Breaking description in depth-averaged models

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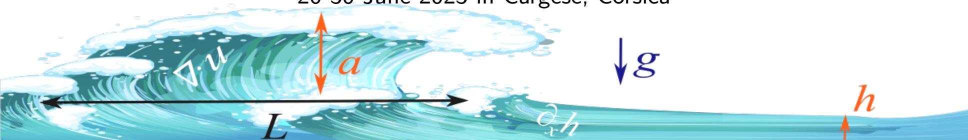


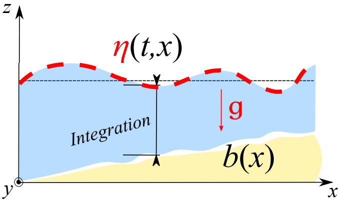
Water waves models

Free-surface incompressible Euler

$t > 0, \vec{x} \in (\mathbb{R}^3, b(\vec{x}) < z < \boldsymbol{\eta}(t, \vec{x}))$

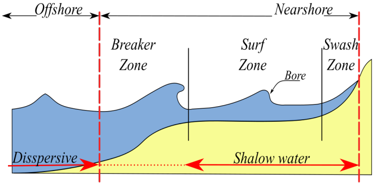
$$\begin{cases} u_t + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \boldsymbol{g} \\ \nabla \cdot u = 0, \quad \boldsymbol{g} = (0, 0, -g) \end{cases}$$

$+$ kinematic and dynamic boundary conditions



Water waves models

$$\mu \equiv \delta^2 = H^2/L^2 \text{ (shallowness)}$$
$$\varepsilon = a/H \text{ (nonlinearity)}$$



Hydrostatic pressure

constant velocity over vertical

$u(t,x,z) = \boldsymbol{v}(t,x)$

$$\begin{cases} \dfrac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{v}) = 0, & \text{(Mass Eq)} \\ \dfrac{\partial h\boldsymbol{v}}{\partial t} + \nabla \cdot \left(h\boldsymbol{v} \otimes \boldsymbol{v} + \dfrac{gh^2}{2}\mathcal{I} + p_{NH} \right) = 0, & \text{(Momentum Eq).} \end{cases}$$

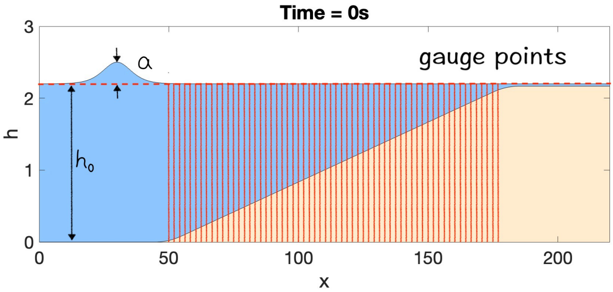
model	**NSWE** $\mathcal{O}(\mu)$	$\mathcal{O}(\varepsilon\mu)$	**SGN** $\mathcal{O}(\mu^2)$
Pressure	$p_{NH} = 0$	**Boussinesq**	$p_{NH} = h^2\ddot{h}/3$
ε	no assump		no assump.
Type	hyperbolic		dispersive

Lannes, **2013**

Breaking waves

Motivation

📖 Hsiao *et al.*, **2008**, $tg\beta = 0.017$



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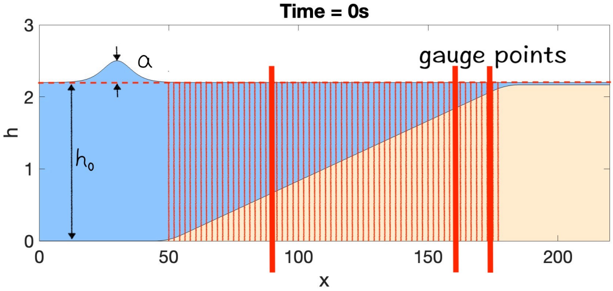

Breaking waves

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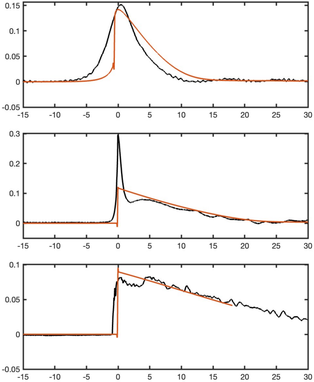
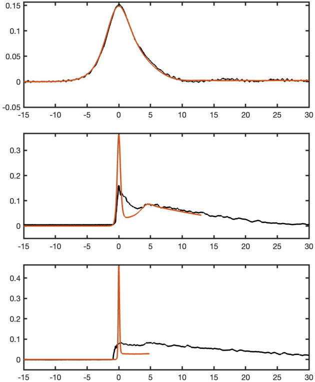
SW



SGN



State of the art

Advances on wave breaking modelling

Artificial dissipative terms

+ Viscous term in (Moment Eq)
+ Convective term in (Mass Eq)

NSWE:

- Packwood&Peregrine, **1981**

Boussinesq :

- Zelt, **1991**
- Wei *et al.*, **1999**

Hybrid method/Switching

− Drop Dispersive terms

Boussinesq type:

- Bonneton *et al.*, **2011**
- Tissier *et al.*, **2012**
- Kazolea *et al.*, **2014**
- Duran&Marche, **2015**

State of the art

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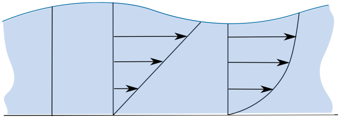
When? Breaking criterion

State of the art

Advances on wave breaking modelling

? Assumption on the velocity profile no valid! $u = U(t,x) + u'(t,x,z)$



(flat) (linear) (polynomial)

NO BREAKING

Hyperbolic framework

- Teshukov, **2007** 2D hyperbolic

Dispersive (conservative!) framework

- Castro&Lannes *et al.*, **2014**
- Richard&Gavriluyk *et al.*, **2015** Dispersive
- $\mathcal{MK}$&Noble, **2016** two-layer flow

ADD DISSIPATION

- Richard&Gavriluyk **2012** Hydraulic jumps
- Gavriluyk *et al.*, JFM, **2016** Breaking waves in two-layer model
- Ivanova&Gavriluyk **2018** Hydraulic jumps

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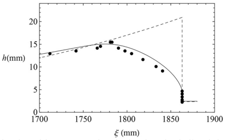

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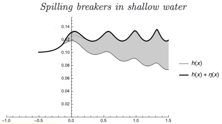

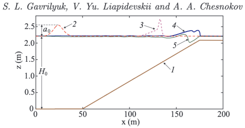

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Breaking waves

Model derivation

The filtering decomposition

$$\boldsymbol{v} = \overline{\boldsymbol{v}} + \boldsymbol{v}^r.$$

Two-dimensional filtered equation

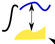
$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{w}}{\partial z} = 0,$$
$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \bar{u}\bar{w}}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial A^r_{xx}}{\partial x} + \frac{\partial A^r_{xz}}{\partial z}\right) + \nu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial z^2}\right),$$
$$\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{u}\bar{w}}{\partial x} + \frac{\partial \bar{w}^2}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left(\frac{\partial A^r_{xz}}{\partial x} + \frac{\partial A^r_{zz}}{\partial z}\right) + \nu\left(\frac{\partial^2 \bar{w}}{\partial x^2} + \frac{\partial^2 \bar{w}}{\partial z^2}\right),$$

$A^r_{xx},\ A^r_{xz},\ A^r_{zz}$ are the deviatoric part of the residual stress tensors which modelled by a turbulent viscosity hypothesis having the form

$$A^r_{xx} = 2\nu_T\frac{\partial \bar{u}}{\partial x},\ A^r_{zz} = -A^r_{xx},\ A^r_{xz} = \nu_T\left(\frac{\partial \bar{u}}{\partial z} + \frac{\partial \bar{w}}{\partial x}\right)$$

Breaking waves

Model derivation



Navier-Stokes $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

Stress tensor is modeled by turbulent viscosity ν_T and shear stress

Mass equation:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0$$

Ox-Momentum equation:

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left(\tilde{h}\tilde{U}^2 + \frac{\tilde{h}^2}{2} + \mu^2 \tilde{h}\int_{\tilde{b}}^{\tilde{\eta}} \tilde{u}'^2 d\tilde{z} + \mu^2 \int_{\tilde{b}}^{\tilde{\eta}} \tilde{p}_N d\tilde{z} - \mu^2 \boxed{2\tilde{\nu}_T \tilde{h}\frac{\partial \tilde{U}}{\partial \tilde{x}}}\right) = -\tilde{p}(b)\frac{\partial \tilde{b}}{\partial \tilde{x}}$$

Oz-Momentum equation:

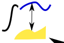
$$\int_{\tilde{b}}^{\tilde{\eta}} \frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{u}\tilde{w}}{\partial x} + \frac{\partial \tilde{w}^2}{\partial z} d\tilde{z} = -\tilde{p}_N(\eta) + \tilde{p}_N(b) + \text{stress tensor terms},$$

Defining $\tilde{\varphi}$ (enstrophy)

$$\tilde{\varphi} := \frac{1}{\tilde{h}^3}\int_{\tilde{b}}^{\tilde{\eta}} \tilde{u}'^2 d\tilde{z}.$$

Breaking waves

Model derivation

Navier-Stokes  $\xrightarrow{\quad}$ $\xrightarrow{O(\mu^2)}$ + dissipation

$$\begin{cases} \dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\boldsymbol{u}) = 0, \\ \dfrac{\partial (h\boldsymbol{u})}{\partial t} + \dfrac{\partial}{\partial x}\left(h\boldsymbol{u}^2 + \dfrac{gh^2}{2} + \dfrac{h^2}{3}\dfrac{D^2 h}{Dt^2} + h^3\varphi \right) = \partial_x \left(h\ \nu_T(x)\ \dfrac{\partial \boldsymbol{u}}{\partial x} \right) + G_b \\ \dfrac{\partial (h\varphi)}{\partial t} + \dfrac{\partial}{\partial x}(h\boldsymbol{u}\varphi) = \nu_T(x) \left(\dfrac{\partial \boldsymbol{u}}{\partial x} \right)^2 - D(x) \ , \quad \dfrac{Dh}{Dt} = \dfrac{\partial h}{\partial t} + \boldsymbol{u}\dfrac{\partial h}{\partial x} \end{cases}$$

$h\mathbf{K} = hu - \frac{1}{3}\nabla\left(h^3 \operatorname{div} u\right)$, $G_b = -gh\dfrac{\partial b}{\partial x}$ (on a mild slope)

Hyperbolic stage : $*$ Godunov's Type Scheme of 2^{nd} order HLL

$*$ Time discretization : RK-2 (Heun's method)

Elliptic stage

Le Métayer *et al.* **2010**

Breaking waves

Model derivation

Navier-Stokes $\xrightarrow{\int} \xrightarrow{O(\mu^2)}$ + dissipation

$$
\begin{cases}
\dfrac{\partial h}{\partial t} + \dfrac{\partial}{\partial x}(h\boldsymbol{u}) = 0, \\[2mm]
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\end{cases}
$$

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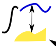
Navier-Stokes  $\longrightarrow \xrightarrow{O(\mu^2)}$ + dissipation

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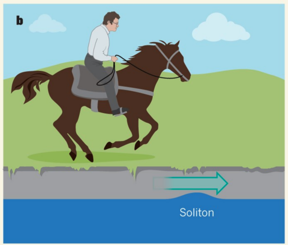
Numerical Simulations

Soliton test & Convergence

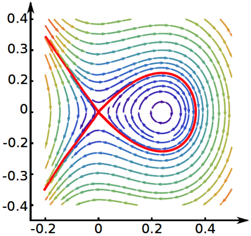
Taking into account nonlinear and dispersive effects yields existence of solitary wave solution.

It is an important physical phenomenon, which can be observed experimentally
(first, discovered and described by John Scott Russell, a naval architect from Glasgow)



Numerical Simulations

Soliton test & Convergence



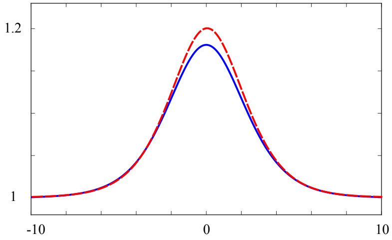
$$h(x,t) = h_0 + h_0\xi(x,t), \quad u(x,t) = c_0\left(1 - h_0/h(x,t)\right)$$

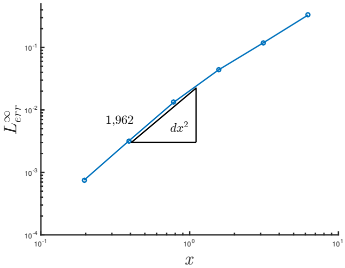
$$\xi(x,t) = \frac{2a\left(Fr^2 - 1 - 3\tilde{\varphi}\right)}{Fr^2 - 1 - (3+\tilde{a}^2)\tilde{\varphi} + \left(Fr^2 - 1 - (3-\tilde{a}^2)\tilde{\varphi}\right)\cosh(\kappa(x - c_0 t - x_0))}$$

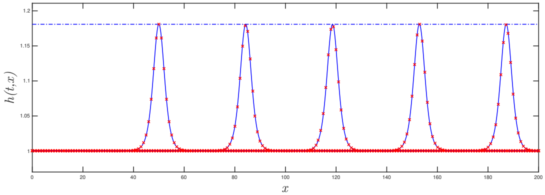
$$\kappa = \sqrt{\frac{3(Fr^2 - 1 - 3\tilde{\varphi})}{Fr^2}}, \; c_0 = \sqrt{g\left(h_0 + \tilde{a} + \tilde{\varphi}(3h_0 + \tilde{a})\right)}$$

Numerical Simulations

Soliton test & Convergence



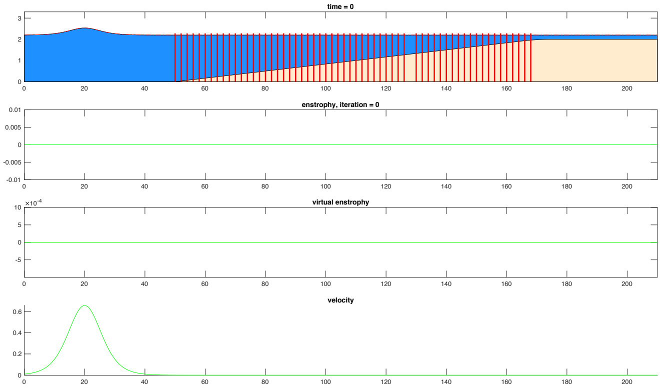




Numerical Simulations

Experimental Data Comparison

Hsiao *et al.*, **2008**, $tg\beta = 0.017$



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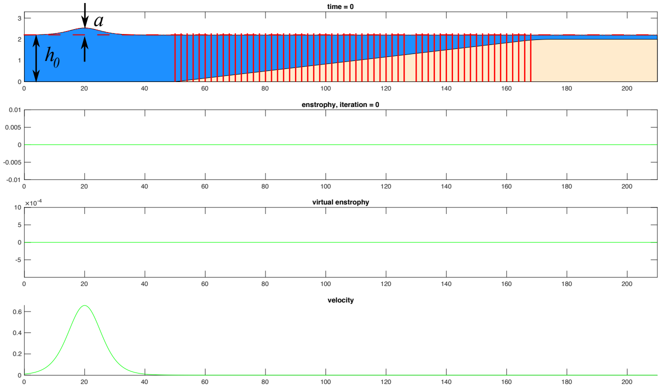

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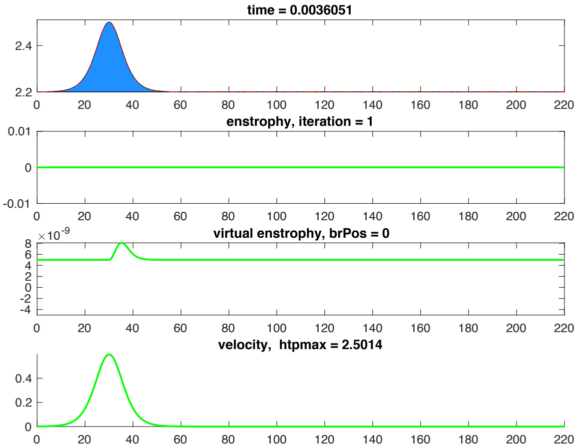
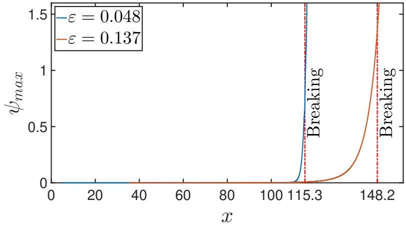


Entropy evolution

Virtual enstrophy : breaking criterion



$$
\begin{cases}
\forall t > t_* \quad \text{SGN} + \dfrac{\partial h\psi}{\partial t} + \dfrac{\partial (hU\psi)}{\partial x} = \dfrac{8h\sqrt{\psi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \psi^{\frac{3}{2}}, \\
t_* : \max_x(\psi(t_*, x)) \geq \psi_0 \\
\forall t > t_* \;\; \dfrac{\partial h\varphi}{\partial t} + \dfrac{\partial (hU\varphi)}{\partial x} = \dfrac{8h\sqrt{\varphi}}{R}\left(\dfrac{\partial U}{\partial x}\right)^2 - C_r h^3 \varphi^{\frac{3}{2}}.
\end{cases}
$$

Entropy evolution

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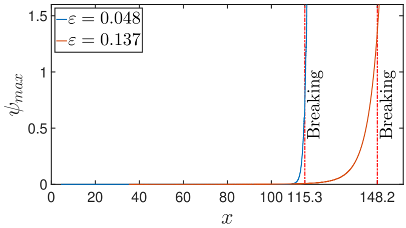
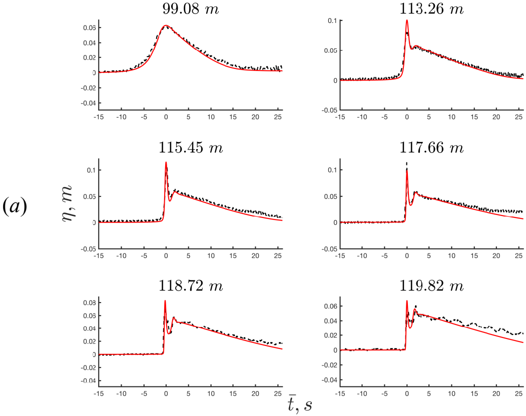


$$\psi_0 = \frac{g}{h_0^*}\widetilde{\psi}_0, \quad \widetilde{\psi}_0 = \begin{cases} \left(0.1 + \dfrac{0.031}{\varepsilon}\right), & \varepsilon > 0.05 \\ 0, & \varepsilon < 0.05 \end{cases}, R = \begin{cases} 1.7, & \mu > 0.05 \\ 6, & \mu < 0.05 \end{cases}$$

Numerical Simulations

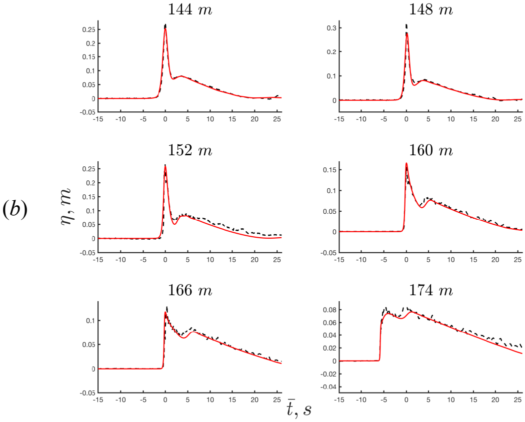
Experimental Data Comparison



$$N_{trial} = 3,\ h = 1.2m,\ \varepsilon = a_0/h_0 = 0.048$$

Numerical Simulations

Experimental Data Comparison



$$N_{trial} = 41,\ h = 2.2m,\ \varepsilon = a_0/h_0 = 0.137$$

Hyperbolic model for breaking waves

Richard, **2021**
(adding new variable for hyperbolic structure)

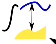
SW eq. SGN eq.

$$u(t,x,z) = U(t,x) + u'(t,x,z)$$
$$p = p_{Hydro} + p_{Nonhydro}$$

Model of Kazakova & Richard

Hyperbolic model

Model derivation: equations for h,U,W



Navier-Stokes $\longrightarrow$ $\xrightarrow{O(\mu^2)}$ + dissipation

Stress tensor is modeled by turbulent viscosity ν_T and shear stress

Mass equation:

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0$$

Ox-Momentum equation:

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h}\tilde{U}^2 + \frac{\tilde{h}^2}{2} + \mu^2 \tilde{h} \left\langle \tilde{u}'^2 \right\rangle + \mu^2 \int_{\tilde{b}}^{\tilde{\eta}} \tilde{p}_N d\tilde{z} - \mu^2 \boxed{2\tilde{\nu}_T \tilde{h} \frac{\partial \tilde{U}}{\partial \tilde{x}}} \right) = -\tilde{p}(b) \frac{\partial \tilde{b}}{\partial \tilde{x}}$$

Oz-Momentum equation:

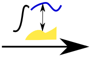
$$\int_{\tilde{b}}^{\tilde{\eta}} \frac{\partial \tilde{w}}{\partial t} + \frac{\partial \tilde{u}\tilde{w}}{\partial x} + \frac{\partial \tilde{w}^2}{\partial z} d\tilde{z} = -\tilde{p}_N(\eta) + \tilde{p}_N(b) + \text{stress tensor terms,}$$

Defining $\tilde{\varphi}$ (enstrophy), $\tilde{W}$, and $\tilde{P}$

$$\tilde{\varphi} := \frac{1}{\tilde{h}^3} \int_{\tilde{b}}^{\tilde{\eta}} \tilde{u}'^2 d\tilde{z}, \; \tilde{W} = \frac{1}{\tilde{h}} \int_{\tilde{b}}^{\tilde{\eta}} \tilde{w} d\tilde{z}, \; \tilde{P} = \frac{1}{\tilde{h}} \int_{\tilde{b}}^{\tilde{\eta}} \tilde{p}_N d\tilde{z}$$

Breaking waves

Model derivation: equations for P and φ

Energy equation  $\xrightarrow{O(\mu^2)}$

$$\frac{h^2}{2}\left(\frac{\partial h\varphi}{\partial t}+\frac{\partial hU\varphi}{\partial x}\right)+\frac{\partial h\langle e_a\rangle}{\partial t}+\frac{\partial hU\langle e_a\rangle}{\partial x}$$
$$=-h\left\langle P^r\right\rangle-\left(hP-2\nu_T h\frac{\partial U}{\partial x}\right)\frac{\partial U}{\partial x}-2\left(P+2\nu_T\frac{\partial U}{\partial x}\right)(W-\dot{b}),$$

where e_a is the acoustic energy and P^r is a dissipative term.

Postulate $\langle e_a\rangle$ in (18) + decouple $\longrightarrow$ equations for P and φ

$$\frac{1}{h}\int_b^\eta e_a dz=\langle e_a\rangle=\frac{P^2}{2a_c^2}$$

where a_c is the constant sound velocity

Hyperbolic model: Full system of equations

Under the mild slope condition (neglect some bottom terms with large order of μ)

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + hP + h^3\varphi\right) = \frac{\partial}{\partial x}\left(2\nu_T h \frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x}$$

$$\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P + 3\nu_T \frac{\partial U}{\partial x}$$

$$\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -a_c^2\left(h\frac{\partial U}{\partial x} + 2W\right)$$

$$\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -\frac{2}{h}\langle P^r\rangle + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{8\nu_T W}{h^2}\frac{\partial U}{\partial x}$$

As $a_c \to \infty$, $W = \dfrac{-h}{2}\dfrac{\partial U}{\partial x}$

With no φ, $P = \dfrac{h\ddot{h}}{3}$ (SGN)

With φ, model of MK&Richard, **2019**

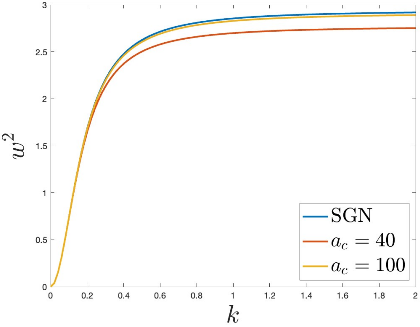
Dispersion relation

The dispersion relation $\omega = \omega(k^2)$ satisfies

$$\frac{h_0^2}{3a_c^2}\omega^4 - \omega^2\left[1 + \frac{h_0^2k^2}{3}\left(1 + \frac{gh_0}{a_c^2}\right)\right] + gh_0k^2 = 0$$

As $a_c \to \infty$, the dispersion relation of our model approaches to that of SGN model

$$\omega^2 = \frac{gh_0k^2}{1 + \frac{h_0^2k^2}{3}}$$



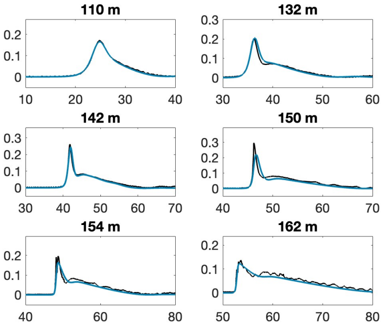
New breaking criteria

Unique treshold $\varphi\frac{\max(0,\eta)}{g} > 0.09,\ R = 1.7$

Blue data is with old criteria depending on initial global μ. Red data is with new local breaking criteria.

Second order accuracy:

- Finite volume MUSCL scheme in space
- IMEX scheme ARS2(2,2,2) in time



Breaking Waves Hyperbolic Model

Axe 1: Model derivation and validation on numerical test cases
– **Wave breaking and dispersion:**
Hyperbolic model with enstrophy descsription
– **Breaking criterion:**
Robust breaking criterion

Axe 2 :(Julien Chauchat, LEGI) Sediment transport coupling
Resolution of Exner equation, nonlinear interaction

Validation:
Implementation with TOLOSA project tolosa-project.com
Validation: Delft3D, XBeach

Hydro: experiences in LEGI (*rip currents*) + Mesurements by SHOM

Morpho: From solitary waves on sand beachs, monochromatic and bichromatic waves