The Binormal Flow: a toy model for turbulence

Cargese 20 de junio, 2023

- Binormal (curvature) flow
  - Vortex Filament Equation
  - Localized Induction Approximation (LIA)
- Schrödinger map
  - Landau Lifshitz (Gilbert)
  - Heisenberg continuous chain for ferromagnetism
- 1 d cubic NLS

- Non linear dispersive PDE's
  - Full line
  - Periodic boundary conditions

  } Hybrid

- Continuation after blow-up
  - blow up $\leftrightarrow$ Long range scattering

- Multifractality & Intermittency

Good toy model for turbulence ?

[redacted]

FLOW CONTROL WITH NONCIRCULAR JETS[1]

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KEY WORDS: vortices, mixing, combustion, entrainment

ABSTRACT

Noncircular jets have been the topic of extensive research in the last fifteen years. These jets were identified as an efficient technique of passive flow control that allows significant improvements of performance in various practical systems at a relatively low cost because noncircular jets rely solely on changes in the geometry of the nozzle. The applications of noncircular jets discussed in this review include improved large- and small-scale mixing in low- and high-speed flows, and enhanced combustor performance, by improving combustion efficiency, reducing combustion instabilities and undesired emissions. Additional applications include noise suppression, heat transfer, and thrust vector control (TVC).

The flow patterns associated with noncircular jets involve mechanisms of vortex evolution and interaction, flow instabilities, and fine-scale turbulence augmentation. Stability theory identified the effects of initial momentum thickness distribution, aspect ratio, and radius of curvature on the initial flow evolution. Experiments revealed complex vortex evolution and interaction related to self-induction and interaction between azimuthal and axial vortices, which lead to axis switching in the mean flow field. Numerical simulations described the details and clarified mechanisms of vorticity dynamics and effects of heat release and reaction on noncircular jet behavior.

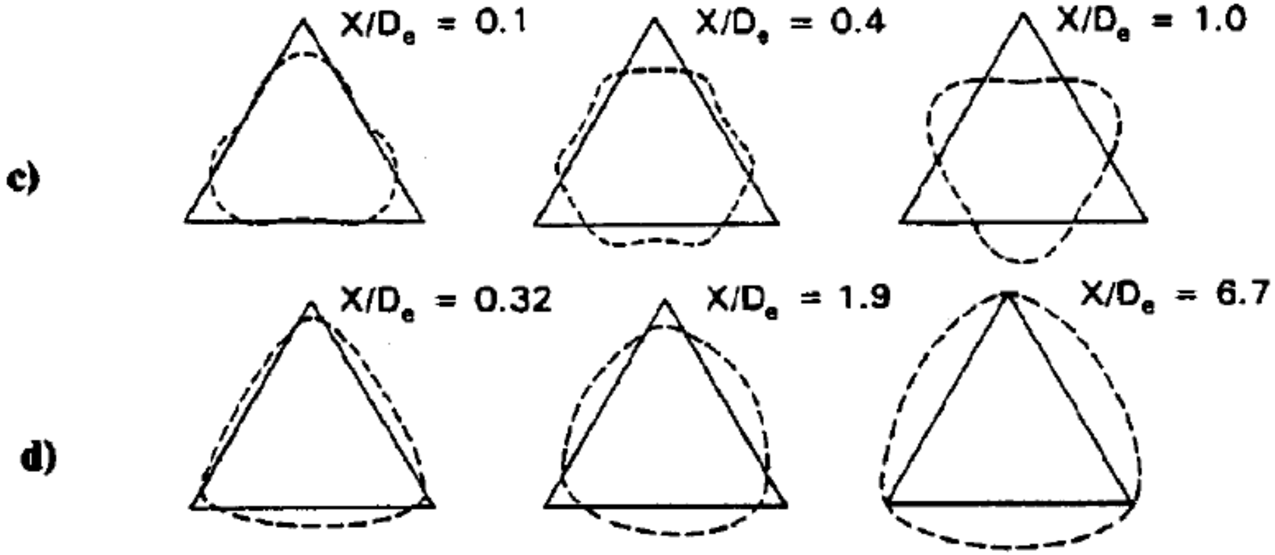


Figure 6 Variation of momentum thickness with axial distance at the vertex and flat sides of the triangular jet: (*a*) orifice jet, (*b*) pipe jet. Corresponding evolution of the jet cross-sections along the axis: (*c*) orifice jet, (*d*) pipe jet. (Koshigoe et al 1989)

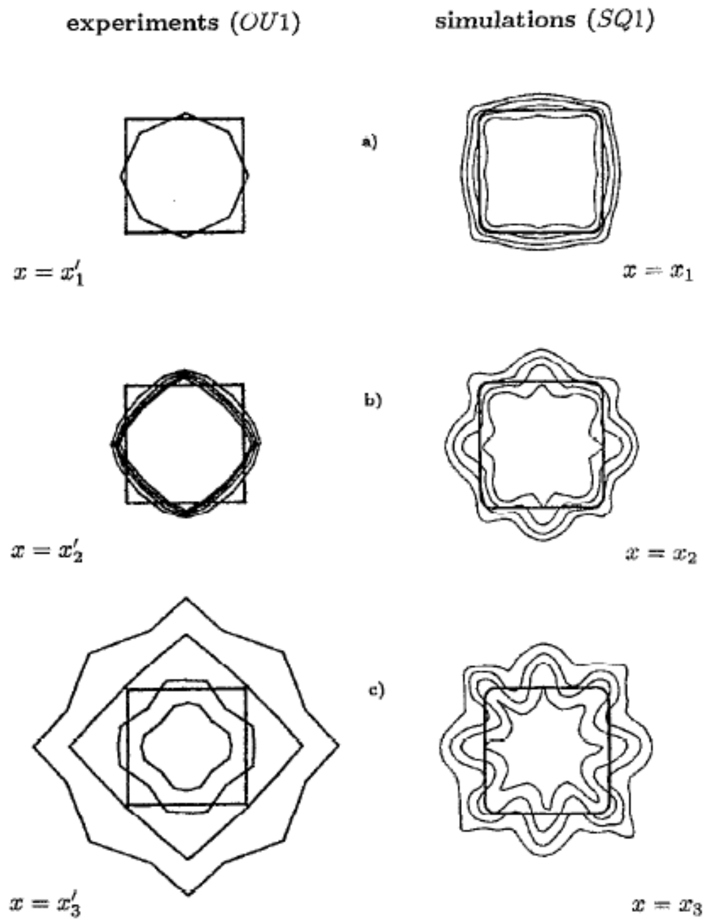
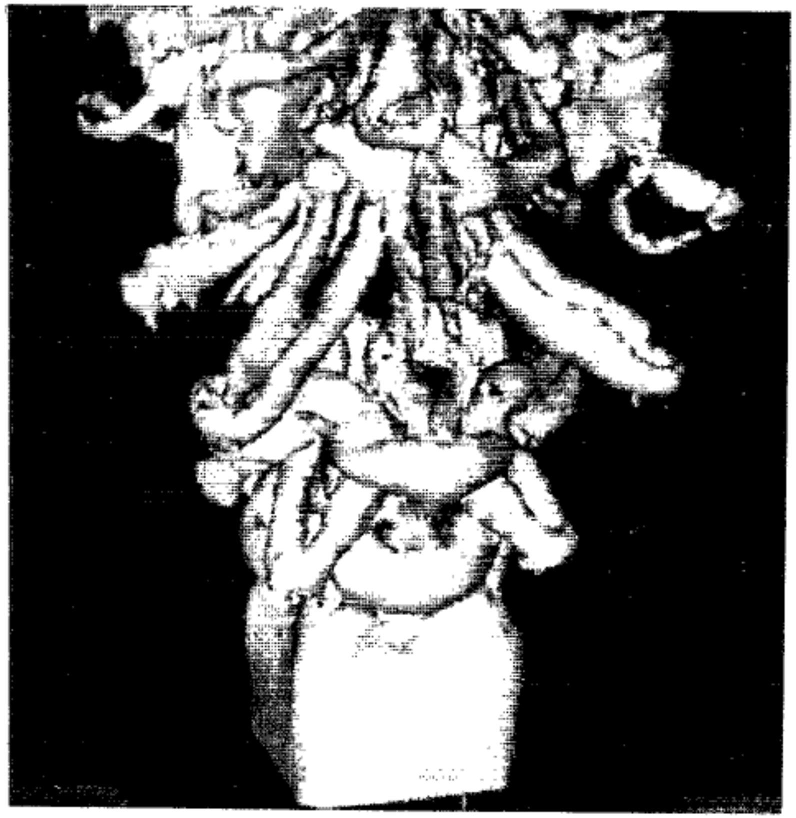
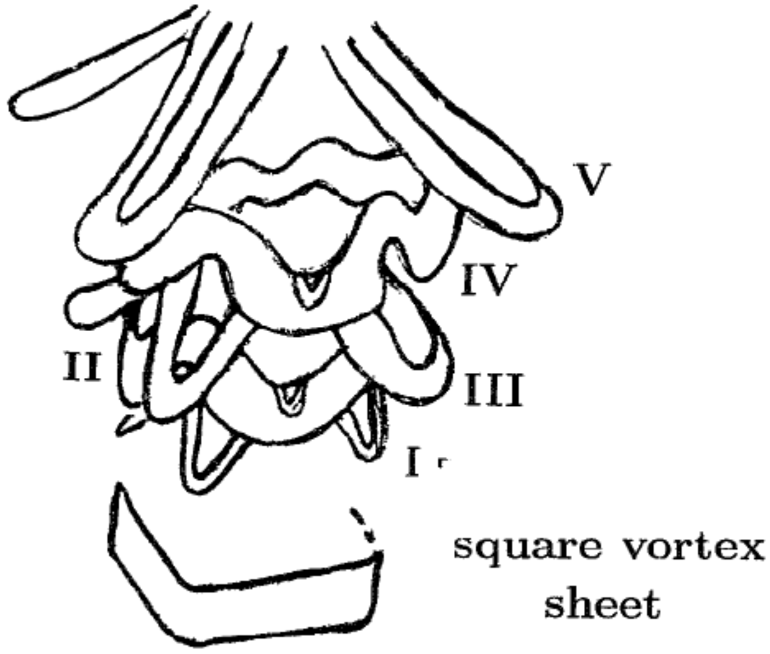
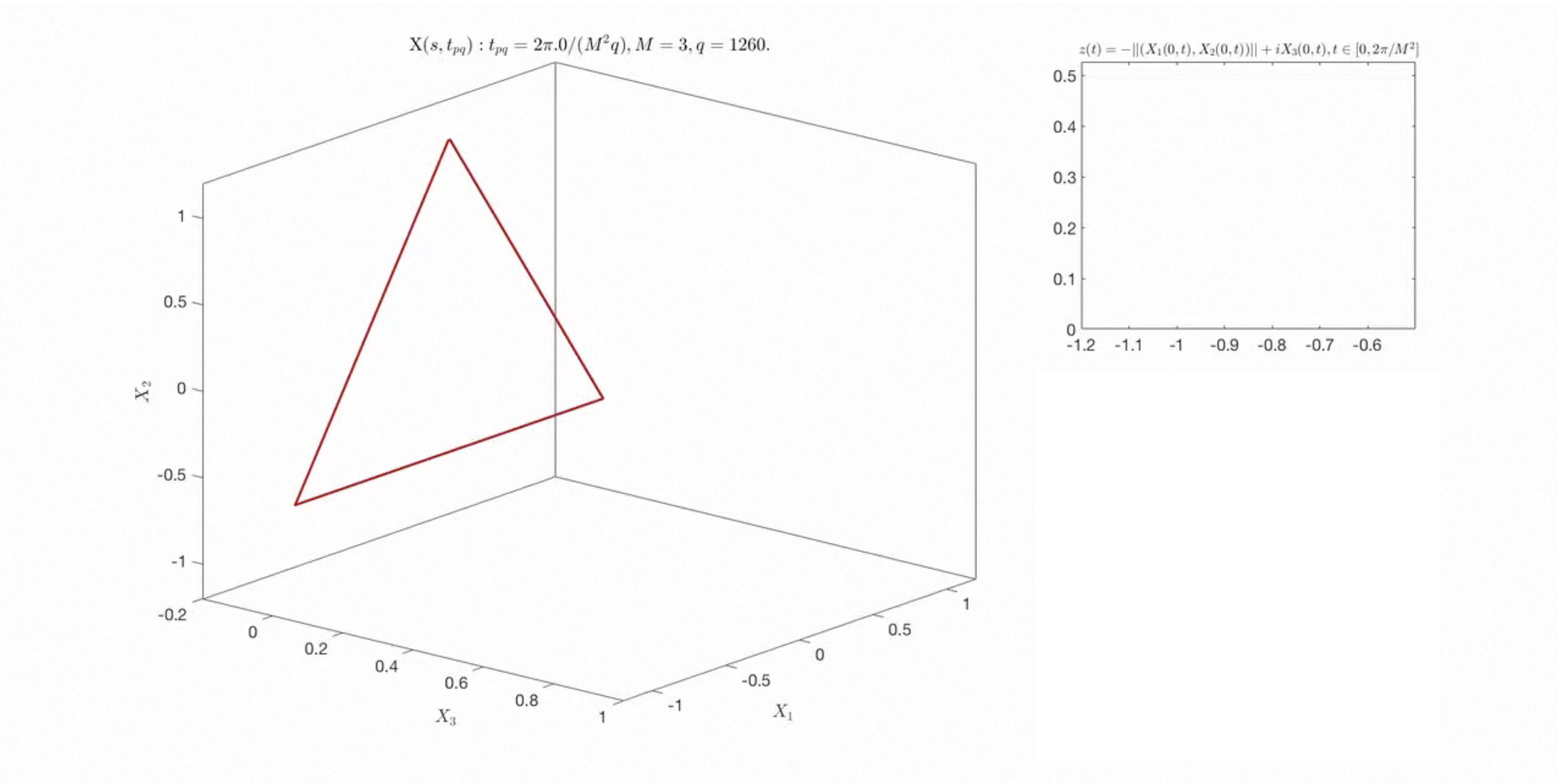


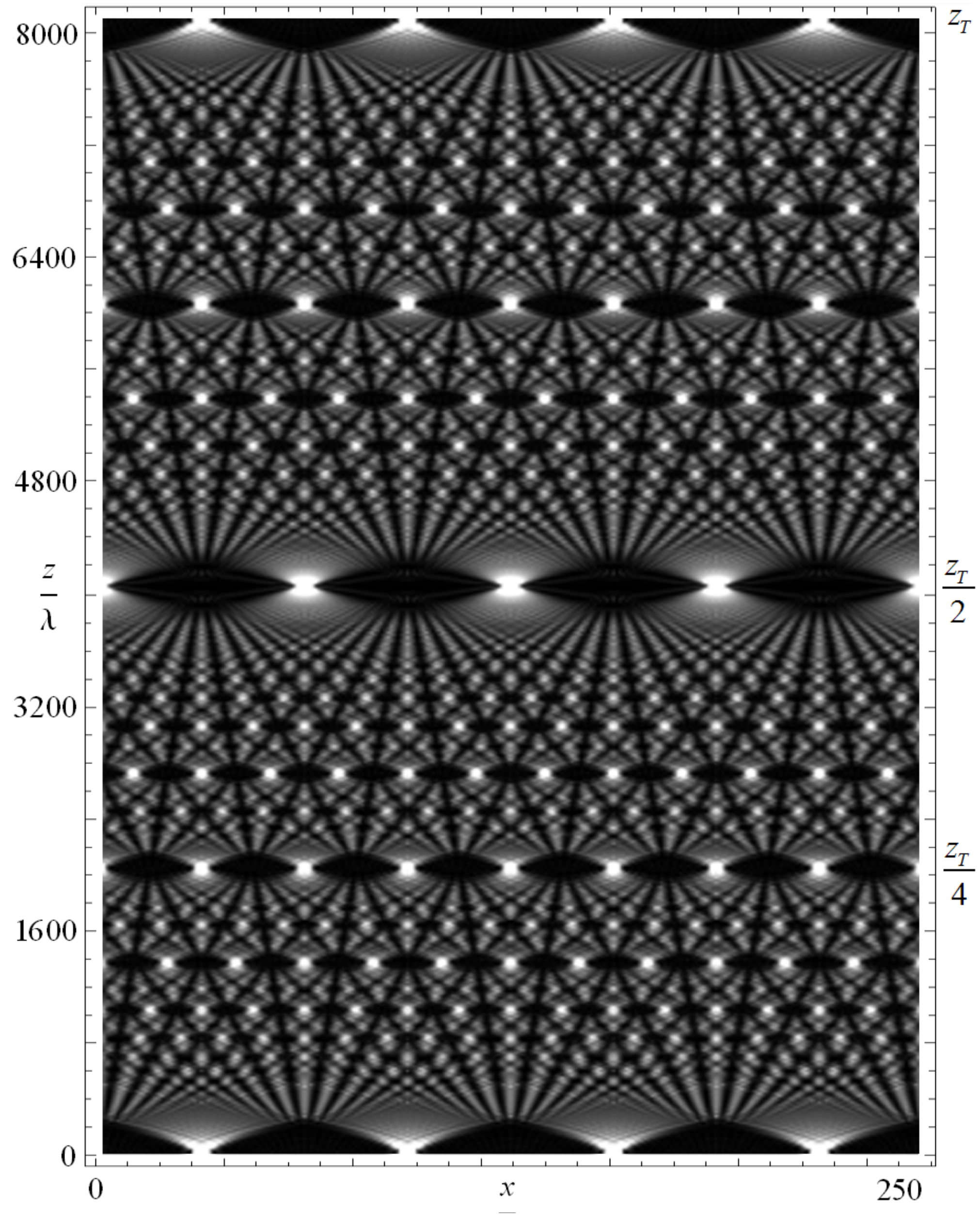
FIG. 10. Axis switching of the jet cross section in terms of isocontours of time-averaged streamwise velocity scaled with its local centerline value

I, III, V : hairpin (braid) vortices
II, IV : deformed vortex rings

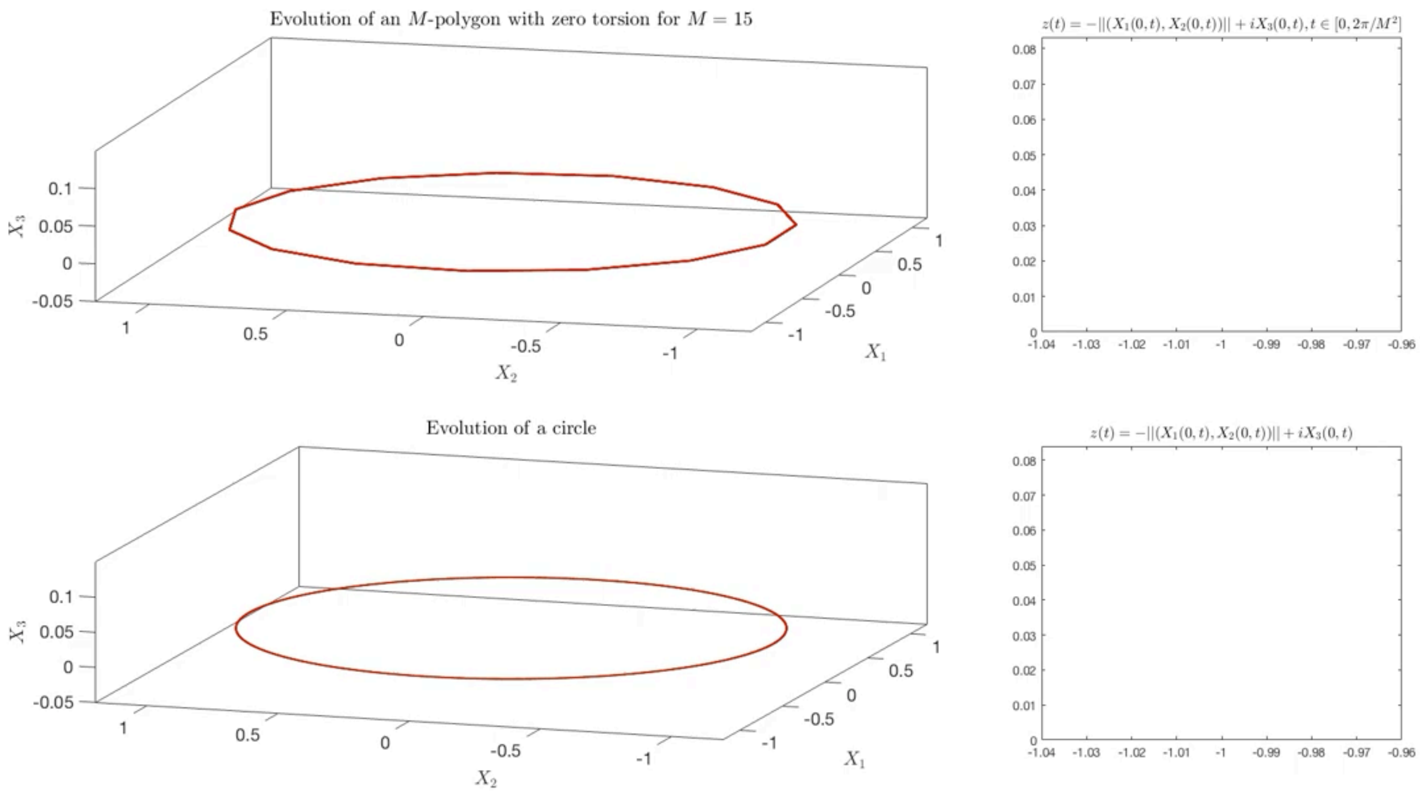


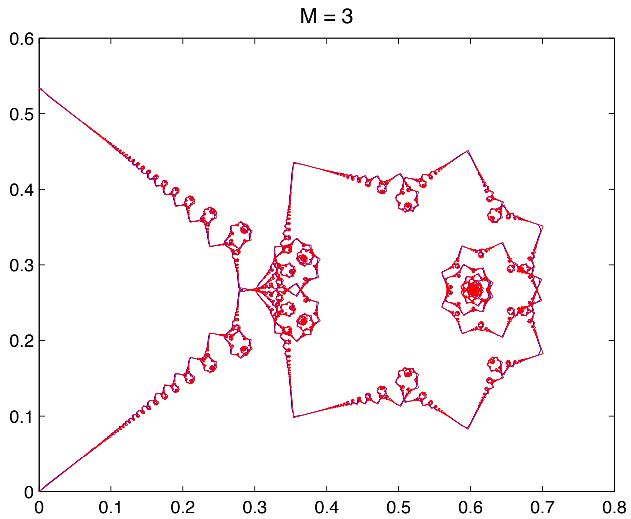


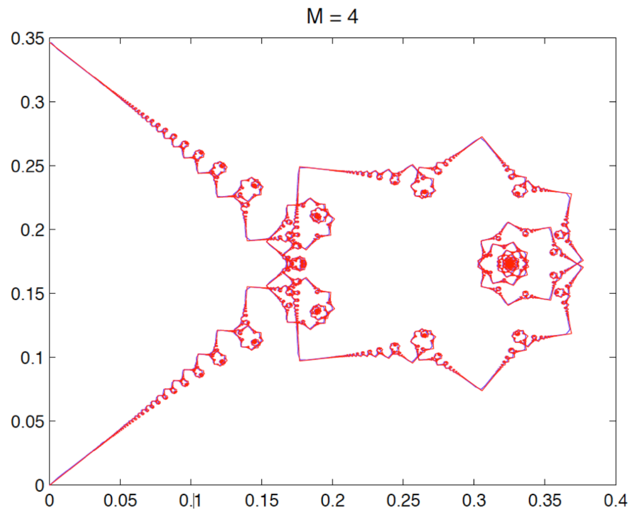

$X(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$
X_2
X_3
X_1
1
0.5
0
-0.5
-1
-0.2
0
0.2
0.4
0.6
0.8
1
-1
-0.5
0
0.5
1
0.5
0.4
0.3
0.2
0.1
0
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6

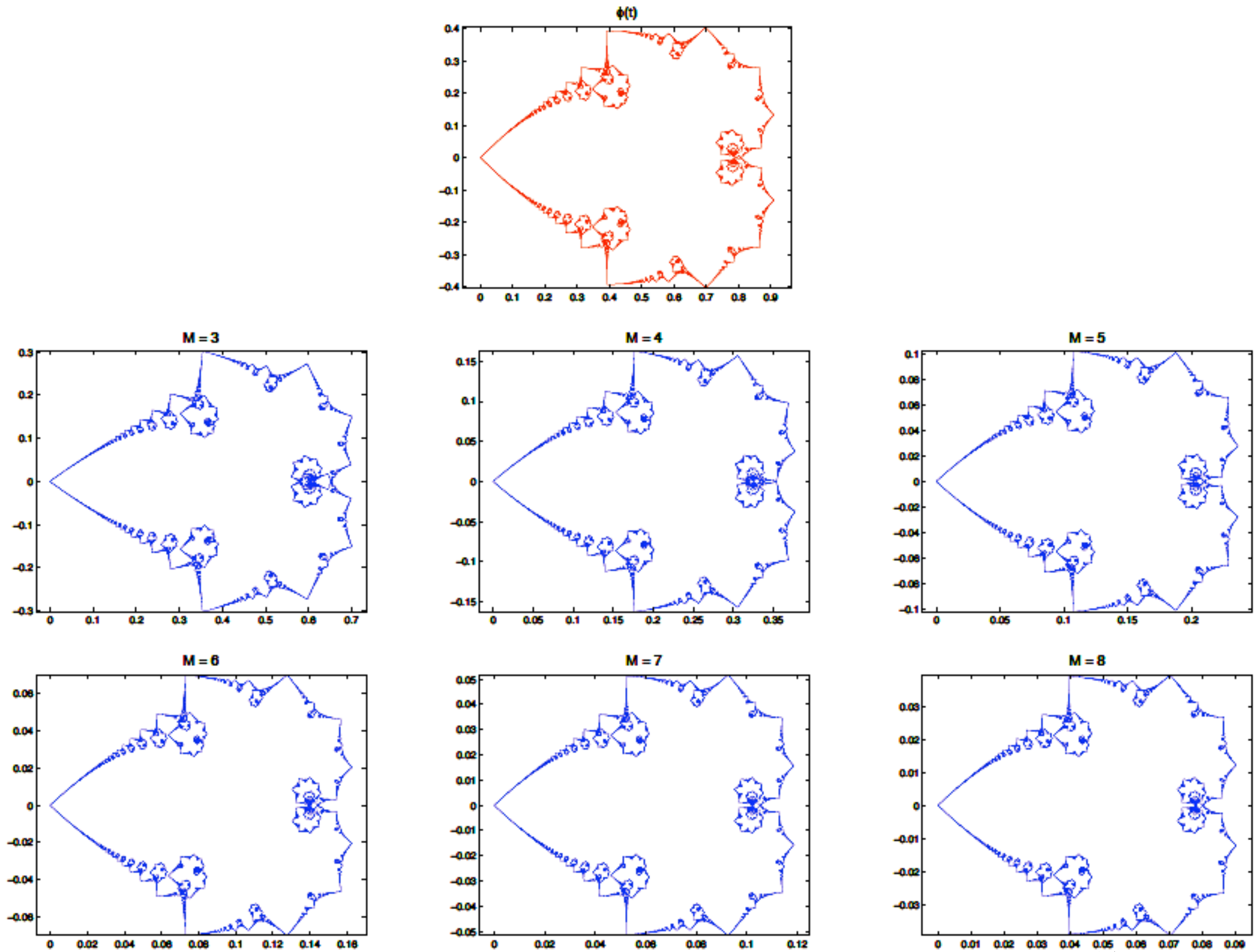


8000
6400
4800
3200
1600
0
$\frac{z}{\lambda}$
0
x
250
z_T
$\frac{z_T}{2}$
$\frac{z_T}{4}$


Evolution of an M-polygon with zero torsion for $M = 15$
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$
Evolution of a circle
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t)$
X_1
X_2
X_3


M = 3
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8


M = 4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4


ϕ(t)
M = 3
M = 4
M = 5
M = 6
M = 7
M = 8

Riemann's function

$$\varphi_R(t) = \sum_{j=1}^{\infty} \frac{\sin(tj^2)}{j^2} \qquad (\sim 1860)$$

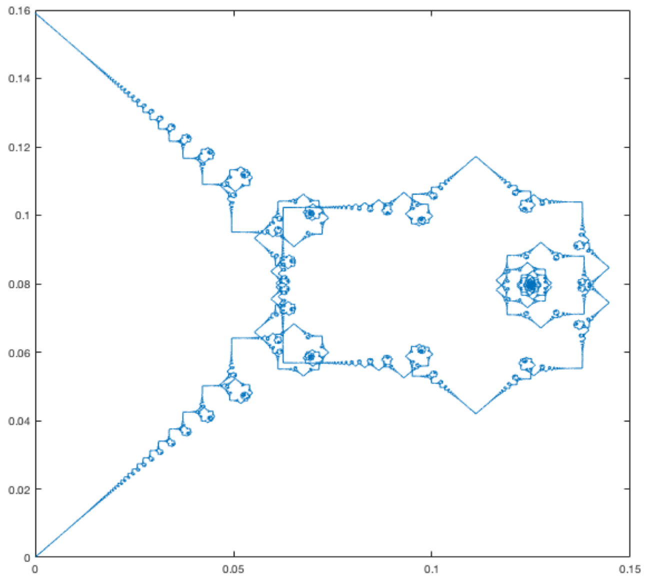
- Hardy 1915 (H–Littlewood circle method)
- Gerver 1960 (Riemann was wrong)

At $t_{p,q} = \pi p/q \quad p, q$ odd, the derivative exists and is $-1/2$

$$\varphi_D(t) = \sum_{j=1}^{\infty} \frac{e^{itj^2}}{ij^2} \qquad \text{Duistermaat 1991}$$

Fractal behavior of the graph.

- Graph on $[0, 2\pi]$ of Riemann's function $\mathfrak{R}(t) = \sum_{j\in\mathbb{Z}} \frac{e^{itj^2}-1}{ij^2}$:



- $\mathfrak{R}$ satisfies the multifractal formalism of Frisch-Parisi (Jaffard 96) is intermittent (Boritchev-Eceizabarrena-Da Rocha 19), its graph has no tangents (at the end Riemann was right!!) and has Haussdorf dimension $\leq \frac{4}{3}$ (Eceizabarrena 19)

- The theorem gives a non-obvious non-linear geometric interpretation for Riemman's function.

This first glimpse of topological fluid flows in experiment raises many questions:

Are all knots unstable?

Is Helicity conserved through reconnections?

Reconnections are also seen in superfluids and plasmas; are the topological dynamics universal?

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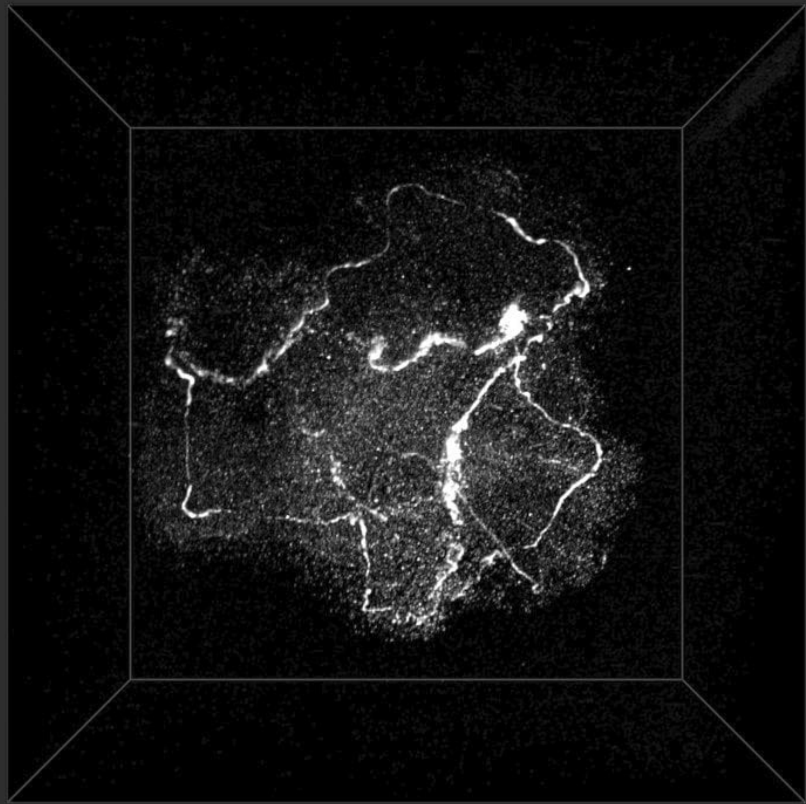


Construction of the Selfsimilar Solutions of the Binormal Flow

1.1 The Binormal Flow

We are interested in the evolution of a vortex filament in 3d. Therefore we consider a fluid that is moving according to Euler Equations such that its velocity field $u(x,t)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$ is irrotational except in a curve. This curve is parametrized by arc length as

$$(s,t) \longmapsto \chi(s,t) \in \mathbb{R}^3 \quad , \quad s \in \mathbb{R}.$$

The velocity field is then obtained using the Biot–Savart integral,

$$u(P) = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{\chi(s,t) - P}{|\chi(s,t) - P|^3} \wedge T(s,t) ds \tag{1.1}$$

with P a point outside of the curve,

$$T(s,t) = \chi_s(s,t),$$

and Γ denoting the strength of the filament.

Typical examples of vortex filaments are:

1. The straight line vortex: $\chi(s,t) = \chi(s) = (0,0,s)$, and from (1.1)
$$u = \frac{\Gamma}{2\pi} \frac{(-y,x,0)}{x^2+y^2}$$
2. The vortex ring $\chi(s,0) = (\cos s, \sin s, 0)$, [Sa].
3. The helical vortex, [LF], [Har], [AlKO].

In order to find the evolution of the filament for later times one has to compute the velocity field for points P in the curve. As we see from (1.1) this can not be easily done due to the singularity of the Biot–Savart integral. Different methods of desingularizing this integral can be found in the literature. In these lectures we shall follow the so called Localized Induction Approximation that it was first proposed by Da Rios in 1906 [DaR] (see also [Ri]).

The consequence of this approximation is that the filament moves in the direction of the binormal with a velocity that is proportional to the curvature. That is to say

$$\left\{ \begin{array}{rcl} \chi_t &=& \chi_s \wedge \chi_{ss} \\ \chi(s,0) &=& \chi_0(s). \end{array} \right. \tag{1.2}$$

Using the Frenet equations and calling (T, n, b) the Frenet frame, we have for (c, τ), the curvature and the torsion of χ, that

$$\begin{array}{rcccc} T_s &=& & cn & \\ n_s &=& -cT & & +\tau b \\ b_1 &=& & -\tau n. & \end{array} \tag{1.3}$$

Hence (1.2) can be written as

$$\left\{ \begin{array}{rcl} \chi_t &=& cb \\ \chi(s,0) &=& \chi_0(s). \end{array} \right. \tag{1.4}$$

A first consequence of (1.2) is that $|T(s,t)| = 1$. In fact, after differentiation in (1.2) we get for the tangent vector the evolution equation

$$\begin{cases} T_t &= T \wedge T_{ss} \\ T(s,0) &= T_0(s). \end{cases} \tag{1.5}$$

Notice that $T \wedge T_{ss} = J\, D_s\, T_s$ with J the complex structure in the unit sphere $\mathbb{S}^2$ and D_s the covariant derivative. As a consequence, (1.5) is also known as the Schrödinger map equation with target $\mathbb{S}^2$. This equation can be easily generalized by changing both the domain of definition and the target. In these lectures we will be interested just in the second possibility, changing $\mathbb{S}^2$ into the Hyperbolic plane $\mathbb{H}^2$: In this case (1.2) becomes

$$\chi_t = \chi_s \wedge_- \chi_{ss}, \tag{1.6}$$

and calling again $\chi_s = T$, (1.5) has to be changed into

$$T_t = T_s \wedge_- T_{ss}, \tag{1.7}$$

with

$$a \wedge_- b = a \wedge \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} b \quad , \quad (a,b) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{1.8}$$

Next we give a fundamental transformation that links the binormal flow (1.2), and its modification (1.6) with the 1d focusing (defocusing in the case of $\mathbb{H}^2$) cubic non–linear Schrödinger equation. In [Has] the “filament” function

$$\psi(s,t) = c(s,t)e^{i\int_0^s \tau(s',t)\, ds'} \tag{1.9}$$

is introduced and proved that if (c,τ) are the curvature and torsion of the filament $\chi(s,t)$ that evolves according to (1.4), then ψ is a solution of

$$\begin{cases} \psi_t &= i\left(\psi_{ss} + \dfrac{1}{2}\left(|\psi|^2 + A(t)\right)\psi\right), \\ \psi(s,0) &= \psi_0(s). \end{cases} \tag{1.10}$$

Here $A(t)$ is real function that it is determined from $\chi(0,t)$.

Notice that the case of a constant solution gives the circle. The example of the helix is easily obtained observing that (1.10) is invariant under the galilean transformations: If $N \in \mathbb{R}$ and ψ is a solution then

$$\psi_N(s,t) = e^{-itN^2+iNs}\psi(s-2Nt,t) \tag{1.11}$$

is also a solution. Finally notice that (1.10) has solitons solutions that can be easily obtained with the ansatz $\psi(s,t) = e^{it\omega}Q(s)$ with ω a real number and Q a real function. This fact together with galilean transformations allow for the existence of solutions such that $|\psi|^2$ behaves as a travelling wave solution. Therefore it is natural to ask about the possibility of existence of this type of solutions in real fluids. This was experimentally proved in [HB], see also [MHR] and more recently [AlKO].

It is interesting to observe that in the case of (1.6) a generalized curvature and torsion can be defined, and as a consequence, a similar equation as (1.10) is obtained with a defocusing non–linear potential (i.e. the negative sign in front of the cubic term instead of the positive one).

As we see Hasimoto transformation involves the Frenet frame. The calculation to obtain (1.10) is slightly more complicated in this way. It is more simple if a parallel frame is used. The argument goes as follows [Ko]. Consider the orthonormal frame (T, e_1, e_2) and the equations

$$\begin{aligned} T_s &= \qquad \alpha e_1 + \beta e_2 \\ e_{1s} &= -\alpha T \\ e_{2s} &= -\beta T. \end{aligned}$$

Then, from (1.5),

$$T_t = T \wedge (\alpha e_1 + \beta e_2)_s = \alpha_s e_2 - \beta_s e_1.$$

As a consequence, we have on one hand that

$$T_{st} = \alpha_t e_1 + \beta_t e_2 + \alpha e_{1t} + \beta e_{2t}, \tag{1.12}$$

and on the other

$$T_{ts} = \alpha_{ss} e_2 - \beta_{ss} e_1 - \alpha_s \beta T + \beta_s \alpha T. \tag{1.13}$$

Hence we need to compute $\langle e_{1t}, e_2 \rangle = -\langle e_{2t}, e_1 \rangle$. Following Hasimoto we compute its s–derivative

$$\begin{aligned} \langle e_{1t}, e_2 \rangle_s &= \langle e_{1ts}, e_2 \rangle + \langle e_{1t}, e_{2s} \rangle \qquad (1.14) \\ &= -\langle \alpha_t T, e_2 \rangle - \langle \alpha T_t, e_2 \rangle - \beta \langle e_{1t}, T \rangle \\ &= -\alpha \langle T_t, e_2 \rangle + \beta \langle e_1, T_t \rangle \\ &= -\alpha\alpha_s - \beta\beta_s = -\frac{1}{2}(\alpha^2 + \beta^2)_s. \end{aligned}$$

The last step is to define the complex function

$$\psi(s,t) = \alpha(s,t) + i\beta(s,t).$$

Then from (1.12)– (1.13)

$$\begin{array}{rcl} \alpha_t + \beta\langle e_{2t}, e_1\rangle & = & -\beta_{ss} \\ \beta_t + \alpha\langle e_{1t}, e_2\rangle & = & \alpha_{ss}, \end{array}$$

and from (1.14)

$$\begin{array}{rcl} \alpha_t + \dfrac{1}{2}\beta\left[(\alpha^2 + \beta^2) + A(t)\right] & = & -\beta_{ss} \\ \\ \beta_t - \dfrac{1}{2}\alpha\left[(\alpha^2 + \beta^2) + A(t)\right] & = & \alpha_{ss}, \end{array}$$

with $A(t)$ a real function. Finally we get

$$\psi_t = (\alpha + i\beta)_t = i\left\{(\alpha_{ss} + i\beta_{ss}) + \frac{1}{2}(\alpha + i\beta)(\alpha^2 + \beta^2 + A(t))\right\}$$

as desired. A similar calculation can be done in the case of (1.7), see for example [DeH].

1.2 Selfsimilar Solutions

In this section we will find the selfsimilar solutions of the binormal flow (1.2). Notice that the flow is invariant under the scaling[1] $\frac{1}{\lambda}\chi(\lambda s, \lambda^2 t)$. Hence we look for solutions that satisfy

$$\frac{1}{\lambda}\chi(\lambda s, \lambda^2 t) = \chi(s,t) \quad \forall\, \lambda > 0.$$

As a consequence such a χ can be written as

$$\chi(s,t) = \sqrt{t}G\left(s/\sqrt{t}\right) \quad t > 0 \tag{1.15}$$

for some $G(s)$. Plugging (1.15) in (1.2) we obtain that G has to be a solution of

$$\frac{1}{2}G - \frac{s}{2}G' = G' \wedge G''. \tag{1.16}$$

Calling $T(s) = G'(s)$ we obtain after differentiation:

$$-\frac{s}{2}T' = T \wedge T''.$$

[1] There are other possible scalings but this is the only one that preserves the arclength.

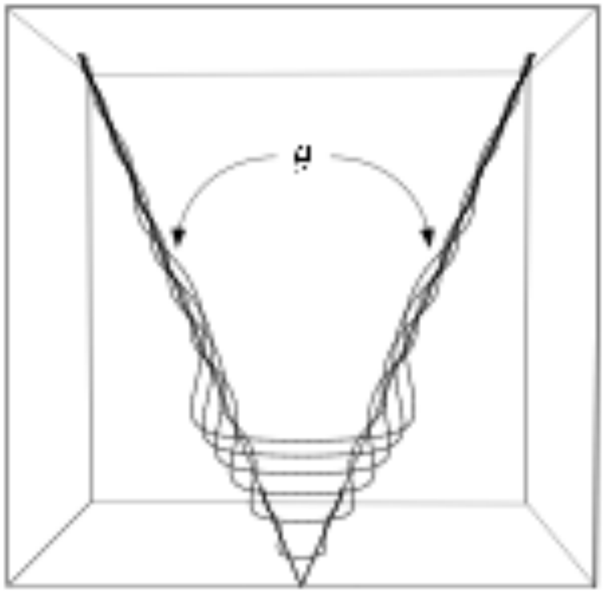
From Frenet equations we get

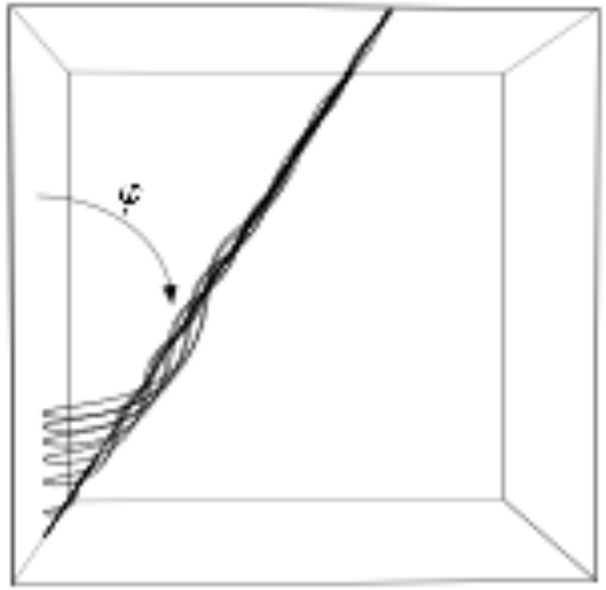
$$-\frac{s}{2}cn = T \wedge \left(c'n - c^2T + c\frac{\tau}{2}b\right).$$

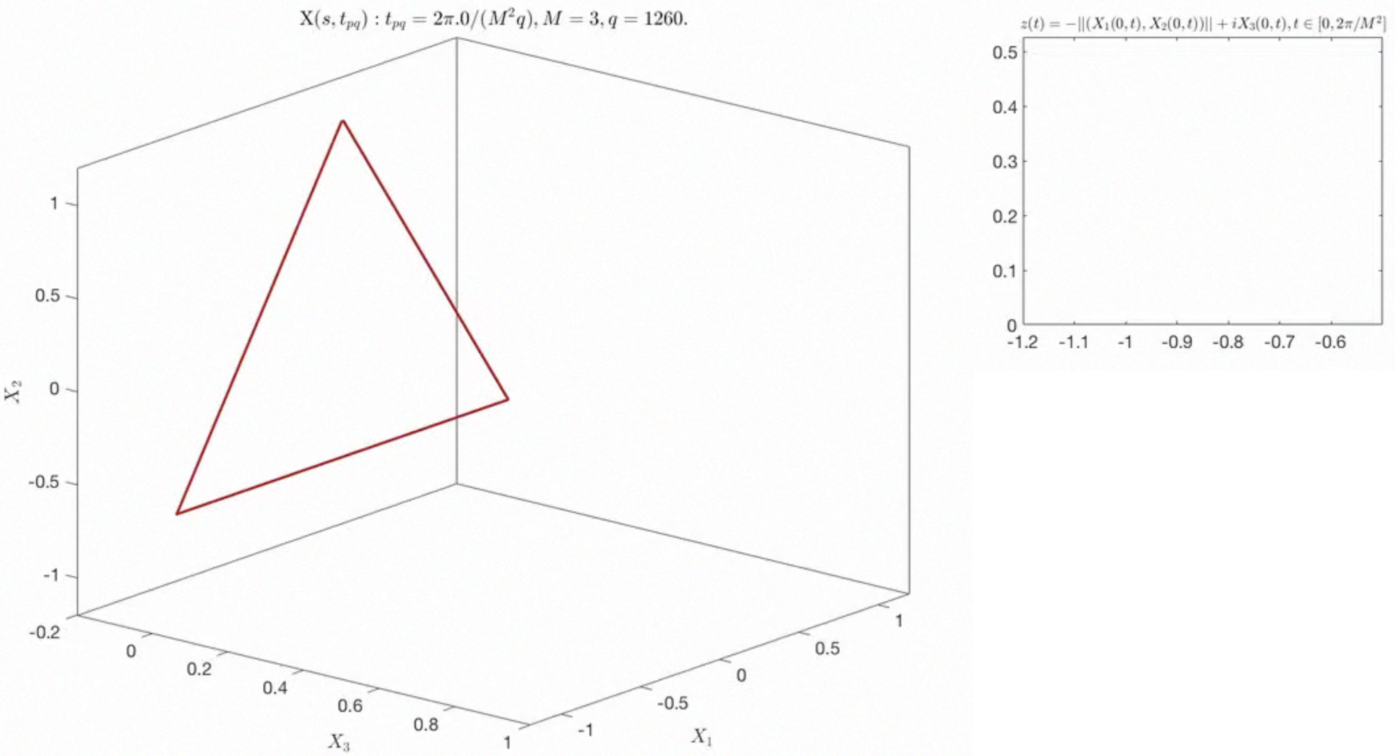
As a consequence ([Bu], [LD])

$$\begin{aligned} c &= a \\ \tau &= \frac{s}{2} \end{aligned}$$

for some $a \geq 0$.







$X(s,t_{pq}) : t_{pq} = 2\pi.0/(M^2q), M = 3, q = 1260.$
$z(t) = -\|(X_1(0,t), X_2(0,t))\| + iX_3(0,t), t \in [0, 2\pi/M^2]$
X_2
X_3
X_1

Therefore if (T_a, n_a, b_a) is the unique solution of Frenet equations (1.3) with $(c, \tau) = (a, s/2)$ and $(T_a(0), n_a(0), b_a(0)) = \mathbb{1}_{3\times 3}$ then G_a is determined by

$$\begin{cases} G_a' &= T_a \\ G_a(0) &= 2a(0,0,1). \end{cases} \tag{1.17}$$

This second identity in (1.17) follows from (1.16) and the fact that $G'(0) \wedge G''(0) = ab_a(0) = a(0,0,1)$.

It is easy to prove that $G_a(s)$ approaches a straight line as s tends to $+\infty$ and similarly for s going to $-\infty$. In fact, from (1.16) we have

$$\left(\frac{G_a}{s}\right)' = \frac{sG_a' - G_a}{s^2} = \frac{-2ab_a}{s^2}.$$

Notice that $|b_a(s)| = 1$. Therefore $\left(\frac{G_a}{s}\right)'$ is integrable in $\pm\infty$ and we can define

$$A_a^{\pm} = \lim_{s\to\pm\infty} \frac{G(s)}{s} \quad , |A_a^{\pm}| = 1.$$

Hence

$$G_a(s) \quad = \quad sA_a^+ + 2as \int_s^\infty \frac{b_a(s')}{s'^2}\, ds' \quad s > 0$$

$$G_a(s) \quad = \quad sA_a^- - 2as \int_{-\infty}^s \frac{b_a(s')}{s'^2}\, ds' \quad s < 0.$$

From these two identities and (1.17) it easily follows that (see [GRV])

$$\left|\chi_a(s,t) - sA_a^+ \mathbb{I}_{[0,\infty)}(s) - sA_a^- \mathbb{I}_{(-\infty,0]}(s)\right| \leq 2a\sqrt{t} \quad t > 0, \tag{1.18}$$

with

$$\chi_a(s,t) = \sqrt{t} G_a \left(s/\sqrt{t}\right), \tag{1.19}$$

and $\mathbb{I}$ denoting the characteristic function. Hence we have found a solution of (1.14) for $t > 0$ with

$$\chi_a(s,0) = \begin{cases} A_a^+ s & s > 0 \\ A_-^+ s & s < 0. \end{cases} \tag{1.20}$$

Notice that the binormal flow (1.2) is reversible in time: if $\chi(s,t)$ is a solution so is $\chi(-s,-t)$. Hence if we give as initial condition at time $t=1$

$$\chi(s,1) = G_a(s)$$

and go backwards in time we obtain from (1.19)– (1.20) that if $A_a^+ \neq A_a^-$ a singularity in the shape of a corner is developed at time $t=0$. Observe that $G_a(s)$ is real analytic because it is a solution of Frenet equations with $(c,\tau) = \left(a, \frac{s}{2}\right)$.

In order to prove that $A_a^+ \neq -A_a^-$, except in the trivial case $a=0$ and $A_0^+ = A_0^- = (1,0,0)$, we need the following geometric lemma.

Lemma 1. *[BV2] Assume* (T_j, n_j, b_j), $j = 1, 2, 3$ *are the components of the Frenet frame of a regular curve* $G(s)$ *with* $s \in \mathbb{R}$ *the arc length parameter and* $(T(0), n(0), b(0)) = \mathbb{1}_{3\times 3}$. *Then*

$$\begin{aligned} T_j(s) &= 1 - \frac{|\theta_j|^2}{2E(0)}, \\ c(s)\,(n_j(s) - ib_j(s)) &= -2\theta_j \overline{\theta}_j', \end{aligned}$$

for $j = 1, 2, 3$ *with* θ_j *the solution of*

$$\theta_j'' + \left(-\frac{c'}{c} + i\tau\right)\theta_j' + \frac{c^2}{4}\theta_j = 0;$$

with

$$\begin{aligned}(\theta_1(0), \theta_1'(0)) &= \left(0, \frac{c(0)}{\sqrt{2}}\right), \\ (\theta_2(0), \theta_2'(0)) &= \left(1, -\frac{c(0)}{2}\right), \\ (\theta_3(0), \theta_3'(0)) &= \left(i, \frac{c(0)}{2}\right),\end{aligned}$$

and

$$E(s) := \left|\frac{\theta_j'}{c}\right|^2 + \frac{|\theta_j|^2}{4} = E(0) = \frac{1}{2}.$$

In our case θ_j has to be a solution of

$$\theta_j'' + i\frac{s}{2}\theta_j' + \frac{a^2}{4}\theta_j = 0. \tag{1.21}$$

This equation can be easily integrated computing the Fourier transform of θ_j. After some lengthy calculations it can be proved see [GRV] that

$$A_a^+ = \left(A_{a1}^+, A_{a2}^+, A_{a3}^+\right) = \left(A_{a1}^+, -A_{a2}^+, -A_{a3}^+\right)$$

and $A_{a1} = e^{-\pi a^2/2}$. Recall that $|A_a^\pm| = 1$. Hence $A_a^+ \neq A_a^-$ if $a > 0$.

In fact, two linearly independent solutions of (1.21) can be taken as follows,

$$\begin{aligned} \beta_1(s) &= \int_{-\infty}^{\infty} e^{i(s\xi+\xi^2)} \frac{d}{d\xi}\left[e^{-i\frac{a^2}{2}\lg|\xi|} \mathbb{I}_{[0,+\infty)}(\xi) \right] d\xi, \\ \beta_2(s) &= \int_{-\infty}^{\infty} e^{i(s\xi+\xi^2)} \frac{d}{d\xi}\left[e^{-i\frac{a^2}{2}\lg|\xi|} \mathbb{I}_{(-\infty,0]}(\xi) \right] d\xi. \end{aligned} \tag{1.22}$$

Notice that the dispersive relation of β_1 and β_2 is

$$\phi(\xi) = \xi^2 - \frac{a^2}{2}\lg|\xi|, \tag{1.23}$$

which indicates that the non–linearity only makes a logarithmic correction to the free evolution.

From (1.22) and Lemma (1) the asymptotic behaviour of G_a and T_a can be obtained. The arguments, although lengthy, are rather elemental and can be found in [GRV], where it is proved the following theorem.

Theorem 1. *[GRV] Given $a \geq 0$ then*

$$\chi_a = \sqrt{t} G_a \left(s/\sqrt{t} \right),$$

with G_a defined by (1.17) *is a real analytic solution of* (1.2) *for $t > 0$. Moreover there exist A_a^+, A_a^-, B_a^+, B_a^- such that*

(i) $\left| \chi_a(s,t) - A_a^+ s \mathbb{I}_{[0,+\infty)}(s) - A_a^- s \mathbb{I}_{(-\infty,0]}(s) \right| \leq a\sqrt{t}$

(ii) *Let* $|s| > \max(2a, 4)$*, then the following asymptotics hold:*

$$\begin{aligned} G_a(s) &= A_a^{\pm} \left(s + 2\frac{a^2}{s} \right) - 4a\frac{n_a}{s^2} + O\left(\frac{1}{s^3} \right), \quad s \to \pm\infty, \\ T_a(s) &= A_a^{\pm} - 2a\frac{b_a}{s} + O\left(1/s^2 \right), \quad s \to \pm\infty, \\ (n_a - ib_a)(s) &= B_a^{\pm} e^{is^2/4 + ia^2 \lg |s|} + O(1/s), \quad s \to \pm\infty. \end{aligned}$$

(iii) $A_a^{\pm} = \left(A_{a1}^{\pm}, A_{a2}^{\pm}, A_{a3}^{\pm}\right)$ *are unitary vectors and*

$$A_{a1}^{+} = A_{a1}^{-} = e^{-\frac{a^2}{2}\pi} \quad ; \quad A_{2a}^{+} = -A_{2a}^{-} \quad ; \quad A_{3a}^{+} = -A_{3a}^{-} \quad ; \quad \langle A_a^{\pm}, B_a^{\pm} \rangle = 0;$$

(iv) $\sin\frac{\theta}{2} = A_{1a}^{\pm} = e^{-\frac{c_0^2}{2}\pi}$ *and* $\sin\phi = \dfrac{A_2}{\sqrt{1 - A_2^2}}$ *with* θ *the angle between* A_a^+ *and* $-A_a^-$*, and* ϕ *the angle between the plane that contains* A_a^+ *and* A_a^- *and the vector* $b_a(0) = (0,0,1)$*.*
Recall that $\chi(0,t) = 2a\sqrt{t}(0,0,1)$*.*

(v) *There exist* a_0 *and* a_1 *with* $0 < a_0 \le a_1 < +\infty$ *such that if* $a < a_0$ *then* χ_a *has no self–intersections and if* $a_1 \le a$*,* χ_a *has finitely many self–intersections.*

Another approach: a complete integrable system

$G_a(s) = X(s)$; $a = c_0$ $\quad + \frac{1}{2} G_a - \frac{1}{2} s G_a' = c_0 b(s)$

$$X'''(s) = T''(s) = (c_0 n)' = c_0 n' = -c_0^2 T + \frac{s}{2} c_0 b(s)$$

$$= -c_0^2 X' + \frac{s}{2}\left(\frac{1}{2} X - \frac{1}{2} s X'\right)$$

$$\boxed{X''' + \left(c_0^2 + \frac{s^2}{4}\right) X' - \frac{s}{4} X = 0}$$

$$X(0) = G_a(0) = \frac{1}{2} c_0 (0,0,1)$$

$$X'(0) = T(0) = (1,0,0)$$

$$X''(0) = c_0 n(0) = c_0 (0,1,0)$$

$X = (x_1, x_2, x_3)$: x_1 odd

x_2, x_3 even

$A_1^+ = -A_1^-$ $\quad A_2^+ = A_2^-$ $\quad A_3^+ = A_3^-$

$$\hat{X}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(s) e^{-is\xi} \mathrm{d}s \quad \Longleftrightarrow \quad X(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{X}(\xi) e^{is\xi} \mathrm{d}\xi, \tag{41}$$

then, $\hat{X}(\xi)$ satisfies

$$\xi \hat{X}''(\xi) + 3\hat{X}'(\xi) + 4\xi^3 \hat{X}(\xi) - 4c_0^2 \xi \hat{X}(\xi) = 0. \tag{42}$$

This is a singular regular equation with indicial equation

$$r(r-1) + 3r = 0, \qquad r = 0 \quad r = -2 \tag{43}$$

$\hat{x}_1 \sim \delta' \quad \xi \sim 0$

Let us define now $\hat{Z}(\xi^2) := \xi^2 \hat{X}(\xi) \iff \hat{Z}(\eta) := \eta \hat{X}(\sqrt{\eta})$, where $\eta = \xi^2 > 0$; note that $\hat{Z}(\xi)$ exists always at $\xi = 0$, because $\xi^2 \delta'(\xi) = 0$, and $\xi^2 \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \ln|\xi| = -1$, so $\hat{Z}(\xi)$ is bounded around the origin. Moreover, from (42), we get

$$\boxed{\hat{Z}''(\eta) + \left(1 - \frac{c_0^2}{\eta}\right) \hat{Z}(\eta) = 0, \quad \eta > 0.} \tag{45}$$

This is a particular case of the so-called Coulomb wave equation (take $l = 0$ in Olver et al. 2010, 33.2(i)). Observe that, if $\hat{X}$ is an analytic solution of (42), i.e., $r = 0$ in (43), then $\hat{X}(0)$ is finite and $\hat{Y}(\eta) := \eta \hat{X}(\sqrt{\eta})$ is analytic in $\eta \geq 0$, with $\hat{Y}(0) = 0$. Also, $\hat{Y}(\eta)$ solves (45), and

$$\hat{X}(0) = \lim_{\xi \to 0} \frac{\hat{Y}(\xi^2)}{\xi^2} = \hat{Y}'(0). \tag{46}$$

Note that $\hat{Y}$ corresponds to the so-called regular Coulomb wave functions [take $l = 0$ in Olver et al. 2010, 33.2(ii)]. Moreover, we have the following result.

Lemma 3.3 *(i) If $\hat{Y}$ solves (45) with $\hat{Y}(0) = 0$, then it is analytic in $\eta \geq 0$, and $|\hat{Y}(\eta)|$ and $|\hat{Y}'(\eta)|$ are bounded in $\eta \in [0, \infty)$.*

(ii) If $\hat{Z}$ solves (45) with $\hat{Z}(0) \neq 0$, then $|\hat{Z}(\eta)|$ is bounded in $\eta \in [0, \infty)$ while $|\hat{Z}'(\eta)|$ is bounded in $\eta \in [\epsilon, \infty)$ for all $\epsilon > 0$. Moreover, for $0 < \eta$ small, we have

$$\hat{Z}'(\eta) = c_0^2 \ln(\eta) + \mathcal{O}(1).$$

Finally, when $\eta \to 0^+$, we have trivially

$$\lim_{\eta\to 0^+} E(\eta) = (\hat{Y}')^2(0) - c_0^2 \lim_{\eta\to 0^+} \frac{\hat{Y}^2(\eta)}{\eta}$$
$$= (\hat{Y}')^2(0) - c_0^2 \lim_{\eta\to 0^+} \frac{2\hat{Y}(\eta)\hat{Y}'(\eta)}{1} = (\hat{Y}')^2(0).$$

Therefore, for some other positive constant C, $|E(\eta)| \le C$, for all $\eta \in [0, \infty)$, from which we conclude the boundedness of $|\hat{Y}(\eta)|$ and $|\hat{Y}'(\eta)|$, and

$$E(\infty) = E_0 + \int_0^{+\infty} \frac{c_0^2}{\eta^2} \hat{Y}^2(\eta)\,\mathrm{d}\eta,$$

which gives (i).

With respect to (ii), it follows from the Frobenius–Fuchs theorem, because the indicial equation for (45) is $r(r-1) = 0$. Therefore, the solutions are $r = 0$ and $r = 1$, which differ in an integer. In fact, the case $r = 0$ leads to the so-called irregular Coulomb wave functions [Olver et al. 2010, 33.2(iii)]. Also, note that, from the above arguments, we know that $\hat{Z}$ is bounded around the origin. Hence, from

$$\hat{Z}'(\eta) = \hat{Z}'(1) - \int_\eta^1 \hat{Z}''(\tau)\,\mathrm{d}\tau,$$

the result can be easily obtained. □

It is straightforward to express $\hat{\mathbf{n}}(\xi)$ and $\hat{\mathbf{b}}(\xi)$ in terms of $\hat{\mathbf{Y}}(\eta)$, $\eta > 0$:

$$\mathbf{T}'(s) = c_0\mathbf{n}(s) \Longrightarrow c_0\hat{\mathbf{n}}(\xi) = i\xi\hat{\mathbf{T}}(\xi) = -\xi^2\hat{\mathbf{X}}(\xi) = -\hat{\mathbf{Y}}(\xi^2) = -\hat{\mathbf{Y}}(\eta),$$

$$c_0\mathbf{b}'(s) = -\frac{s}{2}\mathbf{X}''(s) \Longrightarrow c_0 i\xi\hat{\mathbf{b}}(\xi) = \frac{i}{2}\frac{\mathrm{d}(\xi^2\hat{\mathbf{X}}(\xi))}{\mathrm{d}\xi} = \frac{i}{2}\frac{\mathrm{d}(\hat{\mathbf{Y}}(\xi^2))}{\mathrm{d}\xi} = i\xi\hat{\mathbf{Y}}'(\xi^2)$$

$$\Longrightarrow c_0\hat{\mathbf{b}}(\xi) = \hat{\mathbf{Y}}'(\xi^2) = \hat{\mathbf{Y}}'(\eta),$$

where we have differentiated (39) in the last expression. Then,

$$\boxed{c_0(-\mathbf{n} + i\mathbf{b})^\wedge(\xi) = (\hat{\mathbf{Y}} + i\hat{\mathbf{Y}}')(\xi^2) = (\hat{\mathbf{Y}} + i\hat{\mathbf{Y}}')(\eta).} \qquad (63)$$

Hyperbolic case: $A_1 = e^{c_0^2 \frac{\pi}{2}}$.

$$\widehat{W}'' + \widehat{W}\left(1 + \frac{c_0^2}{\eta}\right) = 0 \Leftrightarrow \eta\widehat{W}'' + \eta\widehat{W} + c_0^2\widehat{W} = 0, \tag{55}$$

with

$$\widehat{W}_1(0) = 0,\ \widehat{W}_1'(0) = \lim_{\eta\to 0}\frac{\widehat{W}_1(\eta)}{\eta} = iA_1c_0^2. \tag{56}$$

On the other hand, the Laplace transform of $\widehat{W}_1(\eta)$,

$$\mathcal{L}(t) = \mathcal{L}\{\widehat{W}_1(\eta)\} = \int_0^\infty \widehat{W}_1(\eta)e^{-t\eta}d\eta,\ t > 0, \tag{57}$$

satisfies

$$t^2\mathcal{L}'(t) + 2t\mathcal{L}(t) + \mathcal{L}'(t) - c_0^2\mathcal{L}(t) = 0. \tag{58}$$

Furthermore,

$$\mathcal{L}(0) = \int_0^\infty \widehat{W}_1(\eta)d\eta = 2\int_0^\infty \xi^3 \hat{X}_1(\xi)d\xi = \int_{-\infty}^\infty \xi^3 \hat{X}_1(\xi)d\xi = iX_1'''(0) = ic_0^2, \tag{59}$$

where we have used the fact that $\hat{X}_1$ is odd. Rewriting (57) as

$$\mathcal{L}(t) = \int_0^\infty \widehat{W}_1(\eta)e^{-t\eta}d\eta = \frac{1}{t}\int_0^\infty \widehat{W}_1'(\eta)e^{-t\eta}d\eta = \frac{\widehat{W}_1'(0)}{t^2} + \frac{1}{t^2}\int_0^\infty \widehat{W}_1''(\eta)e^{-t\eta}d\eta,$$

we have

$$t^2\mathcal{L}(t) = \widehat{W}_1'(0) + \int_0^\infty \widehat{W}_1''e^{-t\eta}d\eta,$$

which, as $t \to \infty$, becomes

$$\lim_{t\to\infty} t^2\mathcal{L}(t) = \widehat{W}'(0). \tag{60}$$

Hence, from (58)-(59), we have an initial value problem whose solution $\mathcal{L}(t)$ satisfies

$$\lim_{t\to\infty} t^2\mathcal{L}(t) = \lim_{t\to\infty} t^2\frac{\mathcal{L}(0)}{1+t^2}e^{c_0^2 \arctan(t)} = ic_0^2 e^{c_0^2\pi/2}.$$

Combining this with (56) and (60), we conclude that

$$ic_0^2 A_1 = \widehat{W}'(0) = ic_0^2 e^{c_0^2\pi/2} \implies A_1 = e^{c_0^2\pi/2}. \tag{61}$$

The above approach works the same for the Euclidean case as well; hence, the corresponding expression for A_1 can be obtained.

An introduction to 1d cubic NLS

Cargese 21 de junio, 2023

$$\partial_t u = i\left(u_{xx} \pm |u|^2 u\right)$$

Examples

$Q_\omega \in \mathbb{R}$ $\quad u = e^{it\omega} Q_\omega(x)$; $\quad i\omega Q_\omega = i\left(Q_\omega'' \pm |Q_\omega|^2 Q_\omega\right)$

$\omega = 1$

Scaling $\lambda > 0$ $\quad u_\lambda(x,t) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$ is also a solution.

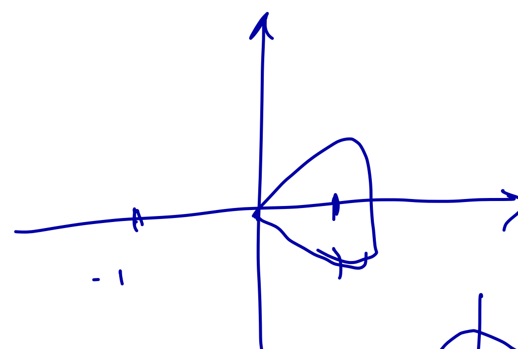
$i\omega^{3/2} Q = i\left(\omega^{3/2} Q'' \pm \omega^{3/2} Q^3\right)$

$Q_\omega = \sqrt{\omega}\, Q\left(\sqrt{\omega}\, x\right)$

$$Q'' - Q \pm Q^3 = 0$$

$Q = \operatorname{sech} x$ soliton

$Q = \operatorname{tgh} x$ kink



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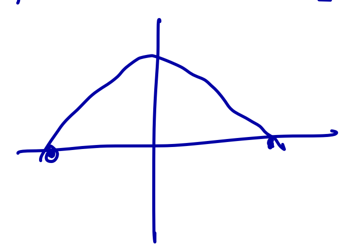

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Simmetries

translations in x and t

Dilations $\quad u_\lambda(x,t) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right) \quad \lambda > 0 \quad u_\lambda(x,0) = \frac{1}{\lambda} u\left(\frac{x}{\lambda}, 0\right)$

Galilean transformations

$N > 0$

$$u(x,t) = e^{-i t N^2 + i N x} u(x - 2Nt, t)$$

$u_N = c_N e^{i \phi_N} \qquad u = c e^{i\phi}$

$$u^N(x,0) = e^{iNx} u(x,0)$$

$c_N = c \qquad \tau_N = \tau + N$

$u(x,0) = c_0 \delta \qquad u_\lambda(x,0) = u(x,0) \qquad \forall \lambda > 0$

$$u^N(x,0) = u(x,0) \qquad \forall N \in \mathbb{R}$$

Exercise: $u_N = u \quad \forall N \qquad u(x,t) = \frac{a}{\sqrt{it}} e^{i\frac{x^2}{4t} \pm i a^2 \lg t}$

The IVP

$$\begin{cases} u_t = i\,(u_{xx} \pm |u|^2 u) \\ u(x,0) = u_0(x) \end{cases}$$

$$u(t) = e^{it\partial_x^2} u_0 \pm i \int e^{i(t-\tau)\partial_x^2} |u|^2 u(\tau)$$

$$\Downarrow \; ?$$

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + \int_0^t \||u|^2 u\|_{L^2}\, d\tau$$

$$\| \qquad \int_0^T \left(\int |u|^6\right)^{1/2} d\tau \leq T^{1/2} \left(\int_0^T |u|^6\, dx\, d\tau\right)^{1/2}$$

$$X = \left\{ F : \sup_{|t| \leq T} \|F\|_{L^2} + \left(\int_0^T |F|^6\, dx\, dt\right)^{1/6} < +\infty \right\}.$$

$$\|F\|_X \leq C\|u_0\|_{L^2} + C T^{1/2} \|F\|^3 \leq \frac{M}{2} + \underbrace{C T^{1/2} M^2}_{1/2} M$$

Given $u_0 \in L^2(\mathbb{R})$ $\exists T$ s.t. $C T^{1/2} \|F\|^2 \leq$

Important remark $\int_0^T \| e^{it\partial_x} u_0 \|_{L^6}^3 \, d\tau$

Lemma $\left(\iint |e^{it\partial_x^2} u_0|^6 \, dx\, dt \right)^{1/6} \le C \left(\int |u_0|^2 \right)^{1/2}$ C optimal is known

Optimizers are

Foschi the Gaussians

$$|e^{it\partial_x^2} u_0|^2 = \iint e^{it(\xi_1^2 - \xi_2^2) + ix(\xi_1 - \xi_2)} \hat{u}_0(\xi_1) \overline{\hat{u}_0(\xi_2)} \, d\xi_1 \, d\xi_2$$

$\xi^2 - \eta^2 =$

$$= \iint e^{it\eta_1 + ix\cdot\eta_2} g \, d\eta_1 \, d\eta_2 \qquad g(\eta_1, \eta_2) = \frac{\hat{u}_0(\xi_1) \overline{\hat{u}_0(\xi_2)}}{2|\xi_1 - \xi_2|}$$

$$\|u\|_{L^6} = \| |u|^2 \|_{L^3}^{1/2} \overset{H-Y}{\le} \left(\iint |g|^{3/2} d\eta_1 d\eta_2 \right)^{2/3 \cdot 1/2} = \left[\iint \frac{1}{(2|\xi_1 - \xi_2|)^{1/2}} |\hat{u}_0(\xi_1)|^{3/2} |\hat{u}_0(\xi_2)|^{3/2} d\xi_1 d\xi_2 \right]^{1/3}$$

$$= \int \left(\int \underbrace{K(\xi_1 - \xi_2) f(\xi_1) d\xi_1}_{L^4} \right) \underbrace{f(\xi_2)}_{L^{4/3}} d\xi_2$$

$$\left(\int \left| \int \frac{1}{|\xi_1 - \xi_2|^{1/2}} f(\xi_1) d\xi_1 \right|^4 \right)^{1/2} \le C \left(\int |f|^{4/3} \right)^{3/4}$$

H-L-S.

Carpere 21 de junio 2023

$$\tau(F) = e^{it\partial_x^2} u_0 \pm i\int_0^t e^{i(t-\tau)\partial_x^2} |F|^2 F \, d\tau$$

$$\|\tau(F)\|_{L^2} \le \|u_0\|_{L^2} + \int_0^T \|F\|_{L^6}^3 \, d\tau$$

$T \le 1$

- $$\left(\int_0^T \|\tau(F)\|_{L^6}^6 \, d\tau\right)^{1/6} \le C\|u_0\|_{L^2} + C\,T^{1/2} \|F\|_6^2 \|F\|$$

$$\le C\,T^{1/2}\left(\|u_0\|_{L^2} + \int_0^T \|F\|_{L^6}^3 \, d\tau\right)$$

Given $u_0 \in L^2$

$$X = \left\{ F(x,t) : \left(\int_0^T \|F(\cdot,\tau)\|_{L^6}^6 \, dx\, d\tau\right)^{1/6} < \overset{M}{\overset{\|}{2C\|u_0\|_{L^2}}} \right\}$$

$\tau : X \to X$

$$\le C\|u_0\|_{L^2} + C\|u_0\|_{L^2}$$

$T^{1/2} M^2 < 1/2$

The solution can be continued for all time because the L^2 norm is preserved. This is because if we differen-

tiate the equation $v = u_x$

$$i \partial_t u_x = \partial_x^2 (\overset{w}{u_x}) + \partial_x (|u|^2 u)$$

$$\partial_x |u|^2 u = \partial_x u \bar{u} u = 2u \partial_x u \bar{u} + u^2 \overline{\partial_x u}$$

$$i \partial_t w = \partial_x^2 w + 2|u|^2 w + u^2 w$$

which can be solved as before. Hence $\partial_x u_0 \in L^2$ then $\partial_x u(x,t) \in L^2$ and the integration by parts

Conservation laws.

(i) $\bar{u}$ $\qquad u_t \bar{u} = i(u_{xx}\bar{u} \pm |u|^2 u \bar{u})$

$$= i\{\partial_x(u_x\bar{u} - u\bar{u}_x) \pm |u|^4\}.$$

Hence $\frac{d}{dt}|u|^2 = -2\,\mathrm{Im}\,\partial_x(u_x\bar{u}) = \text{flux}$

Periodic boundary conditions $L^2([0,2\pi))$

$L^2(\mathbb{R})$

$$\frac{d}{dt}\int |u(x,t)|^2 = 0$$

$$\mathrm{Im}\int \bar{u}u_x \; dx$$

(ii) $x\bar{u}$

(iii) $\bar{u}_t$ $\qquad \int |u_x|^2 \mp \frac{1}{2}|u|^4 \; dx$

What happens if $u_0 \in L^2([0,2\pi))$. $\qquad u(x,0) = u_0(x)$

Assume $\hat{u}_0(k) = \begin{cases} 1 & |k| \leq N. \\ 0 & \text{otherwise}. \end{cases}$

$$u = e^{it\partial_x^2} u_0(x) = \sum_{j=-N}^{N} e^{itj^2 + ijx}$$

$$\|u\|_{L^6}^6 = \int_0^{2\pi}\int_0^{2\pi} |u \bar{u} u|^2 \, dx dt$$

$$u\bar{u}u = \sum_{j_1, j_2, j_3} e^{it(j_1^2 - j_2^2 + j_3^2) + i(j_1 - j_2 + j_3)x} = \sum_{j_1 - j_2 + j_3 = \mathbf{j}} e^{ix\mathbf{j}} \sum e^{it(j_1^2 - j_2^2 + j_3^2)}$$

$$\int_0^{2\pi} |u\bar{u}u|^2 \, dx = \sum_m \Big| \sum_{j_1 - j_2 + j_3 = m} e^{it(j_1^2 - j_2^2 + j_3^2)} \Big|^2$$

$$j_1^2 - j_2^2 + (\mathbf{j} - j_1 + j_2)^2 = j_1^2 - j_2^2 + \mathbf{j}^2 + j_1^2 + j_2^2 - 2j_1\mathbf{j} + 2j_2\mathbf{j} - 2j_1j_2$$

$$= 2j_1^2 - 2j_1j_2 - 2\mathbf{j}(j_1 - j_2)$$

$$= 2j_1(j_1 - j_2) - 2\mathbf{j}(j_1 - j_2) = 2(j_1 - \mathbf{j})(j_1 - j_2) = \mathbf{m}$$

$$= \sum_j \int_0^{2\pi} \left| \sum_m e^{itm} \# r_m \right|^2 dt.$$

$$\# r_m = \mathrm{card}\{ j_1, j_2 : 2(j_1 - j)(j_1 - j_2) = m \}$$

(*) $$= \sum_j \sum_m (\# r_m)^2 \gg \sum_j \sum_m d(k) \quad N^3 \log N = (N^{1/2})^6 \log N.$$

$$(N^{1/2}) = \| u_0 \|_{L^2}$$

Intermittency.

<u>Exercise</u> : Proof (*) knowing that

$$\lim_{N\to\infty} \frac{1}{N^2} \# \left\{ \frac{p}{q} : (p,q)=1 \quad q \le N \right\} = \frac{6}{\pi^2}$$

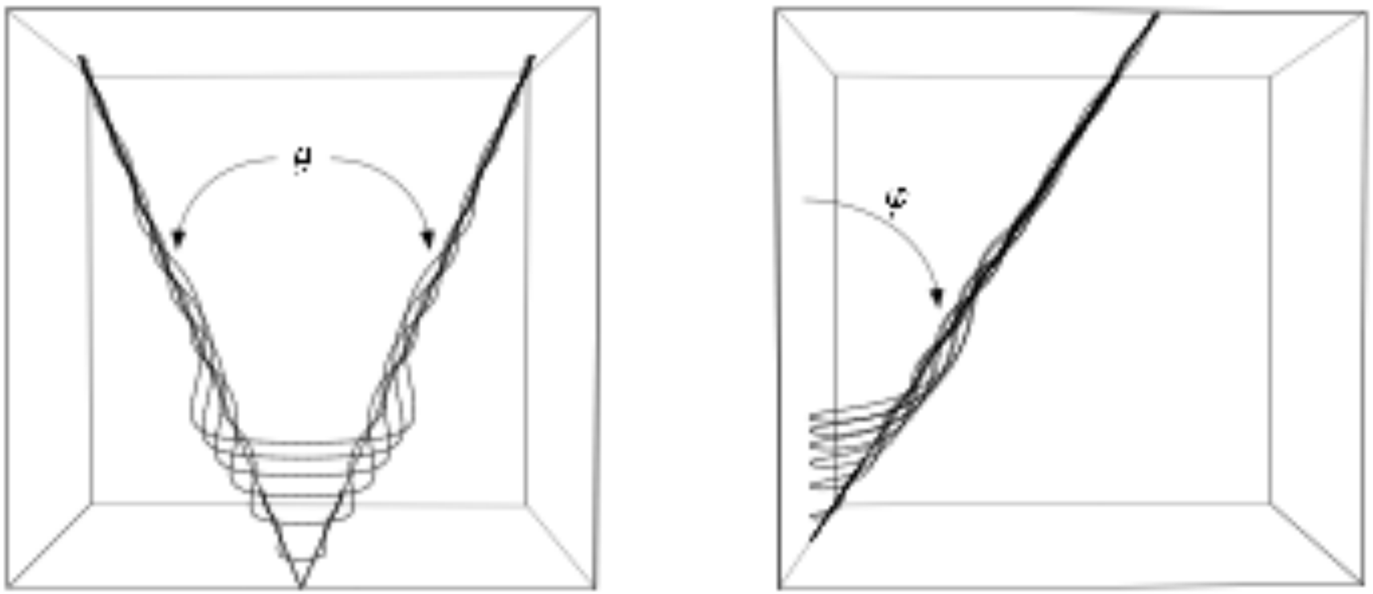
In other words $\# \left\{ \frac{p}{q} \ (p,q)=1 \ q \le N \right\} \simeq N^2$

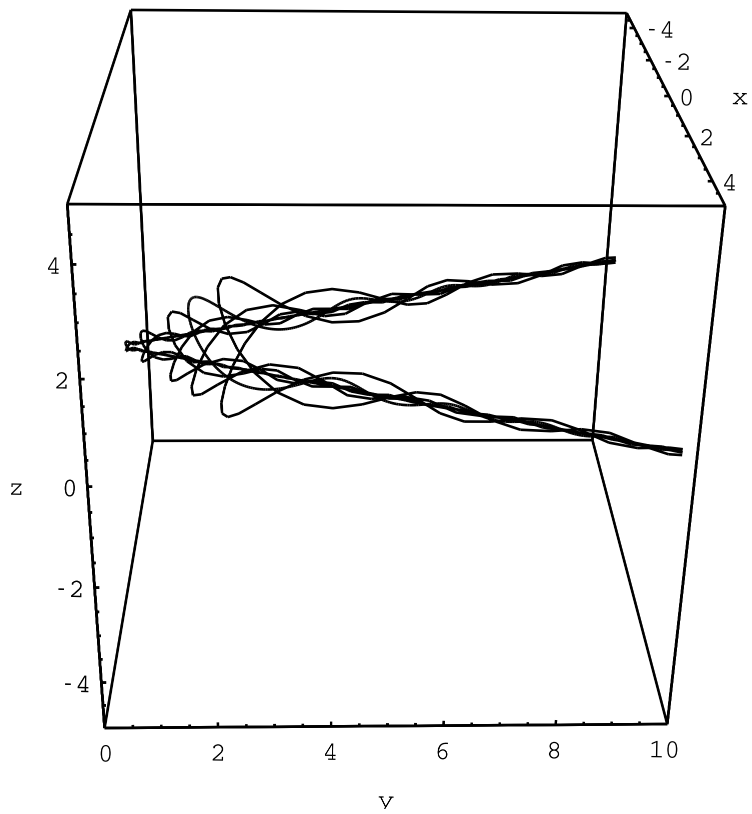
$$2pq = k \qquad \begin{aligned} p &= (j_1 - m) \\ q &= (j_1 - j_2) \end{aligned}$$

$$\sum_{n} \# r_n = \sum_{k \le N^2} d(k) = N^2 \log N^2$$

$$\text{for all } x \geq 1, \sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

Stability of one corner


a.
b.



Theorem.– (with V. Banica) The self-similar solutions are stable. In particular, the creation/annihilation of a corner is stable.

In this lecture we shall present some results concerning the stability of the self–similar solutions that we described in the previous lecture. As we saw the equation (1.2) is transformed thanks to the Hasimoto transformation (1.9) into the 1–d cubic NLS equation (1.10). In this setting the particular solutio χ_a of (1.2) becomes

$$\psi_a(s,t) = \frac{a}{\sqrt{it}} e^{is^2/4t} \qquad 0 < t \quad a \geq 0, \tag{1.24}$$

solution of

$$\begin{cases} \partial_t \psi = i\left(\partial_s^2 \psi + \frac{1}{2}\left(|\psi|^2 + A(t)\right)\psi\right), & A(t) = -\frac{a^2}{t}, \\ \psi(s,0) = a\delta. \end{cases} \tag{1.25}$$

As we see we are considering very rough initial conditions. In fact, the standard arguments to treat NLS equations, Strichartz estimates and Bourgain spaces can not be used in a straightforward way to treat rough data as the δ–function. We refer the reader to the survey paper [BV2] for a detailed study of this question.

An important observation is that neither the initial data in (1.25) nor $\psi_a(s,t)$ given in (1.24) belong to $L^2(\mathbb{R})$, and in fact to any Sobolev space $H^s(|\mathbb{R})$, $s \geq 0$. Therefore although the 1d cubic NLS is a complete integrable system none of the infinitely many conservation laws is useful in this case.

However, there is a natural energy asociated to the family of solutions ψ_a. This energy is easily expressed when we write ψ_a in selfsimilar variables. That is to say, when we perform to the equation (1.25) the so called pseudoconformal transformation. For simplicity of the exposition we will make a simple change of scale that allows us to write (1.25) as

$$iu_t + u_{ss} \pm \left(|u|^2 - \frac{a^2}{t} \right) u = 0. \tag{1.26}$$

Notice that in (1.26) we also considered the defocusing case. This is because in what follows the arguments we use work equally well in both cases.

Then, we define v through the transformation

$$u(s,t) = \frac{e^{is^2/4t}}{\sqrt{t}} \overline{v}\left(\frac{s}{t}, \frac{1}{t}\right), \tag{1.27}$$

so that v becomes a solution of

$$iv_t + v_{ss} \pm \frac{1}{t}\left(|v|^2 - a^2\right) v = 0, \quad 0 < t, \tag{1.28}$$

and ψ_a is transformed into

$$v_a = a \qquad 0 < t, \tag{1.29}$$

that is a particular solution of (1.28). Hence we are interested in the stability of v_a. Notice that (1.27) transforms $t = 0$ into $t = +\infty$. Therefore to construct a solution u of (1.26) in the time interval $[0,1]$ is equivalent to find a solution of v in $[1,\infty]$. Moreover the limit of $u(t)$ at $t \to 0^+$ becomes the limit of $v(t)$ for $t \to \infty$. As a consequence the IVP (1.26) is related to the long time behaviour of $v(t)$, and in particular it will be relevant to know wether or not perturbations of the particular solutions $v_a = a$ scatter. As we know the scattering properties are mainly of two types. Firstly the construction of the Wave Operator and secondly the Asymptotic Completeness of the scattering operator. In this lecture we shall deal with the first question and in the next one with the second one.

The first important remark about (1.28) is that there is a natural energy:

$$E(t) = \frac{1}{2} \int |v_s(t)|^2 \, ds \mp \frac{1}{4t} \int \left(|v(t)|^2 - a^2\right)^2 \, ds \tag{1.30}$$

that satisfies

$$E'(t) = \pm \frac{1}{4t^2} \int \left(|v|^2 - a^2\right) \, ds. \tag{1.31}$$

In the defocusing situation (negative sign in the non–linear potential in (1.28)) this is sufficient to construct a global solution of (1.28) such that $E(t) < +\infty$ and $\|v(t) - a\|_{L^2}$ is finite. Then (1.30), (1.31) give an orbital stability result of the particular solution $v_a = a$. The details can be found in [BV1]. As we saw in Lecture 1 the defocusing case is related to the Schrödinger map onto $\mathbb{H}^2$.

Let us write

$$v = w + a, \tag{1.32}$$

so that if we want v to be a solution of (1.28), w has to solve

$$iw_t + w_{ss} \pm \frac{1}{t}\left(|w+a|^2 - a^2\right)(w+a) = 0. \tag{1.33}$$

The linear part of the potential is

$$\pm \frac{a^2}{t}\left(w + \overline{w}\right). \tag{1.34}$$

The term $\frac{a^2}{t}w$ is easily cancelled using an integrating factor. This suggests to define the new unknown

$$z(s,t) = e^{\mp i a^2 \lg t} w \tag{1.35}$$

that has to be a solution of

$$iz_t + z_{ss} \mp \frac{a^2}{t} e^{\pm 2ia^2 \lg t}\overline{z} + \frac{1}{t}F(z, \overline{z}), \tag{1.36}$$

where $F(z,\overline{z})$ is the non–linear potential, and it involves quadratic and cubic terms in z and $\overline{z}$.

Hence given an asymptotic state u_+ we look for a solution of (1.36) for $t > 1$ such that

$$\left\| z(t) - e^{it\partial_s^2} u_+ \right\|_2 \to 0 \quad \text{as} \quad t \to +\infty.$$

This can be done by finding a fixed point for the operator

$$Az = e^{it\partial_s^2} u_+ \pm i \int_t^\infty e^{i(t-\tau)\partial_s^2} \left(\frac{a^2}{\tau} e^{\mp 2ia^2 \lg \tau} \overline{z}(\tau) + F(z, \overline{z}) \right) d\tau. \tag{1.37}$$

The delicate part is the one in the Duhamel's integral given by the linear term $\overline{z}(\tau)$. We sum and substract $\overline{e^{i\tau\partial_s^2} u_+}$ and therefore if we work in a space of functions $z(s,t)$ such that

$$\left\| z(t) - e^{it\partial_s^2} u_+ \right\|_{L^2} = O(t^{-\alpha}) \tag{1.38}$$

for some $\alpha > 0$, we can expect to obtain a solution $Az = z$ by the classical Picard iteration. Then the key point is to give a good L^2–estimate of the oscillatory integral

$$\int_t^\infty e^{i(t-\tau)\partial_s^2} e^{-i\tau\partial_s^2} \overline{u}_+ \, \frac{d\tau}{\tau^{1 \pm 2ia^2}}. \tag{1.39}$$

$$\int_t^\infty e^{i(t-\tau)\partial_s^2} e^{-i\tau\partial_s^2} \overline{u}_+(s) \, \frac{d\tau}{\tau}$$

$$= \int_t^\infty \int_{-\infty}^\infty e^{-i(t-2\tau)\xi^2 + is\xi} \, \hat{\bar{u}}_+(\xi) \, d\xi \, \frac{d\tau}{\tau}$$

$$= \frac{i}{2t} \int_{-\infty}^\infty e^{it\xi^2 + is\xi} \, \frac{\hat{\bar{u}}_+(\xi)}{\xi^2} + \frac{i}{2} \int_t^\infty \frac{d\tau}{\tau^2} \int e^{-i(t-2\tau)\xi^2 + is\xi} \, \hat{\bar{u}}_+(\xi) \, \frac{d\xi}{\xi^2}$$

where the above calculations make sense as long as $\dfrac{\hat{u}_+(\xi)}{\xi^2}$ is integrable.

Theorem 2. *Let $t > 1$ and $m \in \mathbb{N}^*$. There exists a constant $a_0 > 0$ such that for all $a < a_0$ and for all u_+ small in $\dot{H}^{-2} \cap H^m \cap W^{m,1}$ with respect to a_0, equation (2.5) has a unique solution*

$$v - v_1 \in \mathcal{C}\left([t, \infty), H^m(\mathbb{R})\right),$$

satisfying as t goes to infinity

$$\|v(t) - v_1(t)\|_{L^2} \leq \frac{c}{t^{1/2}} \quad , \quad \left\|\partial_s^k (v - v_1)(t)\right\|_{L^2} \leq \frac{c}{t}.$$

Here v_1 is,

$$v_1 = a + e^{\pm i a^2 \lg t} e^{it\partial_s^2} u_+,$$

$\dot{H}^{-2}$ denotes the set of functions u_+ such

$$\int_{-\infty}^{\infty} |\widehat{u}_+(\xi)|^2 \frac{d\xi}{|\xi|^4} < +\infty,$$

H^m is the usual L^2 Sobolev space, and $W^{m,1}$ is the Sobolev space of integrable functions with m derivatives that are also integrable.

Notice that if given a we take u_+ and its derivatives small enough we can be sure that $|v| > \dfrac{a}{2}$. Therefore after undoing the conformal transformation (2.4) we obtain a solution (2.3) and $\psi = u\left(\frac{s}{\sqrt{2}}, \frac{t}{2}\right)$ solution of

$$\partial_t \psi = i\left(\partial_s^2 \psi \pm \frac{1}{2}\left(|\psi|^2 - \frac{a^2}{t}\right)\right). \tag{2.17}$$

We want to construct a family of curves $\chi(s,t)$ solution of the binormal flow (1.2) that it is close to χ_a. From the properties of v we can find regular functions $c(s,t) > \dfrac{a}{2\sqrt{t}}$, and $\phi(s,t)$ such that

$$\psi(s,t) = c(s,t)e^{i\phi(s,t)}.$$

Here $c(s,t)$ is the curvature while the torsion is

$$\tau(s,t) = \partial_s \phi(s,t).$$

Hence we have to consider as filament funtion

$$\widetilde{\psi}(s,t) = c(s,t)e^{i\int_0^s \tau(s,t)\,ds}, \tag{2.18}$$

that it is a solution of (1.10) with

$$A(t) = \frac{a}{t} + \partial_t \phi(0,t). \tag{2.19}$$

Hereafter we will just consider the focusing problem. The positivity of the curvature allows us to consider the Frenet frame and equations. At this respect it is useful to obtain the intrinsic equations for the (c, τ). They are as follows ([DaR],[Be])

$$\begin{aligned} c_t &= -(c\tau)_s - c_s\tau' \\ \tau_t &= \left(\frac{c_{ss} - c\tau^2}{c}\right)_s + c_s c. \end{aligned}$$

The choice of $A(t)$ is in this case, (see [Has])

$$A(t) = \left(\pm 2\frac{c_{ss} - c\tau^2}{c} + c^2\right)(0, t). \tag{2.20}$$

In order to construct χ we first construct the Frenet frame (T, n, b). This is done in two steps. From (1.5) we can obtain as we did in (1.12)– (1.14) when we considered the parallel frame the equations for (T_t, n_t, b_t):

$$\begin{aligned} T_t &= -c\tau n + c_s b, \\ n_t &= c\tau T + \frac{c_{ss} - c\tau^2}{c} b, \\ b_t &= -c_s T - \frac{c_{ss} - c\tau^2}{c} n. \end{aligned}$$

Observe that at $t = 0$ the term

$$\frac{c_{ss} - c\tau^2}{c}$$

can be obtained from (2.19), (2.20) so that $(T, n, b)(0, t)$ can be constructed starting from and initial condition $(T, n, b)(0, 1)$.

Then we use Frenet equations to obtain $(T, n, b)(s, t)$. Recall that

$$T_t = T \wedge T_{ss}.$$

Finally for a given $\chi(0, 1)$ we define

$$\chi(s, t) = \chi(0, 1) - \int_t^1 cb(0, t')\, dt' + \int_0^s T(s, t)\, ds,$$

that solves $\chi_t = \chi_s \wedge \chi_{ss}$.

The proof that χ also develops a corner at time $t = 0$ is rather technical. The result is the following one.

Theorem 3. *Let $\epsilon > 0$, $t = 1$, $0 < a < a_0$ with a_0 as in Theorem 2. Let u_+ be small in $\dot{H}^{-2} \cap H^3 \cap W^{3,1}$ and $s^2 u_+$ small in H^1 with respect to ϵ and a, and let v be the corresponding solution obtained in Theorem 2. By using the Hasimoto transform, we construct from v a family of curves $\chi(s,t)$ that solves the binormal flow for $1 > t > 0$, and such that there exists a unique χ_0 satisfying*

$$|\chi(s,t) - \chi_0(s)| < c_0\sqrt{t}$$

uniformly in $s \in (-\infty, \infty)$.

Moreover

$$\left|\chi_0(s) - \chi_0(0) - sA_a^+ \mathbb{I}_{[0,\infty)}(s) - sA_a^- \mathbb{I}_{(-\infty,0]}(s)\right| < \epsilon|s|$$

where $A_a^{\pm}$ are two unitary vectors such that the angle θ between A_a^+ and $-A_a^-$ is determined by the relation

$$\sin\frac{\theta}{2} = e^{-\pi a^2/2}.$$

In the previous result we prove the existence of solutions for $t \geq 1$ of the equation

$$iv_t + v_{ss} \pm \frac{1}{t}\left(|v|^2 - a^2\right) v = 0 \tag{3.1}$$

that at infinity behave as

$$a - e^{\pm ia^2 \lg t} e^{it\partial_s^2} u_+$$

where u_+ is any given asymptotic state small in an appropriate space. Our purpose now is to find a solution for $t > 1$ and for any initial condition at time $t = 1$ that belongs to an appropriate function space and that is sufficiently small. We also want to prove the asymptotics of this solution by finding the corresponding asymptotic state u_+. Also we would like to remove the smallness assumption in a

For doing that we have to look at the linearized equation that we obtained in the previous lecture, see (2.13):

$$iz_t + z_{ss} \pm \frac{a^2}{t^{1 \pm 2ia^2}} \overline{z} = 0 \tag{3.2}$$

with initial condition at $t_0 \geq 1$, $z(s, t_0)$.

We need some lemmas on the growth of the Fourier transform of the solution.

Lemma 2. *If z solves* (1.49) *then*

$$|\widehat{z}(t,\xi)| \le \frac{t^{a^2}}{t_0^{a^2}} |\widehat{z}(\xi,t_0| + |\widehat{z}(-\xi,t_0)| \tag{1.49}$$

In particular

$$\|z(t)\|_{\dot{H}^k} \le \left(\frac{t}{t_0}\right)^{a^2} \|u(t_0)\|_{\dot{H}^k},$$

for all $k \in \mathbb{Z}$.

Proof. Using the Fourier transform we have

$$\partial_t |\widehat{z}(\xi,t)|^2 = \mp 2\mathrm{Im}\, \frac{a^2}{t^{1\pm 2ia^2}} \overline{\widehat{z}(\xi,t)\widehat{z}(-\xi,t)},$$

and therefore

$$\partial_t |\widehat{z}(\xi,t)| \le \frac{a^2}{t} |\widehat{z}(-\xi,t)|,$$

and a similar differential inequality for $\partial_t |\widehat{z}(-\xi,t)|$. The lemma then easily follows. $\square$

Lemma 3. *Let $0 < \delta$. If z solves (1.49) then for all $\xi \neq 0$ and $0 < t_0 \leq t$*

$$|\widehat{z}(\xi, t)| \leq \left(c(a) + \frac{c(a, \delta)}{(\xi^2 t_0)^\delta} \right) (|\widehat{z}(\xi, t_0)| + |\widehat{z}(-\xi, t_0)|) \,. \tag{1.50}$$

Sketch of the proof. It is better to go back to the unknown w given in (1.35). So that we write

$$z(s, t) = e^{\mp i a^2 \lg t} w,$$

and w is a solution of

$$i w_t + w_{ss} \pm \frac{a^2}{t}(w + \overline{w}) = 0.$$

We define for $\xi \neq 0$

$$Y_\xi(t) = \widehat{\mathrm{Re}\, w} \left(\xi, \frac{t}{\xi^2} \right) \qquad , \qquad Z_\xi(t) = \widehat{\mathrm{Im}\, w} \left(\xi, \frac{t}{\xi^2} \right).$$

Then

$$\begin{aligned} Y_\xi'(t) &= Z_\xi(t) \\ Z_\xi'(t) &= \left(-1 \pm \frac{2a^2}{t} \right) Y_\xi(t) \end{aligned} \tag{1.51}$$

Theorem 6. *Let* $0 < a$ *and* $\psi_1 \in L^1 \cap L^2$ *small with respect to* a. *Then there exists a unique solution* ψ *of*

$$\begin{cases} \psi_t &= i\left(\psi_{ss} + \frac{1}{2}\left(|\psi|^2 - \frac{a^2}{t}\right)\psi\right) \quad 0 < t < 1 \\ \psi(1) &= \psi_1 + ae^{is^2/4} \end{cases} \tag{1.56}$$

such that

$$\psi(s,t) - \frac{a}{\sqrt{t}} e^{is^2/4t} \in L^\infty\left((0,1), L^2\right) \cap L^4\left((0,1), L^\infty\right).$$

Moreover, there exists $\psi_+ \in L^2$ *such that*

$$\left\| \psi(t) - \frac{a}{\sqrt{t}} e^{is^2/4t} - e^{\pm i a^2 \lg t} e^{it\partial_s^2} \psi_+ \right\|_{L^\infty((0,1),L^2)} \leq c(a,\delta) t^{1/4-\delta} \|\psi_1\|_{L^1 \cap L^2},$$

for any $0 < \delta < 1/4$, *and for* $|x| \leq 2$

$$|x|^{2\delta} |\psi_+(x)| \leq c(a,\delta) \|\psi_1\|_{L^1 \cap L^2}.$$

Theorem 7. *Lt* $0 < a$ *and* $\chi_1(s)$ *a regular curve with* (c_1, τ_1) *the corresponding curvature and torsion. We define*

$$\psi_1(s) = c_1(s)e^{i\int_0^s \tau_1(s')\,ds'} \quad , \quad u_1(s) = e^{-is^2/4}\psi_1(s) - a,$$

and assume that $u_1 \in L^1 \cap H^3$ *and it is small with respect to* a.

Then, there exists a unique regular solution $\chi(s,t)$ *of* (1.2) *for* $0 < t < 1$ *with* $\chi(s,1) = \chi_1(s)$. *Moreover, its curvature and torsion* (c, τ) *satisfy*

$$\left|c(s,t) - \frac{a}{\sqrt{t}}\right| \leq \frac{c(u_1)}{t^{1/4+}} \quad , \quad \left|\tau(s,t) - \frac{s}{2t}\right| \leq \frac{c(u_1)}{t^{3/4+}},$$

and taking $\chi_0(s)$ *as in* (1.58) *then*

$$|\chi(s,t) - \chi_0(s)| \leq c(u_1)\sqrt{t}.$$

Several corners

Carpose 22 de junio, 2023

$i\partial_t u + u_{xx} \pm \left(u^2 - \frac{m}{t}\right)u = 0$

$m = A(t)$

$G_a \quad \psi_a = \frac{a}{\sqrt{t}} e^{i\frac{s^2}{4t}} \qquad A(t) = \frac{a^2}{t}$

- $\psi_a \to V_a$

$$\psi_a(s,t) = \bar{V}_a\left(\frac{s}{t}, \frac{1}{t}\right)$$

- $\frac{s}{t} = x$

$\frac{1}{t} = t$

- $i\partial_t V + \partial_{xx} V \pm \frac{1}{t}\left(|V|^2 - a^2\right)V = 0$

$V_a = a$

$$E(t) = \int |V_x|^2 \mp \frac{1}{t}\left(|V|^2 - a^2\right)^2 dx \qquad E(V_a) = 0$$

- Perturbation of ψ_a : perturbing V_a

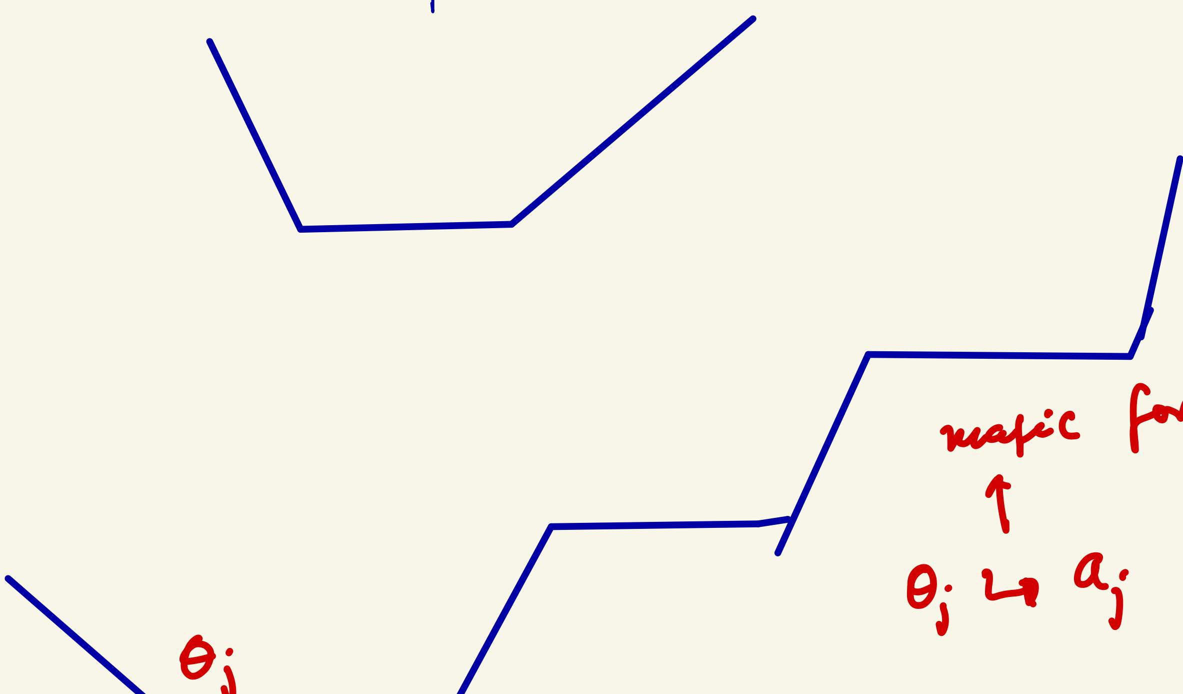
$$V = a + w$$

Solving w

- What happens if $f(s,0)$ has two corners?

Answer: As difficult if it has infinitely many at the same distance and with the corners becoming flat at infinity

- Nonlinearity creates new armonics Kita

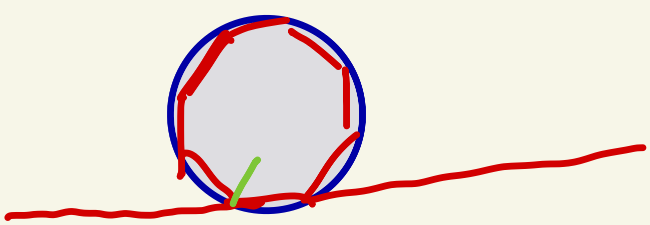


magic formula

$\uparrow$

$\theta_j \rightsquigarrow a_j$

(*) $\sum |a_j|^2 < +\infty$



(**) $\sum |a_j|^2 \langle j \rangle^{2\varepsilon} < +\infty$

Theorem A Given a_j (i.e. θ_j) $_{t=0}$ find the solution

of cubic NLS at $t=1$.

"Easy" assuming for example (**), slightly more complicated assuming (*) with smallness

Geometric problem $\sum |a_j| < +\infty$

$\sum |a_j|^2 \langle j \rangle^2 < +\infty$

Theorem A* Given $f(s,0)$ construct $f(s,t)$

for $|t| \leq 1$

"More delicate": We need to define

the frame at $t=0$ $A_j^{\pm}$ $B_j^{\pm}$

$\boxed{\text{Kita}}$

$$u(x,t) = \sum_{j=-\infty}^{\infty} A_j(t)\, e^{it\partial_x^2} \delta(x-j)$$

$$u(x,t) = \sum_{j=-\infty}^{\infty} A_j(t) \frac{1}{\sqrt{it}} e^{i\frac{(x-j)^2}{4t}}$$

$$= \frac{1}{\sqrt{it}} e^{i\frac{x^2}{4t}} \sum A_j(t)\, e^{i\frac{j^2}{4t} - i\frac{2x}{t}j}$$

$$= \frac{1}{\sqrt{it}} e^{i\frac{x^2}{4t}} \bar{v}\left(\frac{x}{t}, \frac{1}{t}\right)$$

$$i\partial_t v + \partial_{xx} v \pm \frac{1}{t}\left(|v|^2 - M\right) v.$$

with $v(y,\tau) = \sum A_j(1/\tau)\, e^{i\frac{\tau}{4}j^2 - i2jy}$.

$$:= \sum B_j(\tau)\, e^{i\frac{\tau}{4}j^2 - 2ijy}$$

$v(\cdot,\tau)$ is π-periodic

- $\int_0^\pi |v(y,t)|^2 = \sum_j |A_j|^2 = \sum |B_j|^2$ Ricca

Theorem B Given $s>0$ and $v(1) \in H^s([0,\pi])$

$\exists!$ sol'n such that $1 \le \tau < +\infty$

$$\sup_{t} \|v(\tau)\|_{H^s} \le c\,\|v(1)\|_{H^s}.$$

Moreover:

$$\exists \lim_{\tau \to +\infty} |B_j(\tau)|^2 = |a_j|^2$$

$$\lim B_j(\tau) = e^{\pm i |a_j|^2 \log \tau}$$

- Entender en (x,t)
- Entender en ξ
- Blow up

Theorem B* as theorem A* as long as (**) is satisfied $\sum |a_j|^2 \langle j \rangle^2 < +\infty$

• Remark $\sum_j |B_j|^2 = \sum_j |A_j(1)|^2$

Proof. $v = \sum B_j(\tau) e^{i\tau j^2 + iyj}$

$$|v|^2 v = \sum e^{ijy + i\tau j^2} \sum_m e^{itm} \sum_{j_1, j_2 \in r(m)} B_{j_1} \overline{B_{j_2}} B_{j_3}$$

$$\frac{d}{dt} \hat{B}_j = \frac{i}{\tau} \sum_m e^{itm} \sum_{r(m)} B_{j_1} \overline{B_{j_2}} B_{j_3}.$$

resonant set $2m = (j - j_1)(j_1 - j_2)$

$$2\left(\sum |B_j|^2\right) B_j \qquad M = 2\left(\sum |B_j|^2\right).$$

$$\frac{d}{dt} \hat{B}_j = \frac{i}{\tau} \sum_{m \neq 0} \cdots + \frac{i}{\tau} \sum \left(2M - |B_2(t)|^2\right) B_2(t)$$

$$\frac{d}{dt} \hat{B}_j \overline{\hat{B}_j} = \quad \downarrow$$

<u>Remark.</u> (*) system is periodic in "j"

$$B_k(\tau) = B(\tau) = \alpha\, e^{-i|\alpha|^2 \underbrace{\int_{\tau}^{+\infty} \sum_{m\neq 0} \sum_{r(m)} e^{-im\tau} \frac{d\tau}{\tau}}_{\phi_d}}.$$

Hence

$$V_\alpha(y,\tau) = \alpha\, e^{-i|\alpha|^2 \phi_d(\tau)} \sum_j e^{i\tau j^2 + iy\tau}.$$

$$\psi_M(y,\tau) = C_M\, e^{-i|C_M|^2 \phi_d(1/t)} \sum_j e^{i\frac{j^2}{4t} + 2ij/t}$$

Is the "filament" function associated to the polygon of M sides with

$$C_M:\quad \operatorname{sen}\frac{\pi}{M} = e^{-\frac{C_M^2}{2}\pi}$$

The regular polygon

$$t_{pq} = (2\pi/M^2)(p/q)$$

$$\begin{aligned}
\psi(s, t_{pq}) &= \sum_{k=-\infty}^{\infty} e^{-i(Mk)^2 2\pi p/(M^2 q) + iMks} \\
&= \sum_{k=-\infty}^{\infty} e^{-2\pi i(p/q)k^2 + iMks} \\
&= \sum_{l=0}^{q-1} \sum_{k=-\infty}^{\infty} e^{-2\pi i(p/q)(qk+l)^2 + iM(qk+l)s} \\
&= \sum_{l=0}^{q-1} e^{-2\pi i(p/q)l^2 + iMls} \sum_{k=-\infty}^{\infty} e^{iMqks}.
\end{aligned}$$

The generalized quadratic Gauß sums are defined by

$$\sum_{l=0}^{|c|-1} e^{2\pi i(al^2+bl)/c},$$

for given integers a, b, c, with $c \neq 0$.

$$G(-p, m, q) = \begin{cases} \sqrt{q}e^{i\theta m}, & \text{if } q \text{ is odd,} \\ \sqrt{2q}e^{i\theta m}, & \text{if } q \text{ is even and } q/2 \equiv m \mod 2, \\ 0, & \text{if } q \text{ is even and } q/2 \not\equiv m \mod 2, \end{cases}$$

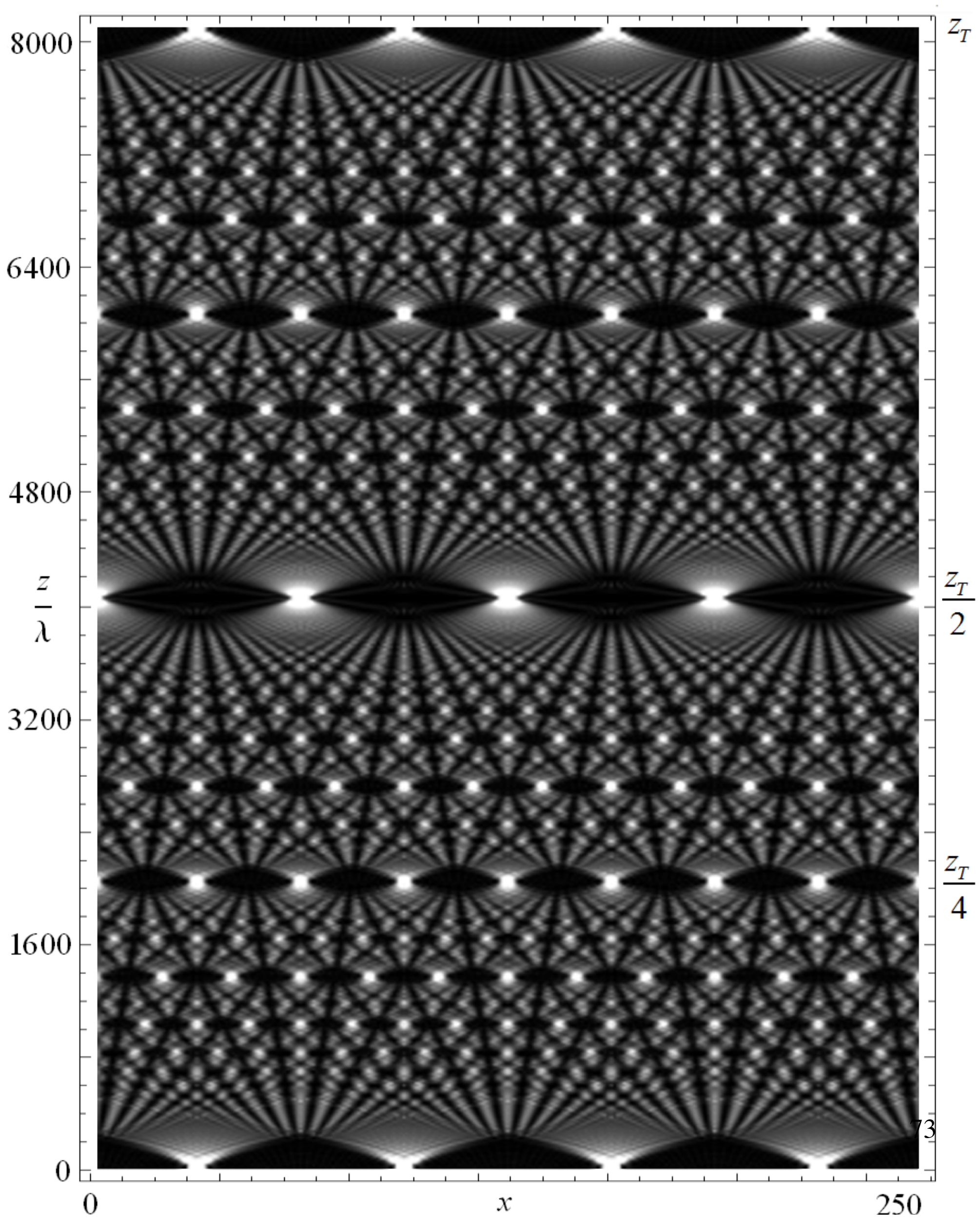
for a certain angle θ_m that depends on m (and, of course, on p and q, too).

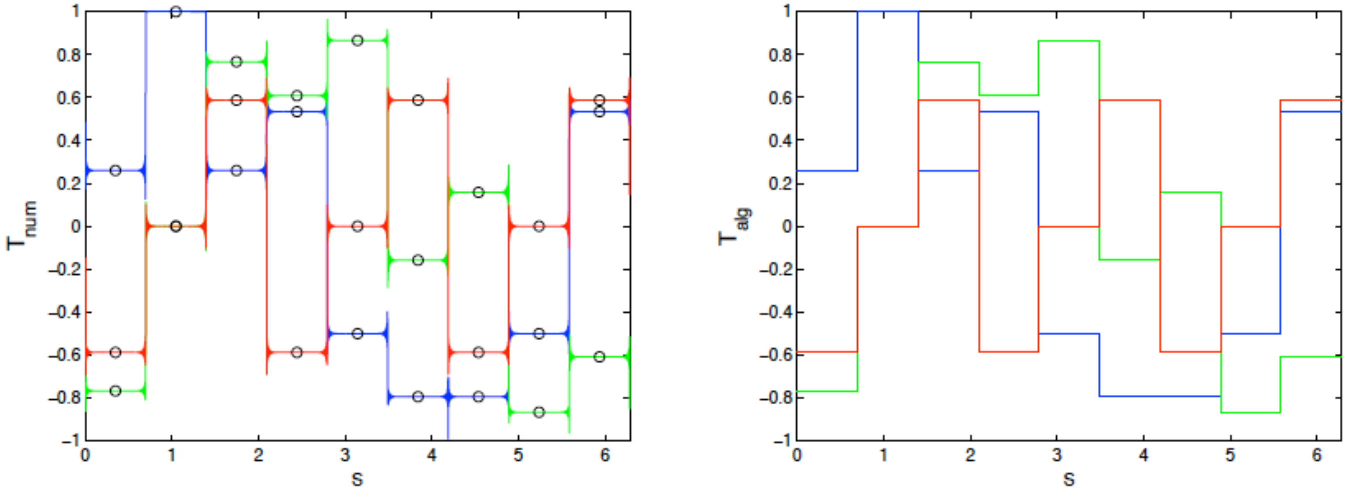
$$t_{pq} = (2\pi/M^2)(p/q)$$

$$\psi(s,0) = \frac{2\pi}{M} \sum_{k=-\infty}^{\infty} \delta(s - \tfrac{2\pi k}{M}).$$

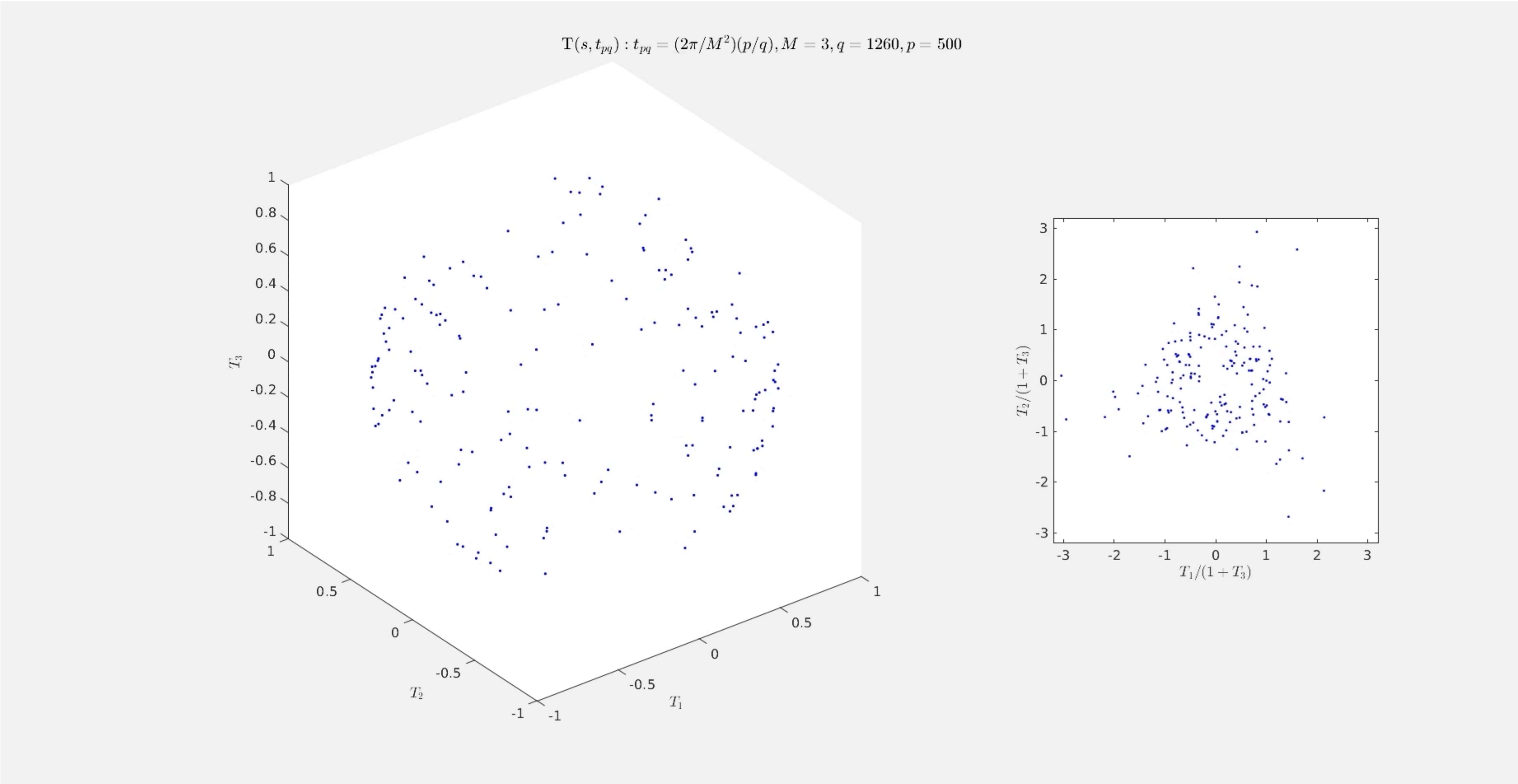
$$\psi(s,t_{pq}) = \frac{2\pi}{Mq} \sum_{k=-\infty}^{\infty} \sum_{m=0}^{q-1} G(-p,m,q)\delta(s - \tfrac{2\pi k}{M} - \tfrac{2\pi m}{Mq})$$

THE TALBOT EFFECT





$\mathbf{T}_{num}$ versus $\mathbf{T}_{alg}$, for $M = 3$, at $T_{1,3} = \dfrac{2\pi}{27}$. T_1 appears in blue, T_2 in green, T_3 in red. In $\mathbf{T}_{num}$, the Gibbs phenomenon is clearly visible. The black circles denote the points chosen for the comparisons.


$\mathrm{T}(s, t_{pq}) : t_{pq} = (2\pi/M^2)(p/q), M = 3, q = 1260, p = 500$
T_3
T_2
T_1
$T_2/(1+T_3)$
$T_1/(1+T_3)$

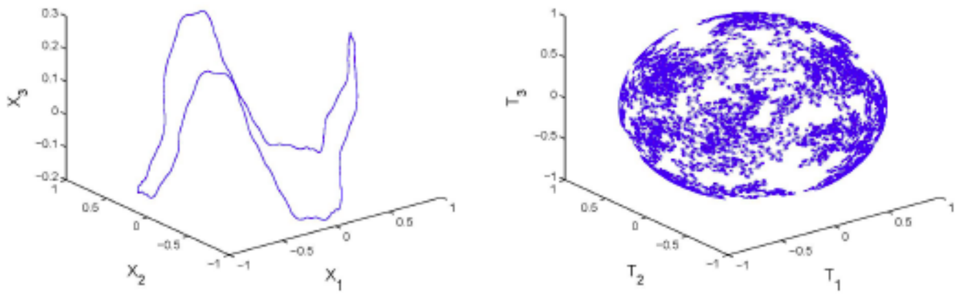


Figure 8: $\mathbf{X}_{alg}$ and $\mathbf{T}_{alg}$, at $t = \frac{2\pi}{9}(\frac{1}{4} + \frac{1}{41} + \frac{1}{401}) = \frac{2\pi}{9} \cdot \frac{18209}{65764}$.

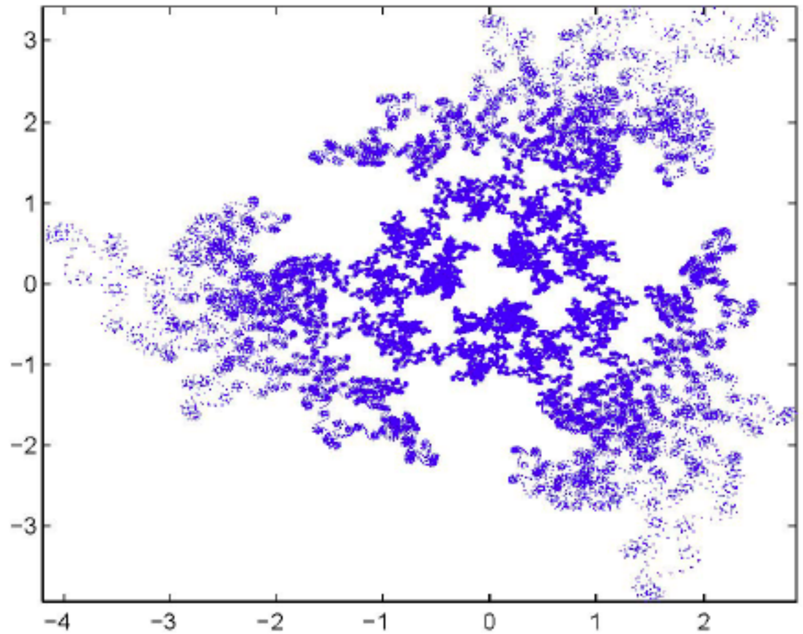
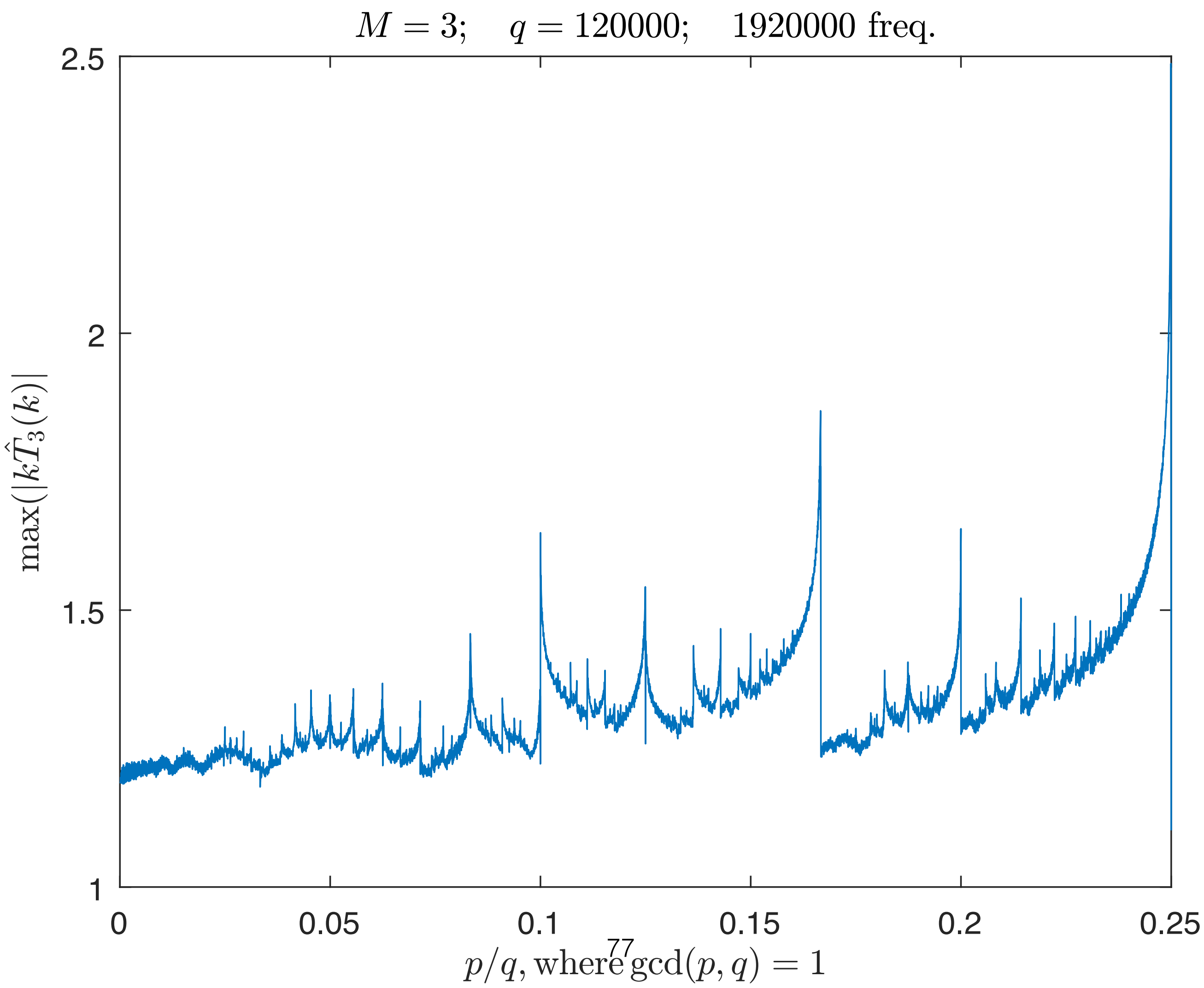


Figure 9: Stereographic projection of the right-hand side of Figure 8.


$M = 3; \quad q = 120000; \quad 1920000$ freq.
2.5
2
1.5
1
0
0.05
0.1
0.15
0.2
0.25
$\max(|k\hat{T}_3(k)|)$
p/q, where $\gcd(p, q) = 1$

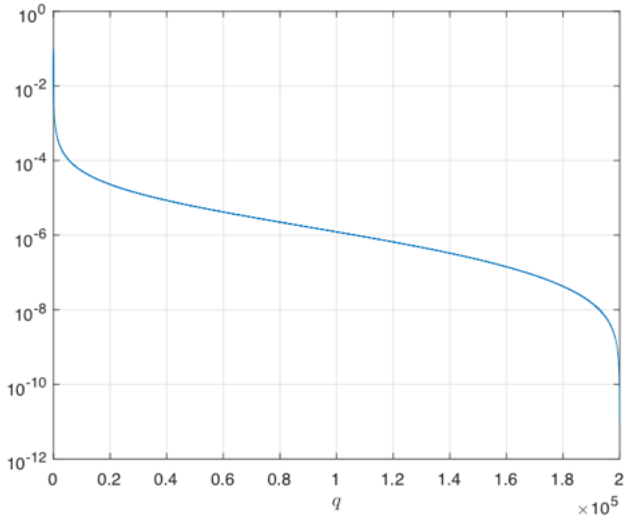


Figure 10: $|\sqrt{2}\max_{t_{pq}} \|\widehat{T_{1,s}}(t_{pq})\|_\infty - a\ln(q) - b|$, for $a = 0.258039752572419$, $b = 0.152992510344641$.