Water Waves and Hamiltonian Partial Differential Equations

Catherine Sulem

University of Toronto

Advanced Summer school in
Mathematical Fluid Dynamics
Cargèse, June 20-30, 2023

Outline

I. The Euler system for free surface water waves

1. Physical derivation of free surface water waves equations
2. Zakharov Hamiltonian formulation and canonical variables
3. The water wave system in terms of canonical variables – The Dirichlet - Neumann operator
4. Canonical variables: Where do they come from?

II. The Dirichlet-Neumann operator (DNO)

1. Basic properties
2. Variational derivatives of Hamiltonian w.r.t. canonical variables
3. Recovering the water wave system in terms of canonical variables – Shape derivative of DNO
4. Recovering conservation laws using Hamiltonian theory
5. Analyticity of DNO w.r.t. surface elevation - Explicit Taylor expansion in powers of surface elevation

III. Hamiltonian transformation theory - Model equations for water waves

1. Examples of Hamiltonian PDEs
2. Calculus of transformations - Examples
3. Derivation of Boussinesq equation
4. Derivation of Korteweg-de Vries equation
5. Interpretation and statement of rigorous results validating this derivation

IV. Birkhoff Normal Forms (BNF) for gravity water waves

1. Setting of the problem
2. Canonical transformations; Birkhoff Normal Forms; Approximate integrability
3. Dispersion relation; Complex symplectic coordinates; Poisson brackets, resonances
4. BNF of 3rd order – A first change of variable
5. A 2nd change of variable - Partial integrability

V. Application of BNF to the derivation of Nonlinear Schrödinger equation

1. Modulational Ansatz
2. Homogenization lemma
3. Derivation of NLS
4. Derivation of higher order NLS – The Dysthe equation
5. Comments

V. Initial value problem

1. Lagrangian/Eulerian formulation
2. The Taylor-sign condition
3. Description of results - local and global wellposedness

VI. Coupling of internal and surface waves

1. The physical problem
2. Euler equations for stratified fluids
3. Scaling regimes - Resonant condition.
4. Coupled KdV-linear Schrödinger system
5. Analysis and interpretation

References

T. Alazard, J.-M. Delort (2015) *Global solutions and asymptotic behavior for two dimensional gravity water waves*, Annales Scientifiques Ecole Normale Supérieure **48**, 1149–1238.

T. Alazard, N. Burq, C. Zuily (2011) *On the water waves equations with surface tension*, Duke Mathematics Journal, **158**, 413–499.

T. Alazard, N. Burq, C. Zuily (2014) *On the Cauchy problem for gravity water waves*, Inventiones Mathematicae, **198**, 71–163.

D. Ambrose, J. Bona, T. Milgrom (2019) *Global solutions and ill-posedness for the Kaup system and related Boussinesq systems*, Indiana Univ. Math. J. **68**, 1173–1198.

J.T. Beale, T. Hou, J. Lowengrub (1993) *Growth rates for the linearized motion of fluid interfaces away from equilibrium*, Comm. Pure Appl. Math. **46**, 1269–1301.

M. Berti (2000) *Nonlinear Oscillations of Hamiltonian PDEs*, Progress in Nonlinear Differential Equations and Their Applications, Volume 74, Birkhaüser.

M. Berti, R. Feola, F. Pusateri (2021) *Birkhoff normal form for gravity water waves*, Water Waves **3**, 117–126.

W. Craig (1985) *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*, CPDE **10**7, 87–1003.

W. Craig, P. Guyenne, C. Sulem (2021) *The Water Wave Problem and Hamiltonian Transformation Theory*, Chapt. 4 in ‘Waves in Flows’, Advances in Mathematical Fluid Mechanics, T. Bodnár, G. Galdi, S., Necasová, (eds.) Springer Nature, Birkhaüser, 113–196.

W. Craig, C. Sulem (1993) *Numerical simulation of gravity waves*, Journal of Computational Physics, **108**, 73–83.

W. Craig, P. Worfolk (1995) *An integrable normal form for water waves in infinite depth*, Phys. D **84**, 513–531.

A. Dyachenko, V.E. Zakharov (1994) *Is free-surface hydrodynamics an integrable system*, Phys. Lett. A **190**, 1994, 144–148.

M. Ifrim, D. Tataru (2016) *Two dimensional water waves in holomorphic coordinates II: global solutions*, Bull. Soc. Math. France **144**, 369–394.

A. Ionescu, F. Pusateri (2015) *Global solutions for the gravity water waves system in 2D*, Invent. Math. **199**, 653–804.

A. Ionescu, F. Pusateri (2018) *Recent advances on the global regularity for wa- ter waves* Philosophical Transactions of the Royal Society A, **376**, 20170089.

D. Lannes (2013). *The Water Wave Problem: Mathematical Analysis and Asymptotics*, Mathematical Surveys and Monographs, Vol. 188, AMS.

P. Olver (1984) *Hamiltonian perturbation theory and water waves. Fluids and plasmas: geometry and dynamics* (Boulder, Colo., 1983), **28**, 231–249. Contemporary Mathematics, AMS, Providence, R.I.

G. Schneider, C. E. Wayne (2000) *The Long-Wave Limit for the Water Wave Problem I. The Case of Zero Surface Tension*, Commun. Pure Appl. Math., Vol. LIII, 1475Ð1535 .

S. Wu (1997) *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Invent. Math. **130**, 39–72

S. Wu (1999) *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc. **12**, 445–495.