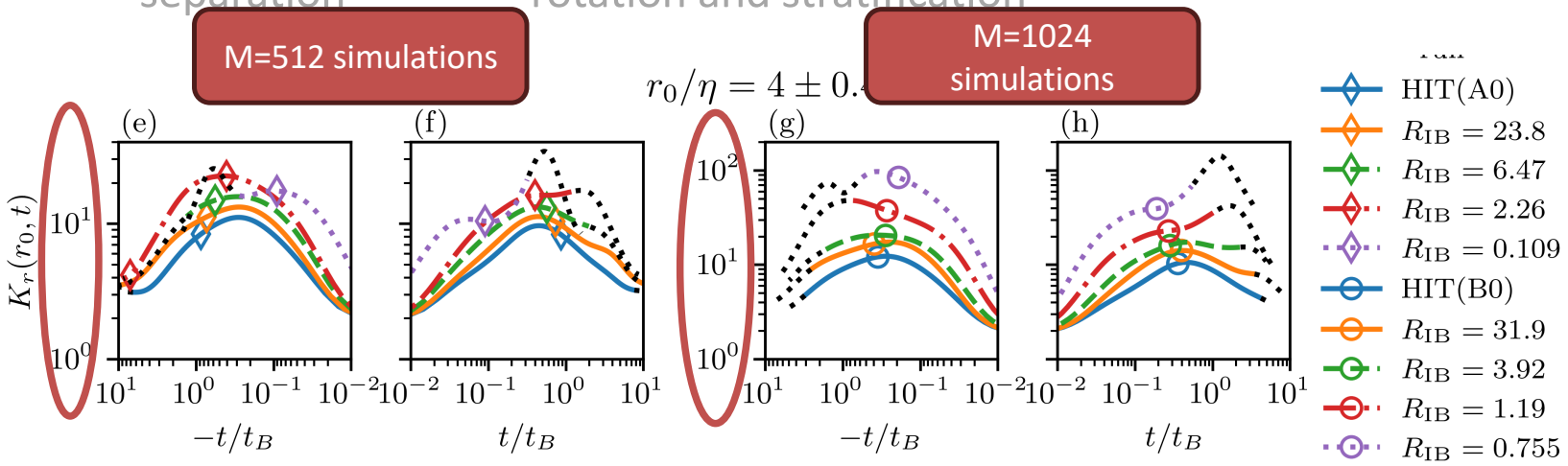
Pair Dispersion Intermittency

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- Pair dispersion kurtosis increases with:

Decreasing initial separation

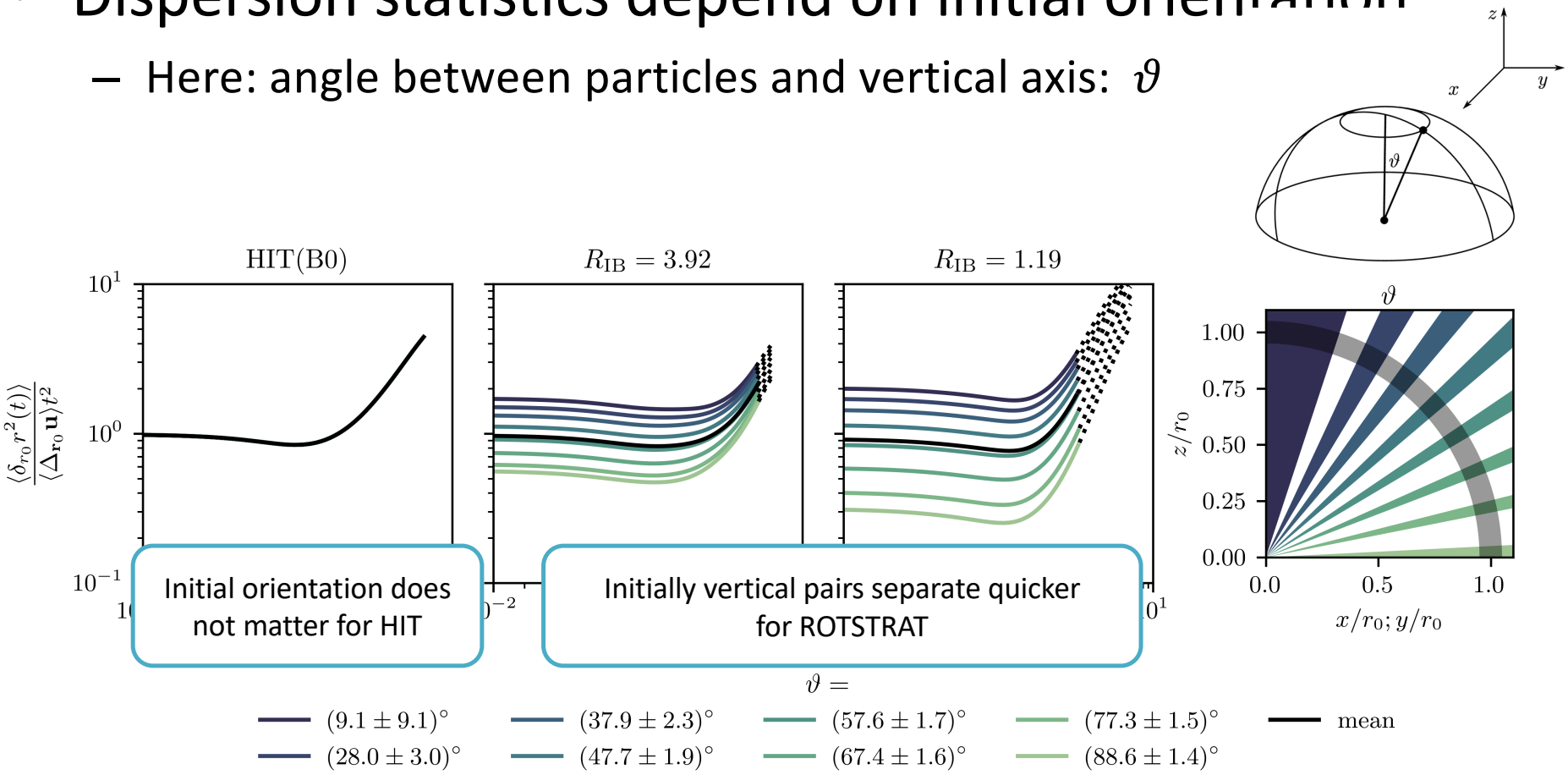
Increasing strength of rotation and stratification

Reynolds Number



Anisotropic Dispersion

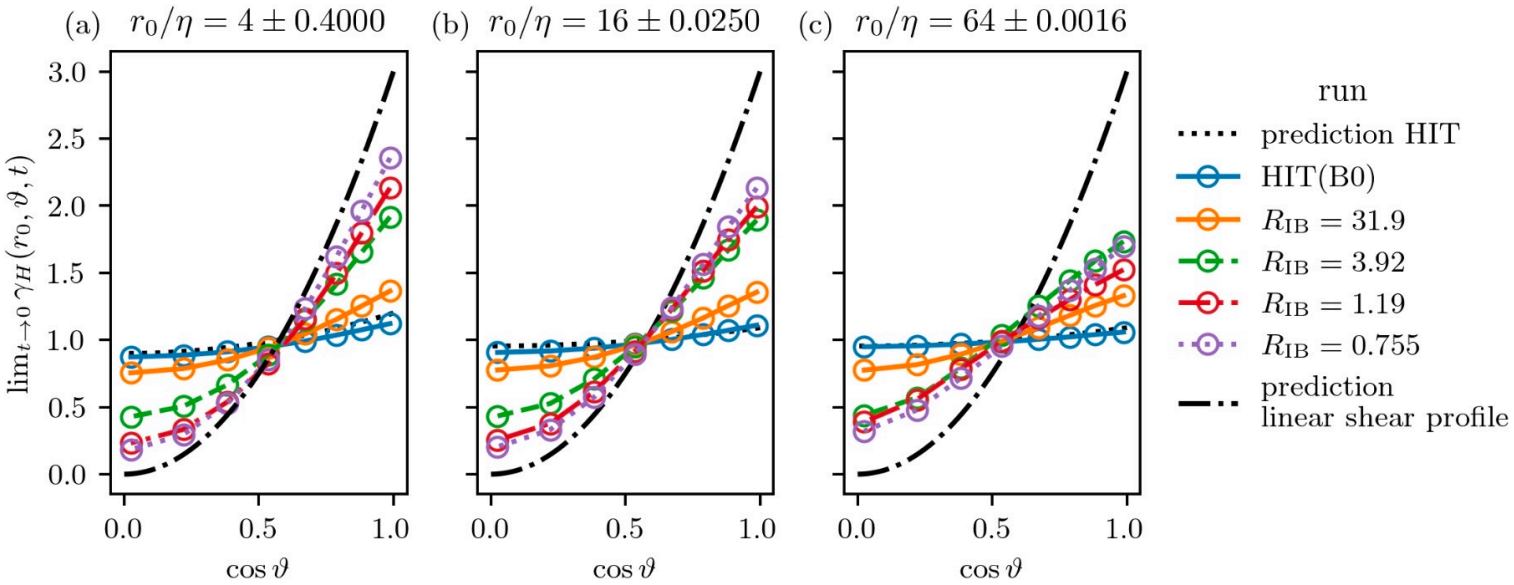
- Dispersion statistics depend on initial orientation
  - Here: angle between particles and vertical axis: ϑ



Shear layers in rotating-stratified flows.

- Observation: The relative motion is more significant when the direction between the pair are better aligned with the vertical.

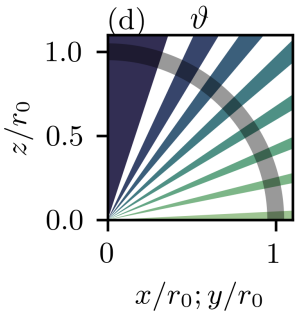
Suggests a shear structure (Lilly, 1963, etc)!

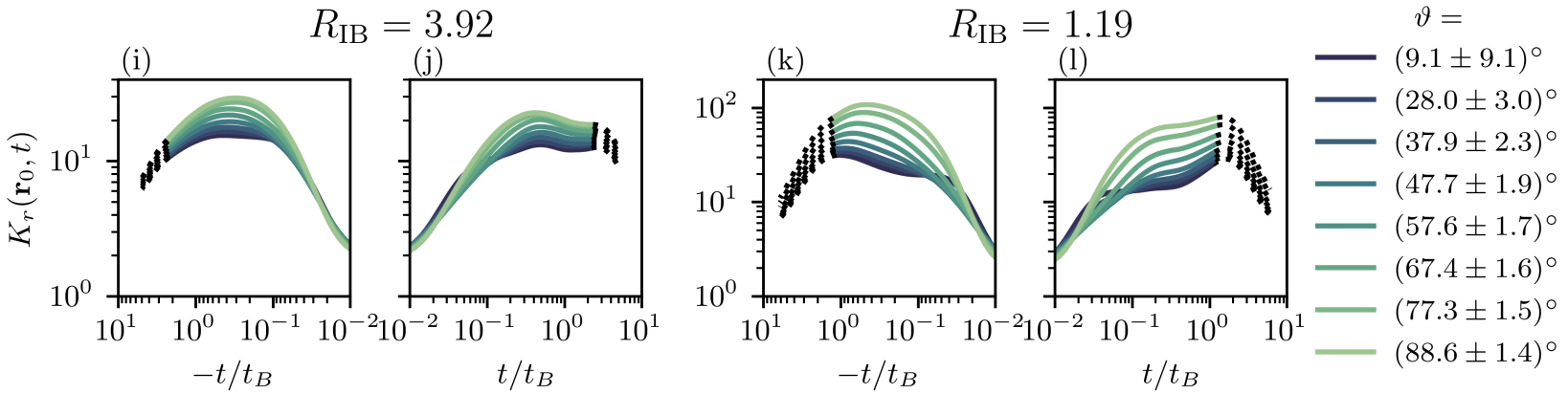


- The dependence on $cos\theta$ gets closer to that of a shear flow when the stratification (N) increases.

Pair Dispersion Intermittency

- Kurtosis of pair separation
  - $K_r(r_0,t) = \frac{\langle [r(t)-\langle r(t)\rangle]^4 \rangle}{\langle [r(t)-\langle r(t)\rangle]^2 \rangle^2}$ Gaussian reference : $K = 3$
- Pair dispersion kurtosis increases with:
  - Initial orientation in the horizontal plane





Conclusions

- Irreversibility of relative dispersion

  => energy exchange between kinetic and potential energy.

- Rotation and Stratification increase pair dispersion intermittency; the more so as Re increases and Fr diminishes.

- Initally horizontal pairs are most intermittent;

- Development of shear structures in the flow

  - An open question: role of rotation?

Thanks for your attention!

Questions?

References:

- S. Gallon *et al.*, PRL **133,** 024101 (2024)
- S. Gallon *et al.*, PRF **10**, 034605 (2025) [Editor's suggestion].
- See also: Buaria et al., PRF **5**, 064801 (2020).

Happy Birthday Keith!



Picture: courtesy of Emmanuel Dormy

Many thanks for your outstanding contributions to the scientific community