Structure and scaling of extreme events in hydrodynamic turbulence

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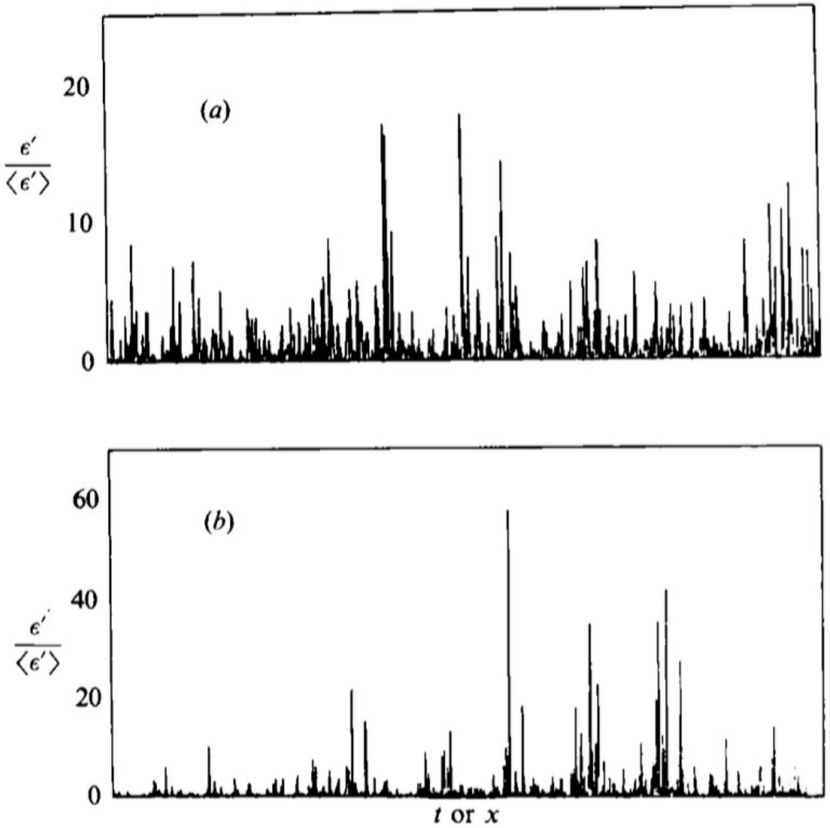
Turbulence and intermittency

- Turbulent flows are known to generate very ***large velocity gradients***, the more so as the Reynolds number increases.



(a) Laboratory experiment;
low Reynolds; $R_\lambda \sim 100$.
(b) Atmospheric boundary layer;
high Reynolds; $R_\lambda \sim 1500$.

Phenomenon known as Intermittency
(see e.g. Batchelor and Townsend, 1949).

Meneveau and Sreenivasan (1991)

Large velocity gradients in turbulent flows: how large ?

ε = kinetic energy dissipation is independent in the zero viscosity limit $\nu \to 0$
[empirical fact; aka "the dissipative anomaly"]

$$\Rightarrow \nu \langle (\partial u)^2 \rangle \sim \varepsilon \sim (U^3/L)$$

The variance of the velocity gradient increases as a function of the Reynolds number:

$$\langle (\partial u)^2 \rangle \propto (\varepsilon/\nu) \simeq (U/L)^2 Re$$

- In addition, the largest values of the gradient grow with Re ***much faster*** than their mean values *[**intermittency**].*

Very large velocity gradients could be generated in the bulk, away from boundaries.

Gradient amplification ?

Vortex stretching

... a very important ingredient in turbulent flows (cf. Tennekes and Lumley) !!!

<u>Equation for the vorticity:</u>

$$\partial_t \boldsymbol{\omega}(\boldsymbol{x},t) + (\boldsymbol{u}.\boldsymbol{\nabla})\boldsymbol{\omega} = (\omega.\boldsymbol{\nabla})\boldsymbol{u} + \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega}(\boldsymbol{x},t)$$

How much stretching can the term: $(\boldsymbol{\omega}.\boldsymbol{\nabla})\boldsymbol{u}_a \equiv \omega_b.S_{ba}$ ($\boldsymbol{S}$ strain-rate tensor) produce?

Dimensionally, $\boldsymbol{S}$ and $\boldsymbol{\omega}$ are both velocity gradients. Superficially,

$$d\omega/dt = S.\ \omega \sim \omega^2$$

Solution: $\omega(t) = \omega(0)/(1 - \omega(0)t)$… *blows up at* $t^* = 1/\omega(0)$

~ Naïve expectation: $/\omega/ \sim |\partial v| \sim 1/(t^*-t)$

Homogeneous isotropic turbulence

Isotropy: fluctuations are invariant under any rotation – invariance ender SO(3).

Homogeneity: the flow properties do not vary in space.

- Conceptually simplest setup to study turbulence. Introduced by Taylor (1936), further development by von Karman-Howarth (1938), Kolmogorov (1941), etc etc.

- In practice, flows are stirred in a very anisotropic manner. However, due to turbulence, the flow develops a wide range of length scales, and at the smallest scales, one postulates (expects, etc) that isotropy is restored.

- Convenient to simulate numerically via *direct numerical simulations (DNS)* of the Navier-Stokes equations.

DNS of turbulent flows:
Constraints on intermittency studies

Issues with any intermittency studies

Large velocity gradient ~ (large velocity difference)/(small distance)
~ 1/(small time)

=> make sure that spatial resolution is good enough

=> ensure proper time resolution

=> Existence of rare events: statistical convergence is an issue.

In this work: use direct numerical simulations (DNS) of the Navier-Stokes equations at very high resolution, based on pseudo-spectral methods.

Studying the velocity gradient tensor: spatial resolution issues

- *How much can one trust the information concerning large velocity gradients from a limited spatial resolution ?*

The usual resolution criterion for DNS using spectral methods involves :

- the smallest length scale in the flow, the Kolmogorov scale: $\eta = (\nu^3/\varepsilon)^{1/4}$
- the highest wavenumber resolved: $k_{max} \approx (\sqrt{2}/3)\, N$, where N is the total number of Fourier modes in any spatial direction.

An accepted criterion for a good simulation : $k_{max}\, \eta \approx 1.5 - 2$

In terms of the smallest length resolved: $\Delta x/\eta\ [\approx 3/(k_{max}\eta)] \approx 2 - 1.5$

Studying the velocity gradient tensor: temporal resolution issues

- Spectral calculations are limited by a Courant-Friedrich-Lewy condition:

$$\Delta t \lesssim \Delta x / |u|$$

The mesh size $\Delta x \sim \eta \sim L\, R_\lambda^{-3/2}$, so $\boldsymbol{\Delta t < (L/U)\, R_\lambda^{-3/2}}$.

=> the time step chosen is therefore much smaller than $\boldsymbol{\tau_K \sim (L/U)\, R_\lambda^{-1}}$, which is the time scale associated with the rms of the velocity gradients.

… however …

Studying the velocity gradient tensor: temporal resolution issues

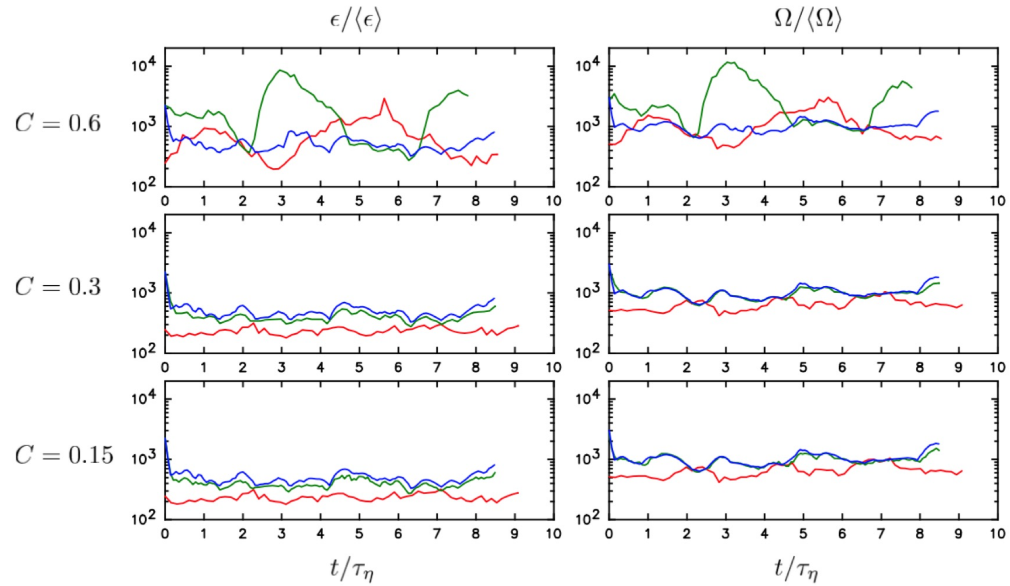
BUT !!! Velocity gradients MUCH larger than 1/ τ_K appear in the flow.

Define: $\Delta t = C\ \Delta x/|u|_{max}$; C = Courant #

=> Stability of the integration algorithm ($C \lesssim 1$) isn't good enough; taking too large a time step may lead to spurious results (Yeung, Pope and Sreenivasan, 2018).

Max values of ε and Ω (≡ $|\omega|^2$) as a function of time

$R_\lambda = 390$



$k_{max}\ \eta = 1.4$ (red)
$= 2.8$ (green)
$= 5.6$ (blue)

Resolution in our study

- Here, use: $Courant\ \# = 0.3$

$k_{max}\,\eta$ larger than 3; or equivalently: $\Delta x/\eta$ **smaller than ~1**

nb: most runs in fact at $k_{max}\,\eta \approx 6$. $(\Delta x/\eta \sim 0.5)$

Use a spectral code (Rogallo 1981) with up to 12288^3 modes/colocation points (a few time steps with 18432^3 modes).

R_λ	N^3	$k_{max}\eta$	T_E/τ_K	T_{sim}	N_s
140	1024^3	5.82	16.0	$6.5T_E$	24
240	2048^3	5.70	30.3	$6.0T_E$	24
390	4096^3	5.81	48.4	$2.8T_E$	35
650	8192^3	5.65	74.4	$2.0T_E$	40
1300	12288^3	2.95	147.4	$20\tau_K$	18

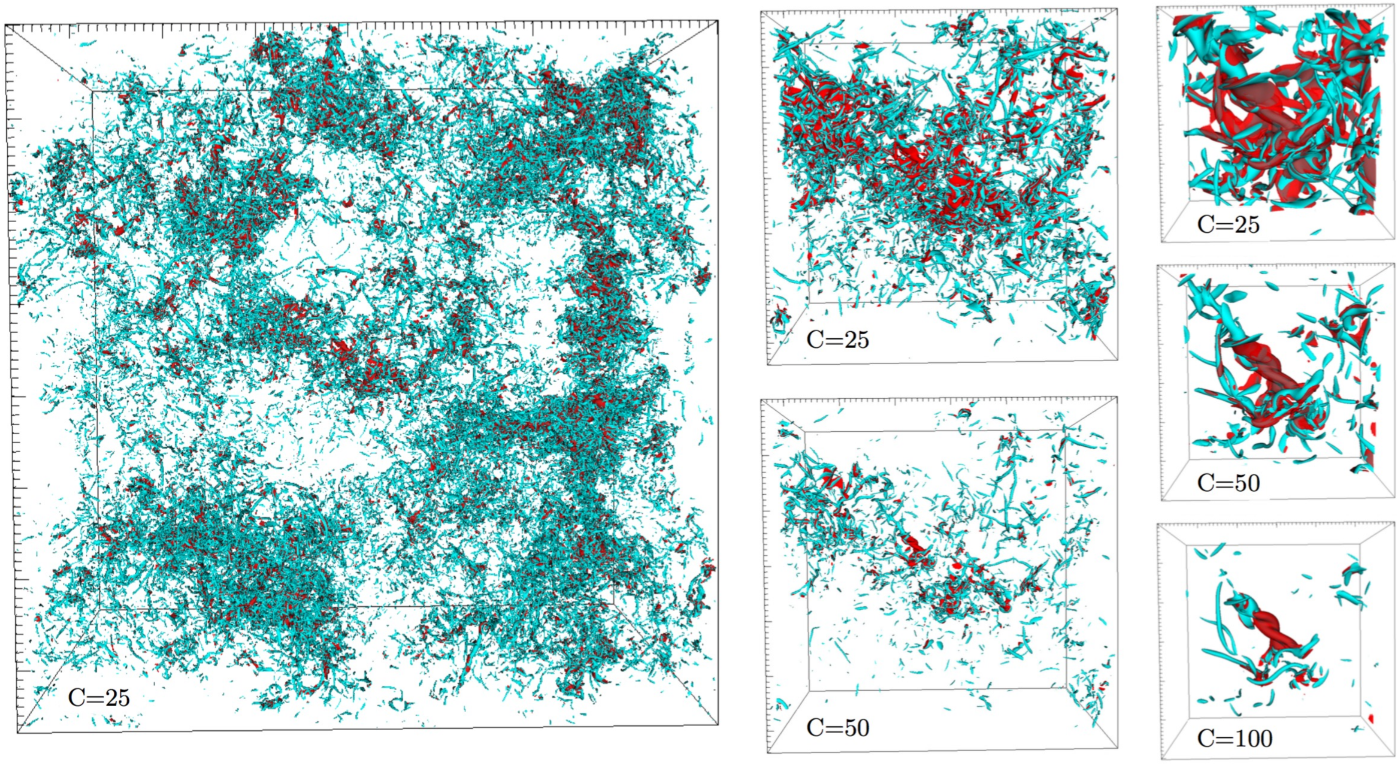
nb: N_s = # of fields considered to construct the statistics.

Intermittency in DNS of turbulent flows

Structure of the regions of intense strain/vorticity

Spatial structure of strain and vorticity

The largest velocity gradient structures are vortex tubes (as found many times before, e.g. Siggia, 1981 (…) Ishihara et al, 2007, 2009).



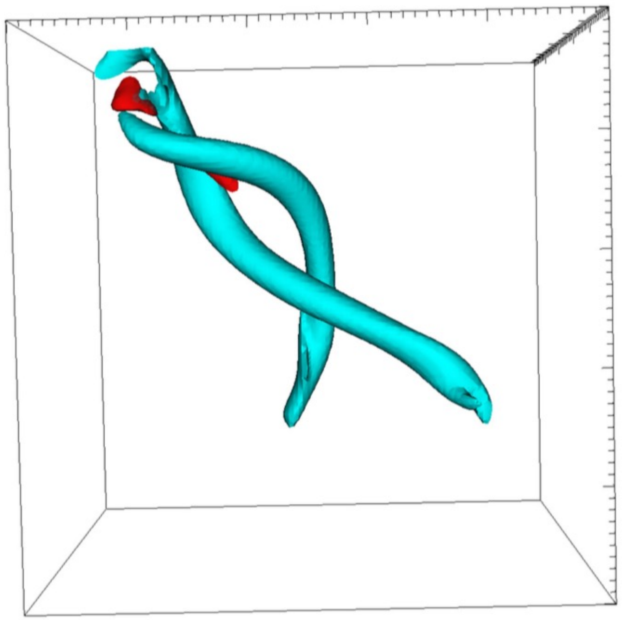
(a) $(2000\eta)^3$ (b) $(600\eta)^3$ (c) $(150\eta)^3$

- Strain (red) is comparatively much Smaller than vorticity (cyan).
- The most intense regions of strain and vorticity are not co-located.

Buaria et al, NJP 2019

Structure of strain and vorticity

- *The most intense events are vortex tubes; possibly in weak interaction.*

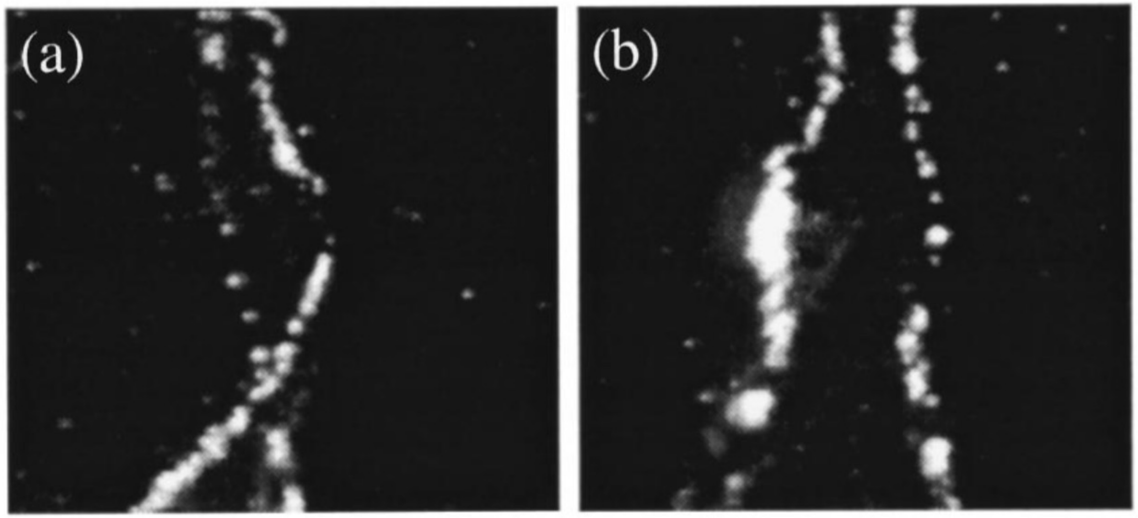


(c) $(50\eta)^3$; C=500 (d) $(25\eta)^3$; C=1500

Blown-up view of the most intense velocity gradient region.

Buaria et al, NJP 2019

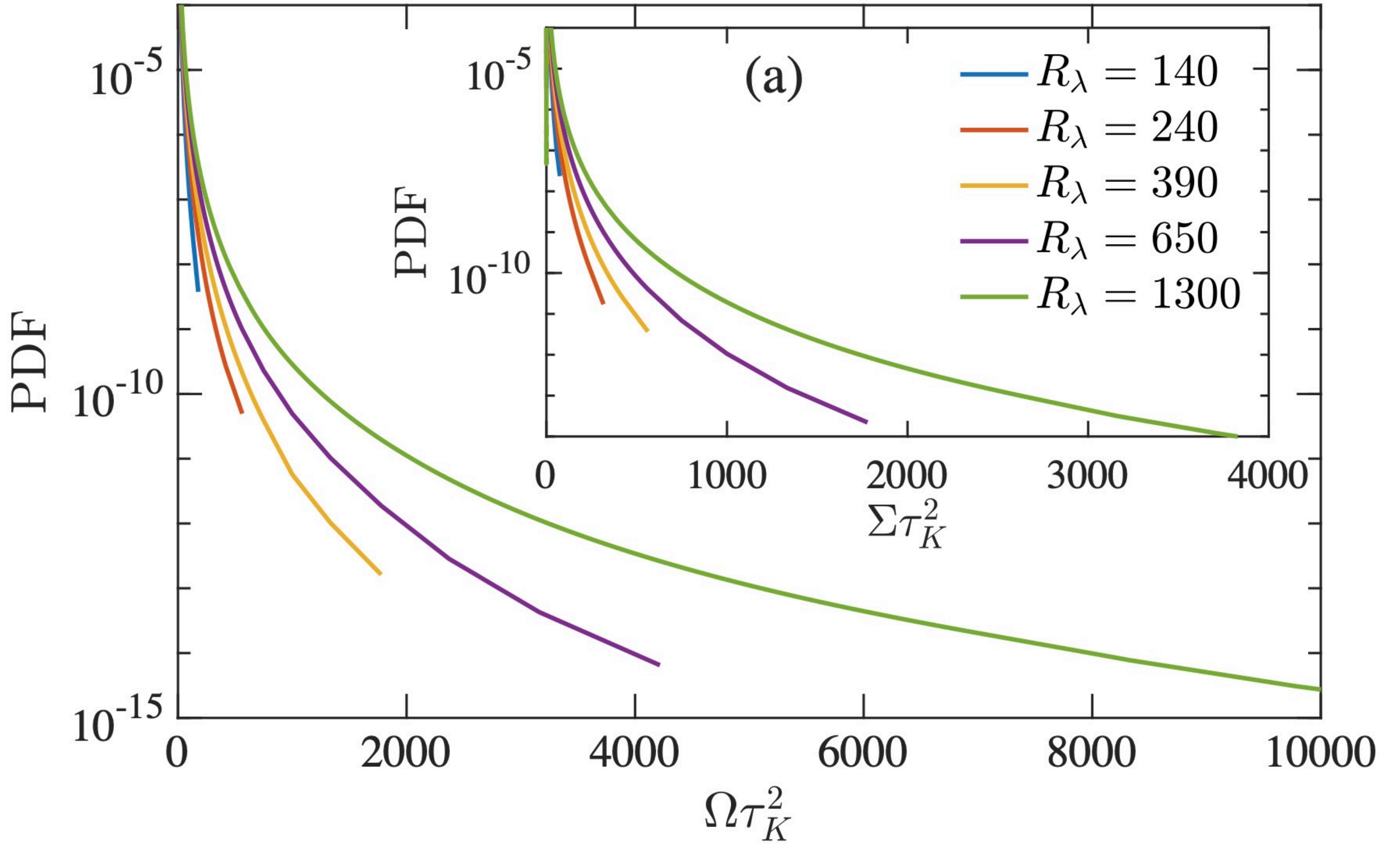
Vortex tubes: not just "in-silico”



LaPorta et al, PoF 2000

- *Visualization of the vortices with cavitation bubbles (see also Douady et al, PRL 1991 etc).*

PDF of $\Omega = \omega^2$ and $\Sigma = 2\,tr(\Sigma^2)$



- The distribution of $\Omega/\langle\Omega\rangle$ become wider when R_λ increases.
- Same conclusion Σ, defined as $\Sigma = 2\ \mathrm{tr}(S^2)$;
- the fluctuations of $\Sigma/\langle\Sigma\rangle$ are slighltly smaller than those of $\Omega/\langle\Omega\rangle$.

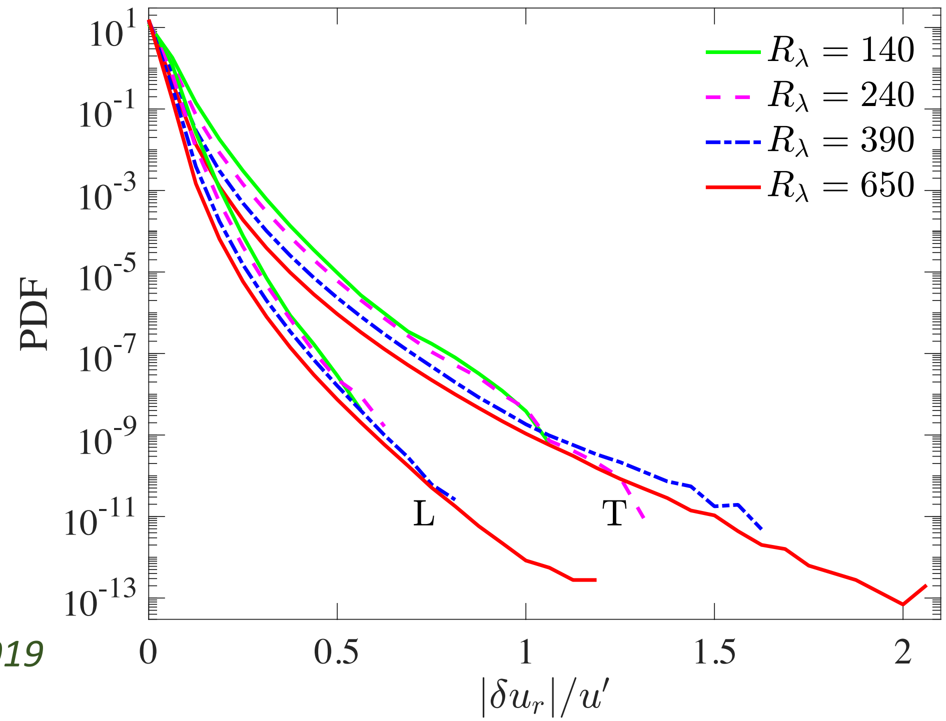
Buaria et al, PRL 2022

Velocity differences at scales $\leq \eta$.

- ***Observation:***

Velocity differences δu_r can be as large as u', the r.m.s. velocity (Jimenez et al, 1993).

If anything, the velocity diff. at $\eta/2$ (and also η) grows when Re_λ increases.



Buaria et al, NJP 2019

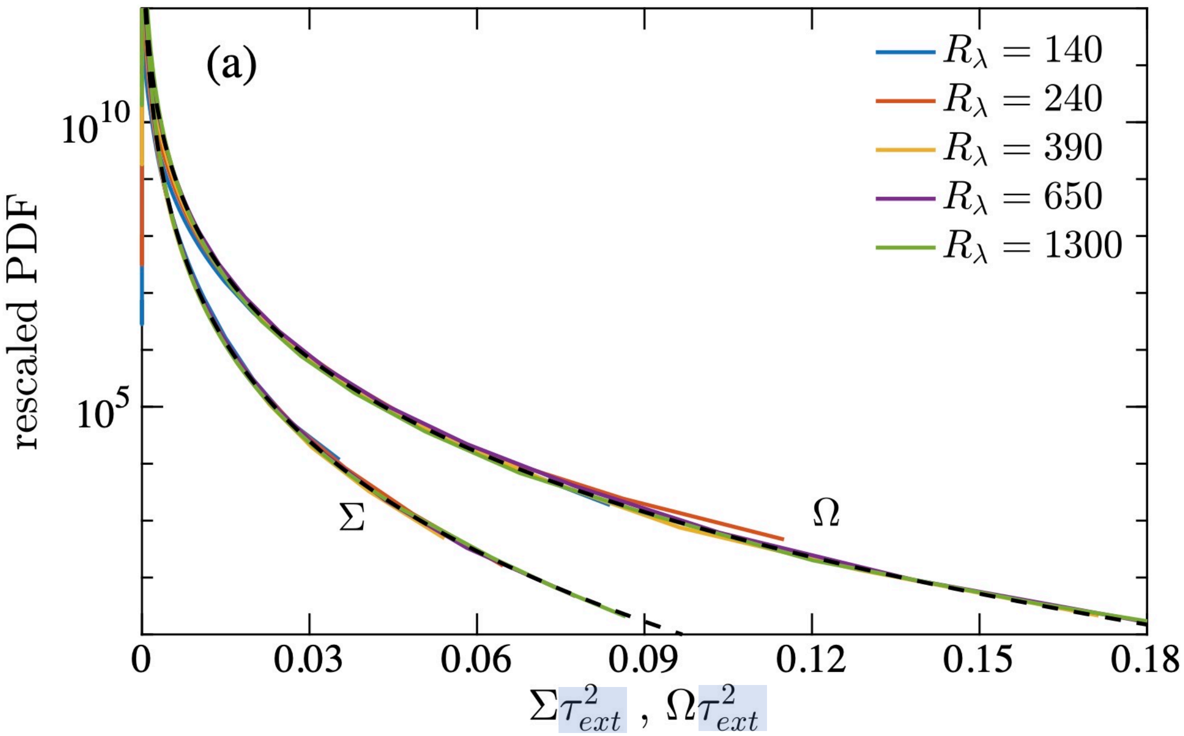
$r = \eta/2$.

T: transverse velocity differences;

L: longitudinal velocity diff.

Two main numerical observations.

Observation 1:
Scaling of the large gradients with Re_λ.



Main result:

PDF of $\Omega\, \tau_K^2$ and $\Sigma\, \tau_K^2$ show tails that rapidly grow when Re_λ increases.

The tails of these PDFs can be very well collapsed by rescaling with $Re_\lambda^{2\beta}$ -- which means that the largest fluctuations of vorticity and strain scale in fact like:

$$\omega, S \propto 1/\tau_{ext} = Re_\lambda^{\beta}/\tau_K$$

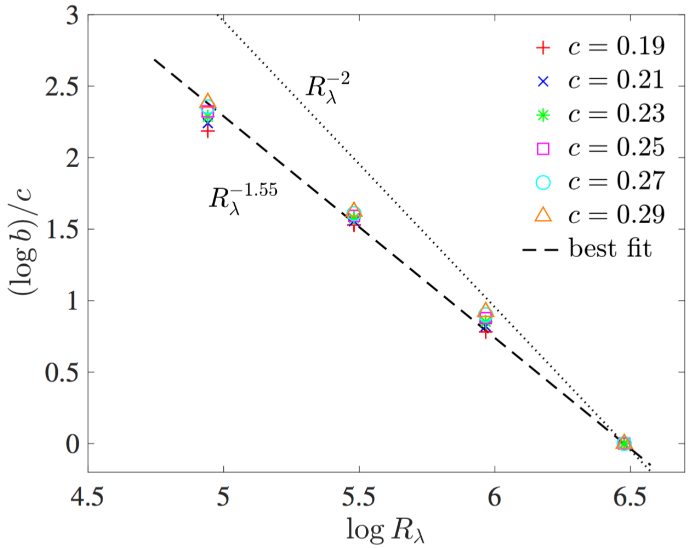
$$\tau_{ext} \sim \tau_K Re_\lambda^{-\beta}$$

Definitions: $\Omega = \omega^2;\ \Sigma = 2\,\mathrm{tr}(S^2)$.

Buaria et al, PRL 2022

Scaling of the PDF: a systematic approach

- Observe that the distribution can be well fitted by stretched exponential functional form: $P(x) = a\ exp(-b\ x^c\ )$ where the exponent $c \sim 1/4$.
- Fit the data with a given value of the exponent c around $1/4$, and look for the dependence of b as a function of R_λ:

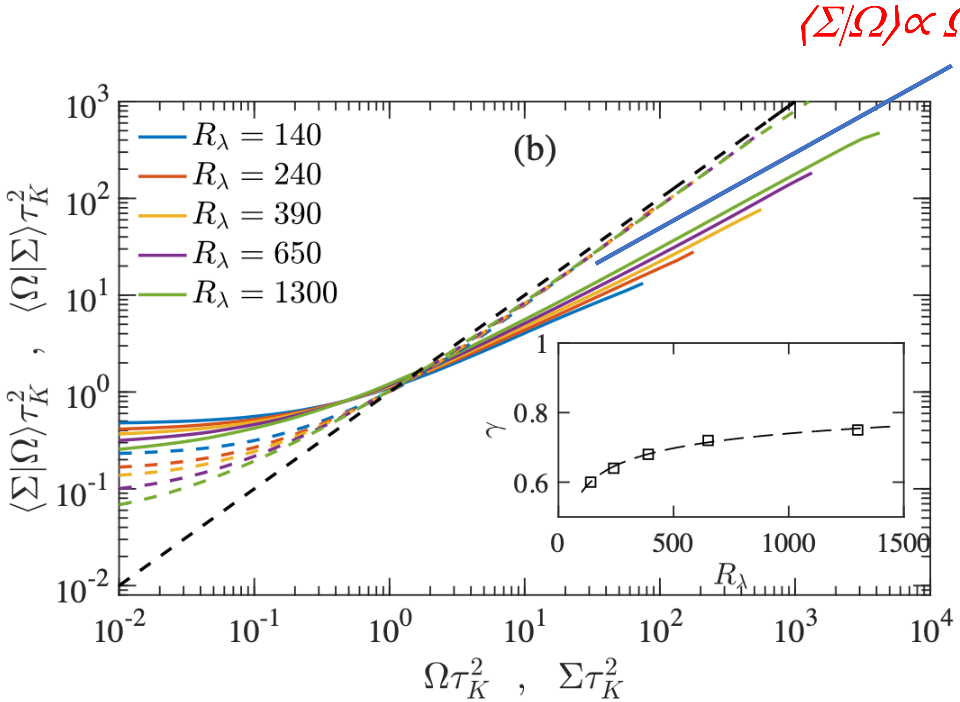


Conclusion:

Values of $b^{1/c}$ are consistent with the observed collapse of the PDFs.

Observation 2:
Strain acting on large vorticity.

- Question: What is the strain for a vortex with a very large vorticity?



Buaria et al, PRL 2022

Answer:

the strain Σ ($= 2\ \mathrm{tr}(S^2)$), conditioned on Ω ($= \omega^2$) grows as $\langle\Sigma|\Omega\rangle \propto \Omega^\gamma$;

Exponent: γ is a function of R_λ (see inset).

$$(1-\gamma) = pR_\lambda^{-q} \ , \qquad p, q > 0$$

$$\log p = -0.033 \text{ and } q = 0.189.$$

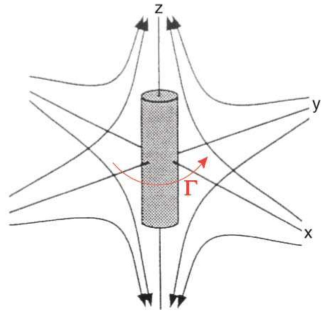
Interpretation

Size of a strained vortex.

S_{out} = straining rate

- The size of a vortex tube results from a balance between viscosity and strain (think of a Burgers vortex):

$$\eta_{Burgers} \sim radius \sim (\nu/S_{out})^{1/2}$$



Hamlington *et al.*, PRE 2008

- A guess for the strain acting on an intense vortex of intensity Ω:

$$\langle \Sigma / \Omega \rangle \, \tau_K^2 \propto (\Omega \tau_K^2)^{\gamma}$$

=> suggests that the size $R(\Omega)$ of an intense vortex, of intensity Ω, is:

$$R(\Omega) = (\nu^2/\langle \Sigma|\Omega \rangle)^{1/4} \sim \eta_K \, (\tau_K^2 \Omega)^{-\gamma/4}$$

Buaria et al, PRR 2021.

Connection with the observed scalings.

<u>The velocity difference across very intense tubes is</u> $\sim u'$:

$$\Omega \sim u'^2/R(\Omega)^2$$

Solve: $\Omega\, \tau_K{}^2 \propto Re_\lambda{}^{2/(2-\gamma)}$

+ identify with earlier definitions: $\beta = 1/(2-\gamma).$

- This agrees <u>quantitatively</u> with our own numerical values !

Corresponding smallest scale: $u'/\eta_{ext} \sim \tau_{ext}$

=> $\eta_{ext} \sim \eta_K\, Re_\lambda{}^{-\alpha}\,; \quad \alpha = \beta - 1/2 = \gamma/\,[2(2-\gamma)]$

See Buaria et al, NJP 2019 and PRL 2022.

R_λ-dependence: implications for the fits of the PDFs

- Postulate that the tails of the PDF behave as: $f_X(x) \approx a \exp(-b x^c)$,
(empirically, c is a fixed number ~ ¼).

- Superposition of the PDFs by rescaling x by $(x R_\lambda^\beta)$: $b^{1/c} \sim R_\lambda^{-2\beta}$

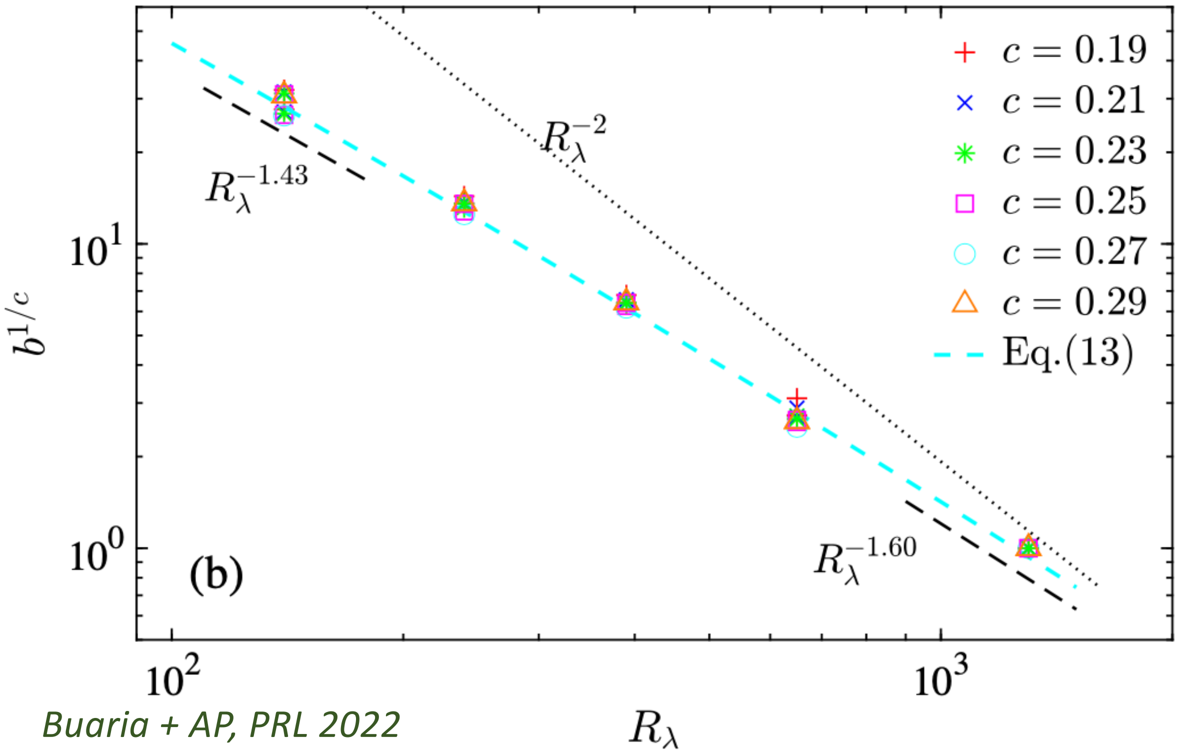
- write the relation above as an ODE:

$$d \ln(b^{1/c})/d \ln(R_\lambda) = -2\beta$$

- Use: $\beta = 1/(2-\gamma)$ with $\gamma = 1 - p R_\lambda^{-q}$

- Solve (…calculate an integral…).

R_λ-dependence of the PDFs



Buaria + AP, PRL 2022

Dashed line: result of the calculation
With the functional fit for γ:

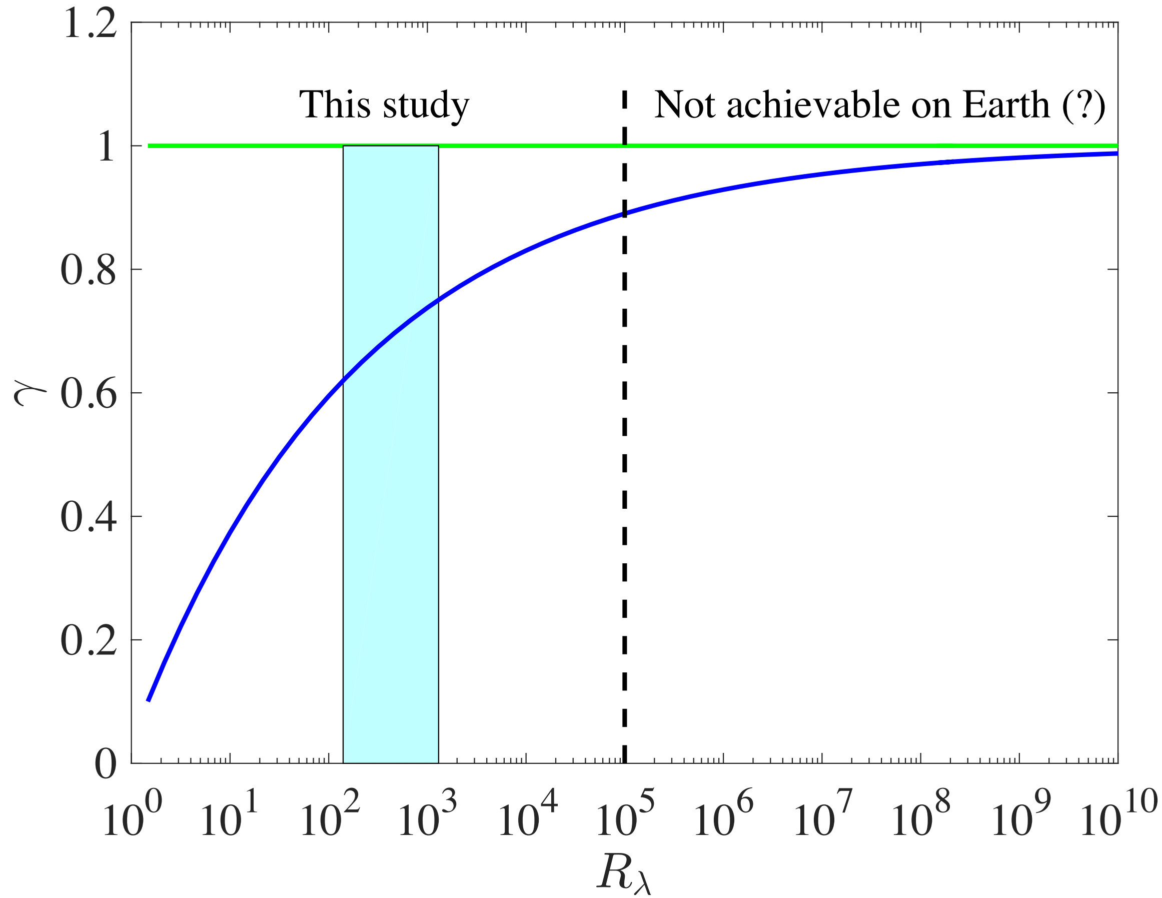
$$\ln(b^{1/c}) = -2\ln(R_\lambda) -$$

$$-\left(\frac{2}{0.189}\right)\ln\left[1+\exp(-0.033-0.189\ln(R_\lambda)\right]$$

✓

Dependence of β on R_λ : plausibly (?) $\beta \longrightarrow 1$ when $R_\lambda \longrightarrow \infty$

How close can one get to the R_λ infinity asymptotic regime ?



IF the fit proposed for γ holds...

... then, the $R_\lambda \to \infty$ regime

(say $\gamma > 0.99$)

is unreachable on Earth !

Analysis of strain: local and nonlocal aspects and implications for the regularity of Navier-Stokes

Nonlocal relation between strain and vorticity

Relation between strain and vorticity is nonlocal (~ Biot-Savart relation) !

$$S_{ij}(\mathbf{x}) = PV \int_{\mathbf{x}'} \frac{3}{8\pi} (\epsilon_{ikl} r_j + \epsilon_{jkl} r_i) \omega_l(\mathbf{x}') \, \frac{r_k}{r^5} \, d^3\mathbf{x}'$$

$$\mathbf{r} = \mathbf{x} - \mathbf{x}'$$

Here: separate the *local (L)* and *nonlocal (NL)* contributions to strain:

$$S_{ij}(\mathbf{x}) = \underbrace{\int_{r>R} [\cdots] \, d^3\mathbf{x}'}_{=S_{ij}^{NL}(\mathbf{x},R)} + \underbrace{\int_{r\leq R} [\cdots] \, d^3\mathbf{x}'}_{=S_{ij}^{L}(\mathbf{x},R)}$$

Decomposition: practical aspects

Performing the Biot-Savart directly is prohibitively costly from a numerical point of view !

An other approach: use the expansion (Hamlington, PRE 2008)

$$S_{ij}^{\rm NL}(\mathbf{x}, R) = \left[1 + \frac{R^2}{10}\nabla^2 + \frac{R^4}{280}\nabla^2\nabla^2 + \ldots + \frac{3R^{2n-2}}{(2n-2)!(4n^2-1)}(\nabla^2)^{n-1} + \ldots\right] S_{ij}(\mathbf{x}) \ .$$

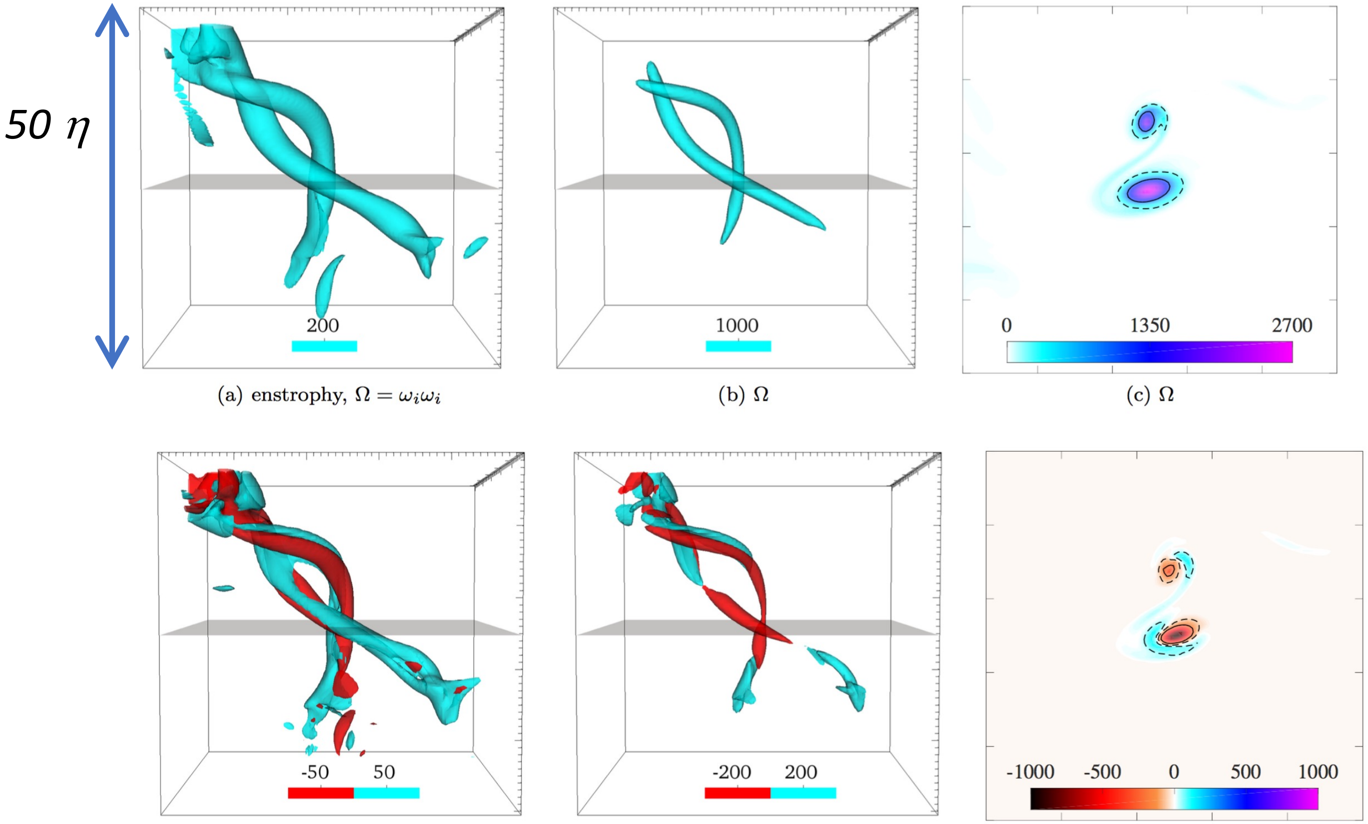
In Fourier space, this can be re-expressed (Buaria et al, Nat Comm. 2020):

$$\hat{S}_{ij}^{\rm NL}(\mathbf{k}, R) = f(kR)\hat{S}_{ij}(\mathbf{k}) \qquad f(kR) = \frac{3\left[\sin(kR) - kR\cos(kR)\right]}{(kR)^3}$$

$\Rightarrow$ Efficient way to compute of S^{NL} ; practical from a numerical point of view.

$\Rightarrow$ Our results agree with the available data from Hamlington *et al.* (Phys. Fluids, 2008).

How much local (self-) amplification for extreme events ?



(a) enstrophy, $\Omega = \omega_i \omega_i$ (b) Ω (c) Ω

(g) enstrophy production based on local strain, $P^L_\Omega = \omega_i \omega_j S^L_{ij}$ (h) P^L_Ω (i) P^L_Ω

Upper row: Enstrophy ($\Omega = \omega^2$)

Lower row: Amplification term:

$$P^L{}_\Omega = \omega . S^L . \omega$$

with $R = 2\ \eta$.

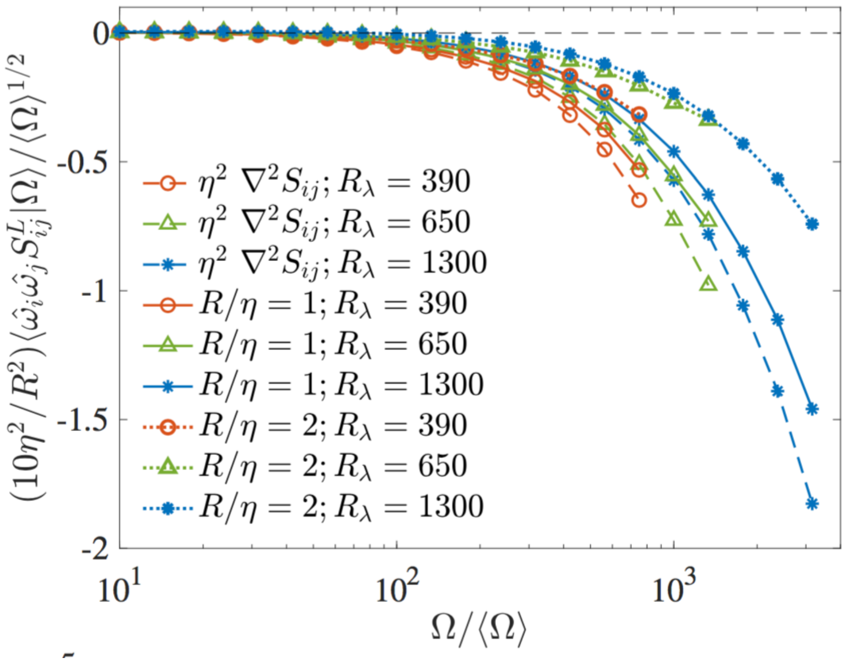
Main observation:

$P^L{}_\Omega$ is *negative* when Ω is large.

Buaria et al, Nat. Comm., 2020

How much local (self-) amplification for extreme events ?

Statistical analysis: compute the conditional average of the production term.



n.b.: It turns out that the production term,

$P^L_\Omega = \omega . S^L . \omega$

is *almost always negative* when

$R \lesssim 2\,\eta$

=> self-attenuation of vorticity.

Buaria et al, Nat. Comm, 2020

Self-attenuation and "helicity"

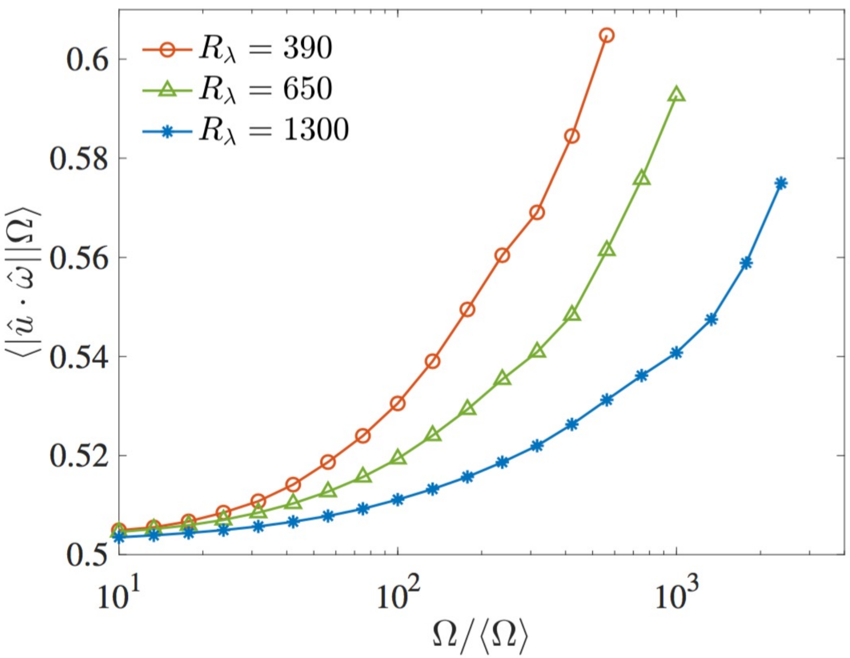


FIG. 3. **Preferential alignment of vorticity and velocity in regions of intense vorticity**. Averaged absolute value of the cosine between velocity and vorticity vectors, conditioned on enstrophy relative to its mean value, at Taylor-scale Reynolds numbers $R_\lambda = 390 - 1300$.

Analysis of the flow (…)

=>

there must be some alignment between vorticity and velocity in the most intense vortex structures, as observed numerically

(also noticed by Choi et al, PRE 2009).

Buaria et al, Nat. Comm., 2020

Lessons for Navier-Stokes regularity

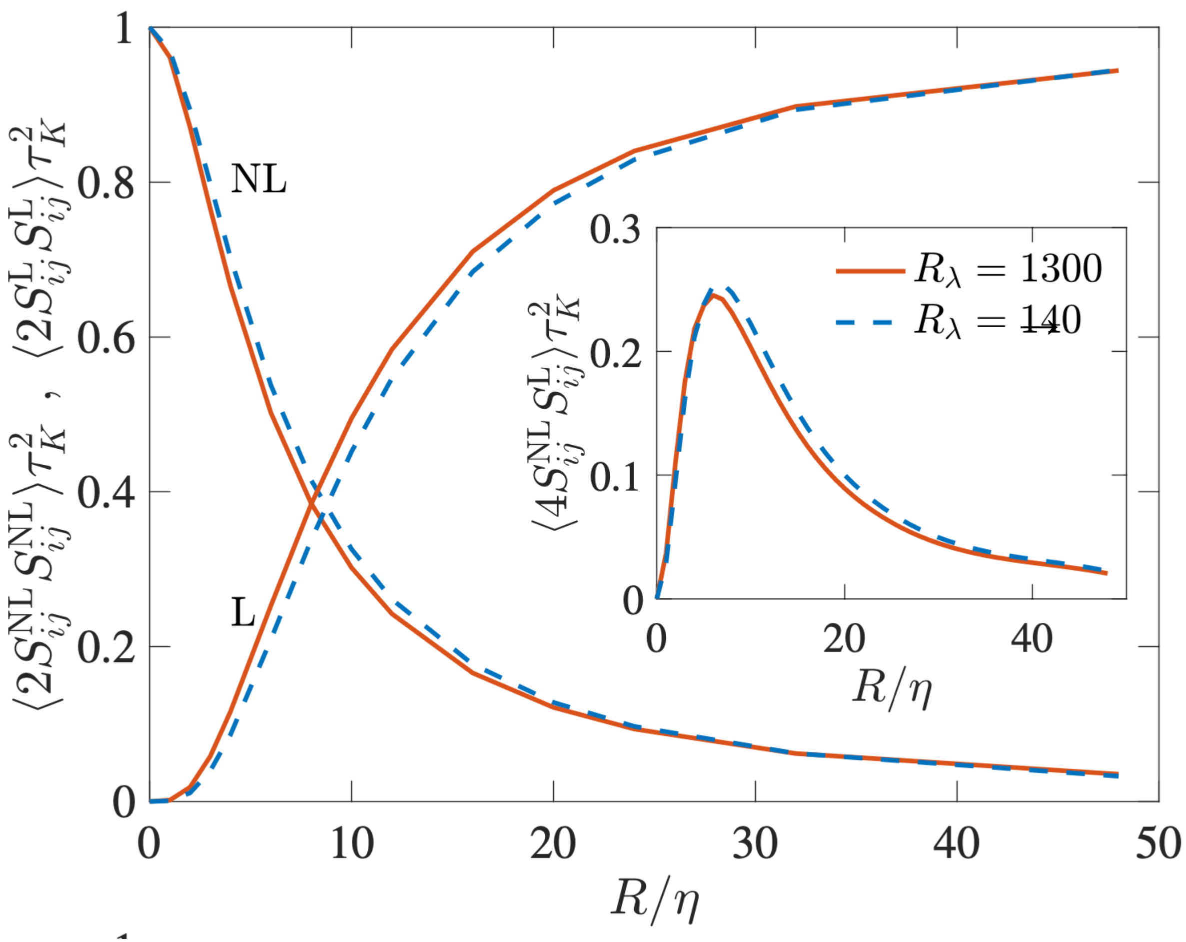
- *Lesson learned*: in DNS, no self-amplification of the most intense vortex structures. On the contrary, “self-interaction” leads to a weakening of the vortex.

  => the very intense events cannot be infinitely strong.

- Understanding the physical origin of the effect, and generalizing the numerical results to “any flow” would be sufficient to show regularity of the Navier-Stokes equations (Constantin *et al. Comm PDE, 1996*).

Decomposition of strain:
implications for the regions of intense velocity gradient.

Local vs. nonlocal strain



<u>*Obvious remarks:*</u>

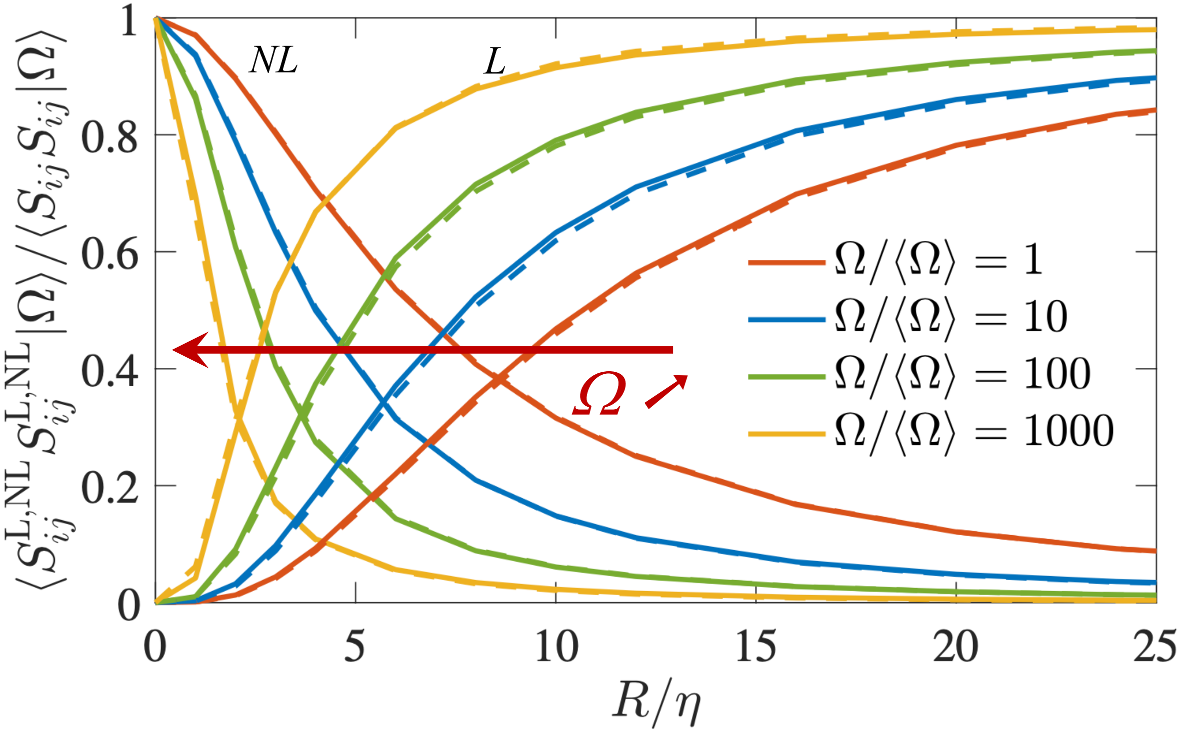
- when $R/\eta \to 0$, $S^{NL} \sim S$, $S^{L} \sim 0$
- when $R/\eta \to \infty$, $S^{NL} \sim 0$, $S^{L} \sim S$

$\Rightarrow$ there exists a scale ρ, at which the two are comparable:

$\rho \sim 10\,\eta$

nb: this scale does not depend very much on R_λ.

Local vs. nonlocal strain: conditioning on vorticity



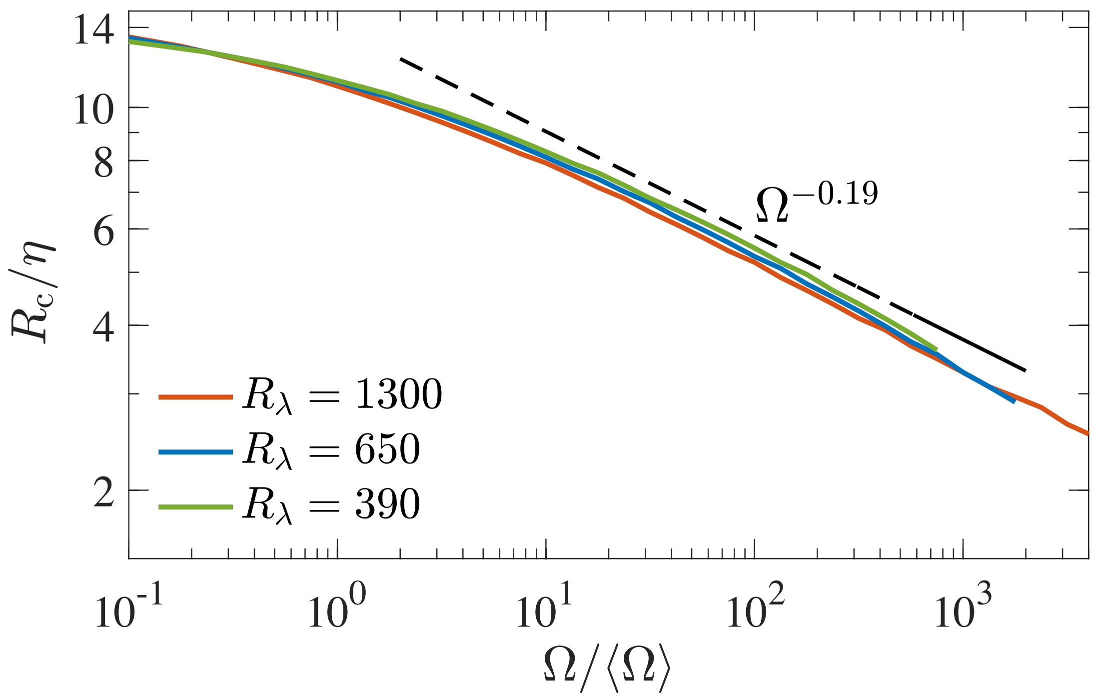
Condition on the intensity of vorticity, Ω.

$\Rightarrow$ When Ω increases, the size $\rho(\Omega)$ at which $|S^{NL}| \sim |S^{L}|$ *decreases*.

Dependence of ρ on Ω?

Full lines: $R_\lambda = 1{,}300$;
Dashed lines: $R_\lambda = 650$.

Dependence of the size on Ω



Plausible scaling of $\rho(\Omega)$:

$$\rho(\Omega) \sim \Omega^{-\gamma/4}.$$

… as expected based on the phenomenological argument

Is there some underlying "self-similarity" ?

With what we know, the answer appears to be negative.

For the most extreme events, the stretching induced by the local contribution of strain changes sign:

$$(\omega \cdot S^L \cdot \omega) \text{ becomes } < 0 \text{ for } \Omega \gtrsim 100.$$

$\sim$ The local strain opposes further growth of vorticity when Ω ***reaches extreme values*** (Buaria et al, Nat. Comm. 2020).

=> This effectively breaks the self-similarity (in Ω) .

Conclusions

- The very large fluctuations of the velocity gradients behave as R_λ^{β}/τ_K, with

$\beta = 0.775 \pm 0.025$ over the range of Reynolds number considered ($140 \leq R_\lambda \leq 650$).

- The largest gradients consist of vortex tubes, with a velocity jump ~ u_{rms}; over a distance ~ $\eta_K R_\lambda^{\alpha}$, with $\alpha = \beta - 1/2$.
- Relation with the straining rate $\langle S^2 | \Omega^2 \rangle \propto \Omega^{\gamma}$; with $\beta = 1/(2-\gamma)$.
- Characteristic size of the vortices seems to emerge: $\rho(\Omega) \sim \Omega^{-\gamma/4}$.

… but only approximate self-similarity …

~ Slow variation of γ; hence β. *What is the $R_\lambda \rightarrow \infty$ limit ??*

Thanks for your attention!

Questions ?

Supercomputing resources:

GCS/JSC (Juqueen/Juwels), XSEDE (Stampede2/Frontera)

- Talk based on the following references:
- D. Buaria, A. Pumir, E. Bodenschatz and P.K. Yeung, New J Phys **21**, 043004 (2019)
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- D. Buaria, A. Pumir and E. Bodenschatz, Phil. Trans. A 380, 20210088 (2022).
- D. Buaria and A. Pumir, in preparation (2023).