Leapfrogging vortex rings for the 3d incompressible Euler equations

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Euler equations for inviscid incompressible fluid of uniform density in $\mathbb{R}^3$:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p,$$
$$\operatorname{div} \mathbf{u} = 0, \quad x \in \mathbb{R}^3, \ t \geq 0.$$

where the velocity field $\mathbf{u}$ and the pressure p are unknowns.

Let

$$\vec{\omega} = \nabla \times \mathbf{u}, \quad \mathbf{u} = \nabla \times \vec{\psi}$$

The **stream-vorticity** formulation

$$\begin{aligned} &\partial_t \vec{\omega} + (\mathbf{u} \cdot \nabla) \vec{\omega} - (\vec{\omega} \cdot \nabla) \mathbf{u} = 0 \ , \\ &\mathbf{u} = \nabla \times \vec{\psi}, \quad \vec{\psi}(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \vec{\omega}(y,t)\, dy. \end{aligned} \tag{SV}$$

Vortex rings. Solutions to Euler equations with vorticity concentrated around travelling circles with **thin** section

Leapfrogging. Interaction of several vortex rings moving in the same direction along the common symmetry axis
Helmholtz (1858)

Axisymmetric No-swirl Euler:

The velocity:

$$\mathbf{u}(x,t) = u^r(r,z,t)\,\mathbf{e}_r + u^z(r,z,t)\,\mathbf{e}_z, \quad x = (r\cos\theta, r\sin\theta, z),$$

where $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)$, $\mathbf{e}_z = (0,0,1)$.

The vorticity $\vec{\omega} = \nabla \times \mathbf{u}$

$$\vec{\omega}(x,t) = \omega^\theta(r,z,t)\,\mathbf{e}_\theta, \quad \text{where } \omega^\theta = \partial_z u^r - \partial_r u^z.$$

The divergence free condition $\nabla \cdot \mathbf{u} = 0$:

$$u^r = -\partial_z \psi^\theta, \quad u^z = \frac{1}{r}\partial_r(r\psi^\theta), \quad -\Delta\vec{\psi} = \vec{\omega}, \quad \vec{\psi} = \psi^\theta \mathbf{e}_\theta$$

in $\Sigma := \{(r,z) \,/\, r > 0, z \in \mathbb{R}\}$.

With the change of variables

$$\omega = \frac{\omega^\theta}{r}, \quad \psi = \frac{\psi^\theta}{r}$$

the Euler equations become

$$\boxed{\begin{array}{l} r\,\partial_t \omega + \nabla^\perp(r^2\psi)\cdot\nabla\omega = 0, \quad -\Delta_5\psi = \omega \quad \text{in } \Sigma, \quad t>0 \\ \partial_r\psi(0,z,t) = 0, \quad \lim_{|(r,z)|\to\infty} \psi(r,z,t) = 0. \end{array}} \qquad (*)$$

where $\nabla^\perp = (-\partial_z, \partial_r)$

$$\Delta_5\psi = \partial_{rr}\psi + \frac{3}{r}\partial_r\psi + \partial_{zz}\psi.$$

Ukhovskii-Yudovich [68], Danchin [07].
Unique global-in-time solution if $\omega^\theta(\cdot,0),\ r^{-1}\omega^\theta(\cdot,0) \in L^1\cap L^\infty(\mathbb{R}^3)$

Vortex rings

A **vortex ring** is a travelling wave solution with **constant speed** c along the z-axis

$$\boxed{\begin{array}{ll} r\,\partial_t\omega + \nabla^{\perp}(r^2\psi)\cdot\nabla\omega = 0, \quad -\Delta_5\psi = \omega & \text{in } \Sigma \\ \partial_r\psi(0,z,t) = 0, \quad \lim\limits_{|(r,z)|\to\infty}\psi(r,z,t) = 0. & \end{array}} \qquad (*)$$

It has the form

$$\omega(r,z,t) = W_0(r, z - c\,t), \quad \psi(r,z,t) = \Psi_0(r, z - c\,t),$$

where W_0 and Ψ_0 solve

$$\boxed{\nabla^{\perp}\left(r^2(\Psi_0 - \frac{c}{2})\right)\cdot\nabla W_0 = 0, \quad -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_0 = W_0} \qquad (**)$$

Remark: If $\Psi_0(r,z)$ satisfies a semilinear equation of the form

$$-[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_0 = f\left(r^2(\Psi_0 - \frac{c}{2})\right) \quad \text{in } \Sigma,$$

for an arbitrary nonlinearity f, then

$$\Psi_0,\ W_0 = f\left(r^2(\Psi_0 - \frac{c}{2})\right)$$

solve

$$\boxed{\nabla^\perp\left(r^2(\Psi_0 - \frac{c}{2})\right)\cdot\nabla W_0 = 0, \quad -[\partial_r^2 + \frac{3}{r}\partial_r + \partial_z^2]\Psi_0 = W_0} \quad (**)$$

Fraenkel (1970, 1972), Maruhn (1957) Vortex rings with **very small** cross section

Assume $(W_\varepsilon, \Psi_\varepsilon)$ is a vortex ring with ε-concentrated vorticity

$$r\,W_\varepsilon(r,z) = \frac{1}{\varepsilon^2} U\left(\frac{x-(r_0,0)}{\varepsilon}\right) \rightharpoonup 8\pi\delta_{P_0}, \quad P_0 = (r_0,0), \quad x = (r,z).$$

We take the **Rosenhead-Kauffmann Scully** vortex.

$$U(y) = \frac{8}{(1+|y|^2)^2}, \quad y \in \mathbb{R}^2$$

The Green's function for $\Delta_5 := \partial_{rr}^2 + \frac{3}{r}\partial_r + \partial_{zz}^2$

$$-\Delta_5 G(x,P_0) = 8\pi\delta_{P_0}, \quad \frac{\partial G}{\partial r}(0,z) = 0, \quad G(x,P_0) \to 0 \quad \text{as } |x| \to \infty$$

Locally around P_0, $G(\cdot, P_0)$ has the expansion

$$G(x,P_0) = \log\frac{1}{|x-P_0|^4}\left(1 - \frac{3}{2r_0}(r-r_0) + O(|x-P_0|^2)\right)$$

The stream-function close to $P_0 = (r_0, 0)$

$$r_0\,\Psi_\varepsilon(r,z) = \log \frac{1}{(\varepsilon^2 + |x - P_0|^2)^2}\Big(1 - \frac{3}{2r_0}(r - r_0)\Big)$$

$$= \big(-4\log\varepsilon - 2\log(1 + |y|^2)\big)\,\big(1 - \frac{3}{2r_0}\varepsilon y_1\big), \quad y = \frac{x - P_0}{\varepsilon}$$

Replacing these approximate expressions into the equation

$$\nabla^\perp\Big(r^2(\Psi_\varepsilon - \frac{c}{2})\Big)\cdot\nabla W_\varepsilon = 2r(\Psi_\varepsilon - \frac{c}{2})\mathbf{e_2}\cdot\nabla W_\varepsilon + r^2\,\nabla^\perp\Psi_\varepsilon\cdot\nabla W_\varepsilon$$

Near $P_0 = (r_0, 0)$, in $y = \frac{x - P_0}{\varepsilon}$

$$\varepsilon^4 \nabla^\perp \Psi_\varepsilon \cdot \nabla W_\varepsilon \approx \underbrace{\nabla^\perp(-2\log(1+|y|^2)) \cdot \nabla U}_{=0} - 4(\log\varepsilon)\,(-\frac{3}{2r_0^2})\,\varepsilon\,\mathbf{e_2} \cdot \nabla U.$$

Combining the terms

$$\varepsilon^4 \nabla^\perp \left(r^2(\Psi_\varepsilon - \frac{c}{2}) \right) \cdot \nabla W_\varepsilon \approx \varepsilon \left[2r_0(\frac{-4\log\varepsilon}{r_0} - \frac{c}{2}) + 6\log\varepsilon \right] \mathbf{e_2} \cdot \nabla U$$

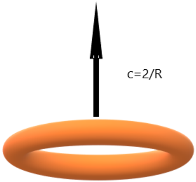
The speed of the vortex ring: (Fraenkel 1970, 1972)

$$c = \frac{2}{r_0} |\log \varepsilon| (1 + o(1)) \quad \text{as } \varepsilon \to 0.$$

Helmholtz (1858), Hill's spherical vortex in Lamb (1932), Norbury (1972-74)

Via constrained variational method: Arnold (1964), Fraenkel-Berger (1974), Benjamin (1976), Friedman-Turkington (1981), Burton (1987), Ambrosetti-Struwe (1989)
Conservation laws: Benedetto-Caglioti-Marchioro (2000), Negrini-Marchioro (1999), Butta-Cavallaro-Marchioro (2022)

If $\tau = |\log \varepsilon| t$, the core of the vortex ring is an ε-tubolar neighborhood of a circle $\gamma(s,t)$ of radius R traveling vertically with constant speed $c = \frac{2}{R}$

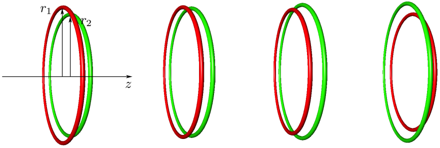


It solves the **bi-normal flow**

$$\gamma_\tau = 2\,(\gamma_s \times \gamma_{ss}) = 2\,\kappa\,\mathbf{b},$$

da Rios' formal computation (1904) for vortex filaments
Dávila, del Pino, Musso, Wei [2021] for helical symmetry
Gutierrez-Rivas-Vega (2003), Banica-Vega (2009,2013,2020,2022)

Leapfrogging: interacting vortex rings



When two vortex-rings interact, Helmholtz predicts the following:

Helmholtz 1858: We can now see generally how two ring-formed vortex-filaments having the same axis would mutually affect each other, since each, in addition to its proper motion, has that of its elements of fluid as produced by the other. If they have the same direction of rotation, they travel in the same direction; the foremost widens and travels more slowly, the pursuer shrinks and travels faster till finally, if their velocities are not too different, it overtakes the first and penetrates it. Then the same game goes on in the opposite order, so that the rings pass through each other alternately.

$$\text{speed} \sim \frac{1}{\text{radius}}$$

Axi-symmetric no-swirl

$$\boxed{\begin{array}{c} r\,\partial_t\omega + \nabla^{\perp}(r^2\psi)\cdot\nabla\omega = 0, \quad -\Delta_5\psi = \omega \quad \text{in } \Sigma \\ \psi_r(0,z,t) = 0, \quad \lim_{|(r,z)|\to\infty} \psi(r,z,t) = 0. \end{array}} \qquad (*)$$

Let $W(r,z,\tau)$, $\tau = |\log\varepsilon|\, t$

$$\omega(r,z,t) \;=\; W(r, z - 2r_0^{-1}\,|\log\varepsilon|\, t, |\log\varepsilon|\, t\,),$$

Problem $(*)$ takes the form

$$\boxed{\begin{array}{c} |\log\varepsilon|\, r\,\partial_\tau W + \nabla^{\perp}(r^2\,(\Psi - r_0^{-1}|\log\varepsilon|)\,)\cdot\nabla W \;=\; 0, \quad -\Delta_5\Psi = W, \\ r > 0, \quad z\in\mathbb{R}, \quad \tau\in[0,T) \end{array}} \qquad (***)$$

Formal Derivation of Leap-Frogging

Assume

$$r\,W(r,z,\tau) = \sum_{j=1}^{k} U_j, \quad U_j = \frac{1}{\varepsilon_j^2(\tau)} U\left(\frac{x - Q_j(\tau)}{\varepsilon_j(\tau)}\right), \quad x = (r,z)$$

This ansatz conserves of circulation. To have conservation of the L^∞-norm for the vorticity we choose

$$Q_j^1(\tau)\varepsilon_j^2(\tau) = r_0\,\varepsilon^2, \quad Q_j(\tau) = \left(Q_j^1, Q_j^2\right)$$

At main order, near $Q_j(\tau)$

$$W(r,z,\tau) = \sum_{j=1}^{k} \frac{1}{r_0\varepsilon^2} U\left(\frac{x - Q_j(\tau)}{\varepsilon_j}\right)$$

Correspondingly: $\Psi_\varepsilon(r,z,\tau) = \sum_{j=1}^{k} \Psi_j$

$$\Psi_j = \sum_{j=1}^{k} \frac{2}{Q_j^1} \log\left(\frac{1}{|x - Q_j(\tau)|^2 + \varepsilon_j^2}\right)(1 - \frac{3}{2Q_j^1}(r - Q_j^1)),$$

Now: fix j and compute the **equation** near Q_j: for $y = \frac{x - Q_j(\tau)}{\varepsilon_j}$

$$\varepsilon^4\, r\,|\log\varepsilon|\,\partial_\tau W \approx -\varepsilon\, r\,|\log\varepsilon|\,\frac{dQ_j}{d\tau}\,\nabla U$$

and

$$\varepsilon^4 \nabla^\perp (r^2 (\Psi - r_0^{-1} |\log \varepsilon|)) \cdot \nabla W \approx \underbrace{\varepsilon^4 \nabla^\perp (r^2 (\Psi_j - (Q_j^1)^{-1} |\log \varepsilon|)) \cdot \nabla W}_{=\varepsilon^2}$$

$$+ \varepsilon^4 \nabla^\perp (r^2 ((Q_j^1)^{-1} - r_0^{-1}) |\log \varepsilon|) \cdot \nabla W + \varepsilon^4 \nabla^\perp (r^2 \sum_{i \neq j} \Psi_i) \cdot \nabla W$$

$$\approx \varepsilon \left[-2 \frac{Q_j^1 - r_0}{r_0} |\log \varepsilon| \, \mathbf{e}_2 + \nabla_x^\perp \sum_{i \neq j} 4 r \log \frac{1}{|x - Q_i|} \right] \cdot \nabla U$$

Combining the terms, we get that the equation near Q_j

$$\left[- r |\log \varepsilon| \frac{dQ_j}{d\tau} - 4 r \sum_{i \neq j} \frac{(Q_j - Q_i)^\perp}{|Q_j - Q_i|^2} - 2 \frac{Q_j^1 - r_0}{r_0} |\log \varepsilon| \, \mathbf{e}_2 \right] \cdot \nabla U(y) \approx 0$$

The location of the centres Q_j of the rings

$$|\log \varepsilon| \frac{dQ_j}{d\tau} = -4 \sum_{\ell \neq j} \frac{(Q_j - Q_\ell)^\perp}{|Q_j - Q_\ell|^2} - 2 \frac{Q_j^1 - r_0}{r_0^2} |\log \varepsilon| \, \mathbf{e}_2$$

It is convenient to use the ansatz

$$Q_j(\tau) = (r_0, 0) + \frac{1}{\sqrt{|\log \varepsilon|}} q_j(\tau), \quad q_j(\tau) = (q_j^1(\tau), q_j^2(\tau)).$$

Neglecting lower order terms in a fixed interval $\tau \in [0, T]$, the limiting system

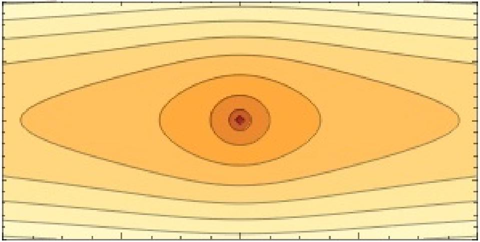
$$\frac{dq_j}{d\tau} = -4 \sum_{\ell \neq j} \frac{(q_j - q_\ell)^\perp}{|q_j - q_\ell|^2} - 2 \frac{q_j^1}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tau \in [0, T]. \quad (L)$$

This is a Hamiltonian system for the energy

$$H_k(q_1, \dots, q_k) = -2 \sum_{i \neq j} \log |q_i - q_j| - \frac{1}{r_0^2} \sum_{j=1}^{k} |q_j^1|^2.$$

For instance for $k = 2$ and restricting ourselves to $q_1 = -q_2 = q$ we arrive at the system

$$\frac{dq}{d\tau} = -2 \frac{q^{\perp}}{|q|^2} - 2 \frac{q^1}{r_0^2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tau \in [0, T].$$



Theorem [Dávila, del Pino, Musso, Wei, 2022]

Let $q(\tau) = (q_1(\tau), \dots q_k(\tau))$ be a solution of System (L) in $[0, T]$. Then there exists a smooth solution $W_\varepsilon(r, z, \tau)$ to $(***)$ such that for certain points $Q_j^\varepsilon(\tau)$ with the form

$$Q_j^\varepsilon(\tau) = (r_0, 0) + \frac{1}{\sqrt{|\log \varepsilon|}} q_j(\tau) + O\Big(\frac{\log(|\log \varepsilon|)}{|\log \varepsilon|}\Big),$$

we have

$$W_\varepsilon(r, z, \tau) = \frac{1}{\varepsilon^2 r_0} \sum_{j=1}^{k} U\left(\frac{x - Q_j^\varepsilon(\tau)}{\varepsilon_j}\right) + \varphi(r, z, \tau), \quad x = (r, z)$$

where U is the bf Kauffmann-Scully vortex

$$U(y) = \frac{8}{(1+|y|^2)^2},$$

$\varepsilon_j^2 = \varepsilon^2 \frac{r_0}{Q_j^1(\tau)}$ and

$$|\varphi(r,z,\tau)| \leq C \sum_{j=1}^{k} \frac{\varepsilon^2 |\log \varepsilon|^{60}}{\varepsilon^2 + \left| x - Q_j^\varepsilon(\tau) \right|^2}, \quad \tau \in [0, T]$$

Remarks

1. Reduced dynamics (L): Dyson (1893), Hicks (1922), Lamb (1932), Lim (1997), Klein-Majda-Damodaran (1995)

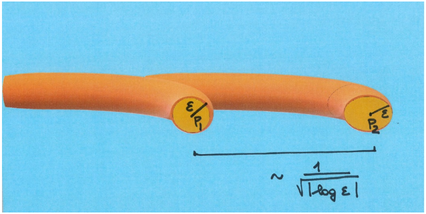
2. Numerical simulations for leapfrogging: Riley-Stevens (1993), Lim (1997), Cheng-Lou-Lim (2015), Alvarez-Ning (2022)

3. Experiment for leapfrogging: Yamada-Matsui (1978)

4. Jerrard-Smets (2018): gave the first mathematical justification of leapfrogging in three-dimensional Gross-Pitaeskii equation

$$iu_t - \Delta u = \frac{1}{\varepsilon^2}(1 - |u|^2)u \quad \text{in} \quad \mathbb{R}^3$$

$u : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2$



Ingredients in the construction:

• Improvement of the approximation in powers of ε: $(w_\varepsilon^*, \psi_\varepsilon^*)$

• Setting up the problem as a coupled system of inner problems near the singularities and and an outer problem more regular (the inner-outer gluing scheme)

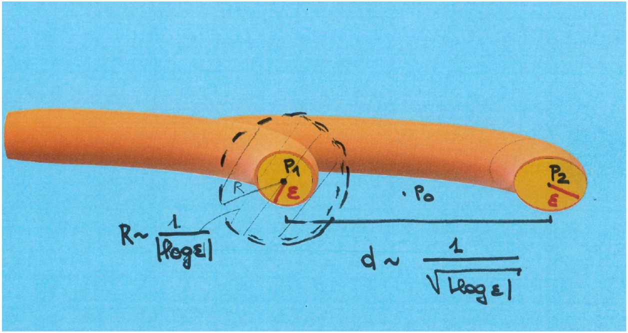
• A priori estimates to solve by a continuation (degree) argument.
Dávila, del Pino, Musso, Wei (2020): point concentration for 2-d Euler equations

Sketch of the proof. We want to solve the equation $S(\omega,\psi)=0$, where

$$S(\omega,\psi) := |\log\varepsilon| r\partial_t\omega + \nabla^{\perp}\left[r^2(\psi - r_0^{-1}|\log\epsilon|)\right]\cdot\nabla\omega = 0,$$
$$-\Delta_5\psi = \omega.$$

Introduce cut-off functions

$$\eta_j(x,t) = \eta(|\log\varepsilon||x-Q_j|), \quad \eta(s) = \begin{cases} 1 & s\le 1 \\ 0 & s\ge 2 \end{cases}$$



The inner-outer gluing scheme

$$\psi(x,t) = \psi_\varepsilon^* + \sum_{j=1}^{2} \frac{\eta_j}{r_j}\psi_j(\frac{x-Q_j}{\varepsilon_j}, t) + \psi^{out}(x,t)$$

$$\omega(x,t) = w_\varepsilon^* + \sum_{j=1}^{2} \frac{\eta_j}{r_j\varepsilon_j^2}\phi_j(\frac{x-Q_j}{\varepsilon_j}, t) + \phi^{out}(x,t)$$

where $-\Delta_5\psi = \omega$, $\phi_j = -\Delta_{5,j}\psi_j$

$$\Delta_{5,j}\psi_j := -[\Delta_y\psi_j + \frac{3\varepsilon_j}{r_j+\varepsilon_j y_1}\partial_1\psi_j], \quad y = \frac{x-Q_j}{\varepsilon_j}.$$

The problem becomes

$$S(\omega,\psi) = 0 \quad \text{if} \quad \begin{cases} E_j^{in}[\phi_j, \psi_j, \psi^{out}, Q] = 0, \quad j = 1,2, \\ E^{out}[\phi^{out}, \psi^{out}, \phi_j, \psi_j, Q] = 0 \end{cases}$$

A simplified version of E_j^{in}

$$
\begin{aligned}
E_j^{in}(y,t) :=\ & \varepsilon_j^2|\log\varepsilon|\partial_t\phi_j + \varepsilon_j|\log\varepsilon|\nabla\phi\cdot\partial_t Q_j \\
& + \nabla^\perp\left((1+\frac{\varepsilon_j}{2r_j}\,y_1)\,\Gamma_0\right)\cdot\nabla\phi_j + \nabla^\perp\left((1+\frac{2\varepsilon_j}{r_j}\,y_1)\,\psi_j\right)\cdot\nabla U \\
& + \nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j + r_j\psi^{out})\right)\cdot\nabla\phi_j \\
& + \eta_j e_{final}^*, \qquad y\in B(0,\varepsilon^{-1}|\log\varepsilon|^{-1}),\ t\in[0,T)
\end{aligned}
$$

where $\Gamma_0(y) = -2\log(1+|y|^2)$, $y = \frac{x-Q_j}{\varepsilon_j}$, $Q_j = (r_j, z_j)$, and e_{final}^* is the final error

$$
e_{final}^* = \varepsilon_j^4 S(w_\varepsilon^*, \psi_\varepsilon^*)(\varepsilon_j y + Q_j)
$$

A simplified version of E^{out}

$$E^{out}(x,t) := |\log\varepsilon|\, r\, \phi_t^{out} + \nabla_x^{\perp}(r^2(\Psi^0 - r_0^{-1}|\log\varepsilon|)) \cdot \nabla_x \phi^{out}$$

$$+ \sum_{j=1}^{2} \left[r\, |\log\varepsilon|\, \partial_t \bar{\eta}_{j1} + \nabla_x^{\perp}(r^2(\Psi^0 - r_0^{-1}|\log\varepsilon|)) \nabla \bar{\eta}_{1j} \right] \frac{\phi_j}{\varepsilon_j^2 r_j}$$

$$+ (1 - \sum_{j=1}^{2} \eta_{j1}) S(w_\varepsilon^*, \psi_\varepsilon^*) = 0 \quad r > 0,\ z \in \mathbb{R},\ t \in [0,T)$$

To decouple the inner and outer problems, we need the inner functions ϕ_j to decay as ρ becomes large

For the **inner problem** we solve in $\mathbb{R}^2$

$$\varepsilon^2|\log\varepsilon|\phi_t - \nabla^\perp\Gamma_0\cdot\nabla(\Delta\psi + f'(\Gamma_0)\psi) + E(y,t) = 0, \quad \phi(y,0) = 0$$
$$-\Delta\psi = \phi \quad \text{in } \mathbb{R}^2\times[0,T]$$

A central ingredient is an L^2-a priori estimate:

Lemma: A priori estimates If ϕ is a solution and satisfies certain orthogonality conditions, then the following estimate holds

$$\boxed{\|\phi(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)} \le C\varepsilon^{-2}|\log\varepsilon|^{-\frac{1}{2}} \sup_{t\in[0,T]} \|E(\cdot,t)U^{-\frac{1}{2}}\|_{L^2(\mathbb{R}^2)}}$$

The **inner problem**

$$\varepsilon^2 |\log \varepsilon| \phi_t - \nabla^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + f'(\Gamma_0)\psi) + Q(\phi) + e^*_{final} = 0,$$
$$\phi(y,0) = 0, \quad -\Delta_5 \psi = \phi \quad \text{in } \mathbb{R}^2 \times [0,T]$$

The whole construction works if we get to an approximation $(w^*_\varepsilon, \psi^*_\varepsilon)$ with a final error

$$e^*_{final} = \frac{\varepsilon^5 |\log \varepsilon|^\beta}{1+\rho^3}, \quad \beta > 0$$

How do we improve the approximation? Recall

$$w_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{\varepsilon_j^2 r_j} U(\frac{x-Q_j}{\varepsilon_j}), \quad Q_j = (r_j, z_j)$$

$$\psi_\varepsilon^0(x,t) = \sum_{j=1}^{2} \frac{1}{r_j} \log \frac{1}{(\varepsilon_j^2 + |x-Q_j|^2)^2} \left[1 - \frac{3}{2r_j}(r - r_j) \right]$$

The points

$$Q_i = (r_0,\ 0) + \frac{1}{\sqrt{|\log \epsilon|}} b_i(t) + a_j(t)$$

with $|a_j(t)|_{L^\infty(0,T)} \leq \varepsilon^2$. So: $|Q_1 - Q_2| \sim \frac{1}{\sqrt{|\log \epsilon|}}$.

Improvement of the error near Q_j

The inner equation:

$$
\begin{aligned}
&\varepsilon_j^2|\log\varepsilon|\partial_t\phi_j + \varepsilon_j|\log\varepsilon|\partial_t Q_j\cdot\nabla\phi_j \\
&+\nabla^\perp\left(\Gamma_0 + \frac{\varepsilon_j}{2r_j}\,y_1\,\Gamma_0\right)\cdot\nabla\phi_j + \nabla^\perp(\psi_j + \frac{2\varepsilon_j}{r_j}\,y_1\,\psi_j)\cdot\nabla U \\
&+\nabla^\perp\left((1+\frac{\varepsilon_j}{r_j}y_1)^2(\psi_j + r_j\psi^{out})\right)\cdot\nabla\phi_j + e_0 \sim 0
\end{aligned}
$$

where e_0 is the initial error

$$
e_0 = \frac{\varepsilon^2|\log\varepsilon|}{1+\rho^4}E_2, \quad \rho = |y|, \quad y = \frac{x-Q_j}{\varepsilon_j}.
$$

Let $y = \frac{x - Q_1}{\varepsilon_1}$, $\quad y = \rho e^{i\theta}$. Assume the error

$$E(y,t) = \sum_n E_n(\rho,t)e^{in\theta}, \ E_n(\rho,t) = \int E(\rho e^{i\theta},t)e^{in\theta}d\theta.$$

If $E_0 = 0$: we solve the elliptic equation

$$\nabla^{\perp}\Gamma_0 \cdot \nabla\phi + \nabla^{\perp}\psi \cdot \nabla U + E = 0, \quad -\Delta\psi = \phi.$$

Since $U = e^{\Gamma_0}$, the problem becomes

$$-\nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + U\psi) + E = 0$$

In polar coordinates $y = \rho e^{i\theta}$ we see that

$$\mathcal{L}[\psi] := \nabla_y^{\perp}\Gamma_0 \cdot \nabla(\Delta\psi + U\psi) = \frac{4}{1+\rho^2}\frac{\partial}{\partial\theta}[\Delta_y\psi + U\psi]$$

The operator $\mathcal{L}$ (for Liouville) is highly degenerate:

- All radial functions are in its kernel
- Kernel of

$$\Delta_y\psi + U\psi = 0, \quad \psi \in < 2 + \nabla\Gamma_0 \cdot y, \partial_{y_1}\Gamma_0, \partial_{y_2}\Gamma_0 >$$

We can use $\mathcal{L}$ for E_n, only for $n \geq 1$. We need to adjust the position of the points when $n = 1$.

We **cannot** use $\mathcal{L}$ to improve E_0. In this case we solve the ODE

$$\varepsilon_j^2 |\log \varepsilon| \partial_t \phi_3 + E_0 = 0.$$

To get **spacial decay**, we solve with the transport equation

$$\begin{aligned}\mathcal{T}(\phi_6) := & \,|\log \varepsilon|\, \varepsilon_j^2 \partial_t \phi_6 \\ & + \nabla^\perp \left(\Gamma + \frac{\varepsilon_j}{2r_j}\, y_1\, \Gamma \right) \cdot \nabla \phi_6\end{aligned}$$

Scheme for the inner approximation

$$e_0 = \frac{\varepsilon^2}{1+\rho^4} E_2 \underset{\mathcal{L}}{\Rightarrow} e_1 = \frac{\varepsilon^3}{1+\rho^3} E_1 \underset{\mathcal{L}\& a_j}{\Rightarrow} e_2 = \frac{\varepsilon^4}{1+\rho^2} E_0$$

$$\underset{ODE}{\Rightarrow} e_3 = \frac{\varepsilon^3}{1+\rho^3} \sin\theta \underset{\mathcal{L}\& a_j}{\Rightarrow} e_4 = \frac{\varepsilon^4}{1+\rho^2} E_2 \underset{\mathcal{L}}{\Rightarrow} e_5 = \frac{\varepsilon^5}{1+\rho} E_0$$

$$\underset{\mathcal{T}}{\Rightarrow} e_6 = \frac{\varepsilon^3}{1+\rho^5} E_1 \underset{\mathcal{L}\& a_j}{\Rightarrow} e_7 = \frac{\varepsilon^4}{1+\rho^4} E_0 \underset{ODE}{\Rightarrow} e_8 = \frac{\varepsilon^3}{1+\rho^5} \sin\theta$$

$$\underset{\mathcal{L}\& a_j}{\Rightarrow} e_9 = \frac{\varepsilon^4}{1+\rho^4} E_2 \underset{\mathcal{L}}{\Rightarrow} e_{10} = \frac{\varepsilon^5}{1+\rho^3} E_0$$

Thanks for your attention