Introduction to the mathematical description of water waves

Additional notes to the lecture course given at
the Summer School in Cargèse (June 2023)

by

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Preamble

The lectures will focus on the classical problem of inviscid water waves, and the application of asymptotic (parameter) methods to their study. To this end, we first describe the underlying assumptions that lead to the governing equations, and then a few simple, special problems are presented. These lay the foundations for a more extensive discussion of various flow problems of both practical and mathematical interest, which will be covered in the second half of the course. The overall aim is to show the complexity of flows associated with the theory of water-wave propagation, many of which can be accessed by invoking asymptotic methods.

These notes are to be read in conjunction with the lectures, the aim being to provide some additional information and background to the material described in the PowerPoint presentations.

Lecture 1a: Governing equations, boundary conditions, non-dimensionalisation and scaling

A fluid is a material that cannot, in general, withstand any force without change of shape. This property of a fluid should be compared with what happens to a solid: this can withstand a force, without any appreciable change of shape – until it fractures!

We work with water – a *liquid* – which is virtually incompressible: the density of water increases by about $0\cdot5\%$ under a pressure of 100 atmospheres. Water (and all conventional fluids) are viscous. However, in the context of water waves, the rôle of viscosity is weak; the length and time scales over which viscosity becomes important for water waves (moving in the oceans and rivers, for example) is typically very much greater than the scales on which the waves develop, evolve and interact. This observation is applicable even for turbulent flow, although then we must treat any background, established flow-field as being some appropriate (ensemble) average. It is observed, for example, that gravity waves (with wavelengths of tens of metres, or more) moving across the oceans, typically lose about $1/10^{\text{th}}$ of their amplitude due to viscous action, in travelling 1000km. On the other hand, surface-tension waves (ripples), which are typically 1-2 cms in wavelength, decay appreciably in a minute or two.

The continuum hypothesis: The first task is to introduce a suitable, general description of a fluid, and then to develop an appropriate (mathematical) representation of it. This involves regarding the body of fluid on the large (macroscopic) scale, i.e. consistent with the familiar observation that water appears to fill completely the region of space that it occupies: we ignore the existence of molecules and the ‘gaps’ between them (which would constitute a microscopic or molecular model). This crucial idealisation, which regards the fluid as *continuously distributed* throughout a region of space, is called the *continuum hypothesis*.

On this basis, at every point (particle), we may define a set of functions that describe the properties of the fluid at that point:

$\mathbf{u}'(\mathbf{x}',t')$ – the velocity vector (a vector field)
$p'(\mathbf{x}',t')$ – the pressure (a scalar field)
$\rho'(\mathbf{x}',t')$ – the density (*ditto*),

where $\mathbf{x}'=(x',y',z')$ is the position vector (expressed here in rectangular Cartesian coordinates, but other coordinate systems may sometimes be required). Further, we will assume that our fluid is maintained at constant temperature. There will, however, be an exception to this when we examine a flow structure with a thermocline (Lecture 5b) where we have different constant temperatures above and below this line. In addition, we introduce the body-force vector, $\mathbf{F}'(\mathbf{x}')$, defined per unit mass; for constant acceleration of gravity, this is $\mathbf{F}'=(0,0,-g')$. Here, t' is time and we usually write $\mathbf{u}'=(u',v',w')$. Note that both p' and ρ' are defined at a point, with no preferred orientation: they are *isotropic*. Further, these three functions are certainly to be continuous (C^0) in both $\mathbf{x}'$ and t'. Comment: applied mathematicians tend to allow the resulting governing equations to determine, after the event, the necessary requirements on the class of functions under consideration. (We see, therefore, that for an inviscid fluid we require $\mathbf{u}', p'$ and ρ' all to be of class C^1; for a viscous fluid, this has to be extended to include $\mathbf{u}'\in C^2$.)

Streamlines: definition of a streamline is

$$\frac{\mathrm{d}\mathbf{X}'}{\mathrm{d}s}=\mathbf{u}'(\mathbf{X}',t') \quad (\mathbf{X}'\in C^1).$$

In Cartesian components, this is the set of three coupled, ordinary differential equations

$$\frac{\mathrm{d}x'}{\mathrm{d}s}=u',\ \frac{\mathrm{d}y'}{\mathrm{d}s}=v',\ \frac{\mathrm{d}z'}{\mathrm{d}s}=w' \ \text{(all at fixed } t'\text{)}$$

or, more conveniently, a pair of equations, e.g. $\frac{\mathrm{d}y'}{\mathrm{d}x'}=\frac{v'}{u'},\ \frac{\mathrm{d}z'}{\mathrm{d}x'}=\frac{w'}{u'}$. [This is often expressed in the symmetric form $\frac{\mathrm{d}x'}{u'}=\frac{\mathrm{d}y'}{v'}=\frac{\mathrm{d}z'}{w'}$.] Note that, in 2-space (x',y'), we simply have $\frac{\mathrm{d}y'}{\mathrm{d}x'}=\frac{v'}{u'}$ (because there is no variation, and no flow, in the z'-direction).

Particle path: this is the path, $\mathbf{x}'=\mathbf{X}'(t')$, followed by a point (particle) as it moves in the fluid with a given velocity, i.e. $\frac{\mathrm{d}\mathbf{X}'}{\mathrm{d}t'}=\mathbf{u}'$; this is pure kinematics, providing a determination of $\mathbf{X}'(t')$ given $\mathbf{u}'(\mathbf{X}',t')$.

Note: A *steady flow* is one for which the velocity field is independent of time, and then the families of streamlines and particle paths coincide, because

$$\frac{\mathrm{d}\mathbf{X}'}{\mathrm{d}s'} = \mathbf{u}'(\mathbf{X}') \text{ and } \frac{\mathrm{d}\mathbf{X}'}{\mathrm{d}t'} = \mathbf{u}'(\mathbf{X}') \text{ define the same families of curves.}$$

The kinematic condition: Points in a surface of the fluid – at the free surface or on the bottom – remain in that surface for all time (which is the appropriate condition to invoke in the absence of mixing). Let such a (C^1) surface be $S(\mathbf{x}', t') = 0$, then points remain in the surface if S maintains the value 0 as the particles move; thus we require $\frac{\mathrm{D}S}{\mathrm{D}t'} = 0$. This constitutes a necessary condition for the evolution of the surface. (An alternative derivation, which provides a sufficiency argument, is based on the observation that, because particles cannot cross the boundary, the normal velocity *of* the surface must equal the normal velocity of the particles (points) that sit *in* the surface.) For the free surface, represented by $S = z' - h'(x', y', t') = 0$ (written in rectangular Cartesian coordinates), we then obtain the condition

$$\frac{\mathrm{D}S}{\mathrm{D}t'} = \left(\frac{\partial}{\partial t'} + u' \frac{\partial}{\partial x'} + v' \frac{\partial}{\partial y'} + w' \frac{\partial}{\partial z'} \right)(z' - h')$$

$$= w' - (h'_{t'} + u' h'_{x'} + v' h'_{y'}) = 0,$$

i.e. $$w' = (h'_{t'} + u' h'_{x'} + v' h'_{y'}) \text{ on } z' = h'(x', y', t').$$

Subscripts here denote partial derivatives. On the bottom ($z' = b'(\mathbf{x}'_\perp)$) – which we take to be a fixed, rigid boundary – we have $S = z' - b'(\mathbf{x}'_\perp)$, and so the application of the kinematic condition (again, written in Cartesians) gives

$$w' = u' b'_{x'} + v' b'_{y'} \text{ on } z' = b'(x', y').$$

N.B. Using $b' = b'(\mathbf{x}'_\perp, t')$ we can model undersea earthquakes (i.e. marine quakes).

The dynamic condition: This prescribes the stresses at the free surface of the fluid. With zero viscosity, this becomes simply the normal stress (pressure) which we usually take to be the constant pressure of the atmosphere at the surface.

Well-posedness: We seek a solution for $z' \in \left[b'(\mathbf{x}'_\perp), h'(\mathbf{x}'_\perp, t') \right]$ and $\mathbf{x}'_\perp \in D$ (which may be a finite or infinite domain, e.g. $0 \le y' \le d'$, $-\infty < x' < \infty$ for waves that propagate to infinity in a finite-width channel). Now an important and fundamental technical issue is the well-posedness of this problem. By this we mean that, at some non-zero time, $t' = T' > 0$, and given initial data at $t' = 0$, then

- a solution exists
- the solution is unique
- the solution depends continuously on the initial data.

It is well beyond our aims and remit in this applied mathematical course to discuss this aspect of the problem. Suffice it to report that, at least for irrotational flow, well-posedness has been proved for suitably smooth initial data but, for general (rotational) flows, the question is still open. Of course, there is good evidence based on the applicability of the results, various familiar approximations to the equations and the large number of numerical studies, that we have a well-posed problem – hardly a rigorous proof! Nevertheless, it does give us some confidence that we do have a useful model for water waves, even if a suitable proof eludes us, at present. We proceed, in these presentations, on the assumption that we are working with a well-posed problem. Indeed, in terms of the connection to initial data, this is less critical: we obtain either steady solutions or determine the initial data, after the event, consistent with the solution constructed.

The vorticity: An important property of a fluid flow, both in terms of what is observed in real flows and what is relevant in making theoretical headway, is the vorticity; this provides a measure of the local spin or rotation exhibited by fluid elements. It is defined by

$$\boldsymbol{\omega}' = \nabla' \wedge \mathbf{u}'$$

(i.e. $\boldsymbol{\omega}' = \operatorname{curl} \mathbf{u}'$, sometimes written $\boldsymbol{\omega}' = \nabla' \times \mathbf{u}'$ or $\boldsymbol{\omega}' = \operatorname{rot} \mathbf{u}'$), and one simple observation follows directly. If the flow is restricted to motion and variation in only 2-space – (x', y') say – then we see that

$$\boldsymbol{\omega}' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'} \right) \wedge \left(u'(x', y', t'), v'(x', y', t'), 0 \right) = \left(0, 0, \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} \right):$$

the vorticity possesses a component in only the *third* (z') direction. (Note that this holds for unsteady flow.)

Now, it is observed that many real flows have almost zero vorticity almost everywhere. However, observations of some classes of water waves indicate that this is clearly not the situation; for example, waves breaking at a shoreline exhibit a highly turbulent mixing process (air is mixed with water), where vorticity plays a significant rôle and, usually, the kinematic condition is disrupted. Nevertheless, the simplifications that zero vorticity provide are a good starting point in the study of water waves, and many types of waves do propagate in essentially tranquil conditions. (In recent years, there has been considerable interest in the development of rigorous theories of water waves which accommodate some – often quite general – vorticity in the flow field, but one that is geometrically simple, e.g. 2D and periodic.) If we accept that zero vorticity is a reasonable assumption, at least for a class of 'ideal' waves, then we may set $\boldsymbol{\omega}' \equiv \mathbf{0}$. This is called – obviously – *irrotational* flow; then we have

$$\boldsymbol{\omega}' = \nabla' \wedge \mathbf{u}' \equiv \mathbf{0} \quad \Rightarrow \quad \mathbf{u}' = \nabla' \phi'$$

for arbitrary C^1 functions $\phi'(\mathbf{x}', t')$; the function ϕ' is called the *velocity potential* (dimensional here). Flows with vorticity are often called *rotational* flows. If, in addition to the flow being irrotational, it is also incompressible (as we certainly have for water), then

$$\nabla' \cdot \mathbf{u}' = 0 \text{ and so } \nabla'^2 \phi' = 0:$$

Laplace's equation for $\phi'(\mathbf{x}', t')$. Consequently, if we are able to determine ϕ' which satisfies Laplace's equation and also satisfies the given boundary conditions, then $\mathbf{u}'\ (= \nabla'\phi')$ is known; Euler's equation then gives, by direct integration, the pressure $p'(\mathbf{x}', t')$.

The stream function: Important progress is afforded by the restriction to two spatial dimensions ((x', y') say – the conventional choice – or we might select (x', z') in the water-wave context), but the flow can be unsteady. For an incompressible flow, we have

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0,$$

using rectangular Cartesian components (but other systems are possible); let us introduce an arbitrary $\psi(x', y', t') \in C^2$ such that

$$u' = \frac{\partial \psi'}{\partial y'} \text{ and then } v' = -\frac{\partial \psi'}{\partial x'}.$$

Consider lines $\psi'(x', y', t') = k(t')$ where t' plays the rôle of a parameter; we assume that this relation defines $y' = y'(x', t') \in C^1$, and then we form (all at fixed t')

$$\frac{\mathrm{d}}{\mathrm{d}x'}\psi'\left(x', y'(x', t'), t'\right) = 0 \text{ i.e. } \frac{\partial \psi'}{\partial x'} + \frac{\partial \psi'}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x'} = 0 \text{ or } \frac{\mathrm{d}y'}{\mathrm{d}x'} = -\frac{\partial \psi'/\partial y'}{\partial \psi'/\partial x'} = \frac{v'}{u'}.$$

But this last statement is the definition, in two spatial dimensions, of the streamlines, defined at an instant in time. Thus lines $\psi'(x', y', t') = k(t')$, at fixed t', are the streamlines; consequently, we call ψ' the *stream function* (dimensional here).

Further, let us now suppose that this 2D flow is also irrotational, then we obtain

$$\boldsymbol{\omega}' = \left(0, 0, \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}\right) = (0, 0, -\nabla'^2\psi') = \mathbf{0}.$$

Thus ψ' also satisfies Laplace's equation (in 2D); cf. the result for the velocity potential. In summary, therefore, we have, for two-dimensional, incompressible, irrotational flow

$$u' = \frac{\partial \phi'}{\partial x'} = \frac{\partial \psi'}{\partial y'} \text{ and } v' = \frac{\partial \phi'}{\partial y'} = -\frac{\partial \psi'}{\partial x'},$$

which are the Cauchy-Riemann relations relating ϕ' and ψ'. Thus $\exists w(Z,t') \in C^1$ such that

$$w(Z,t') = \phi'(x',y',t') + \mathrm{i}\,\psi'(x',y',t') \;\; (Z = x' + \mathrm{i}y'),$$

and then the techniques of complex analysis become available.

In the case of a 2D flow that is incompressible only – so not necessarily irrotational, but still unsteady – we have

$$\nabla'^2\psi' = -\omega'(x',y',t') \text{ where } \boldsymbol{\omega}' = \left(0,0,\omega'(x',y',t')\right),$$

which provides an equation for ψ', given the vorticity in 2D.

With surface tension: The nondimensional surface-kinematic-boundary-condition, with surface tension included (but approximated for small $\varepsilon\delta$), is

$$p = \eta - \delta^2 W_e\left(\nabla^2{}_{\perp}\eta + O(\varepsilon^2\delta^2)\right) \;\text{ on }\; z = 1 + \varepsilon\eta(\mathbf{x}_{\perp},t),$$

where $W_e = \dfrac{\Gamma'}{\rho' g' h_0'^2}$, a *Weber number*, with Γ' the surface-tension coefficient (force/unit length). This formulation is based on the property that the pressure jump across the surface, which possesses a surface tension, is proportional to the local Gaussian curvature.

Lecture 1b: <u>Energy equation and integrals of the motion, and two classical problems: one linear and one nonlinear</u>

Energy equation: The relevant vector identity is

$$\mathbf{u} \wedge (\nabla \wedge \mathbf{u}) = \nabla\left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) - (\mathbf{u}\cdot\nabla)\mathbf{u} \quad \text{(and } \nabla \wedge \mathbf{u} = \boldsymbol{\omega} \text{ is the vorticity).}$$

Irrotational, steady flow: We then have

$$\frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + \Omega' = \text{constant},$$

and the constant here is the *same constant everywhere*; only for rotational flow is the constant different on different streamlines.

Energy integral: Starting with

$$\frac{\partial \mathbf{u}'}{\partial t'} + \nabla'\left(\frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + \Omega'\right) = \mathbf{u}' \wedge \boldsymbol{\omega}'$$

we dot each side with $\mathbf{u}'$ (then $\mathbf{u}'\cdot(\mathbf{u}'\wedge\boldsymbol{\omega}')=0$), set $\Omega'=g'z'$ and add

$$\left(\frac{1}{2}\mathbf{u}'\cdot\mathbf{u}'+\frac{p'}{\rho'}+g'z'\right)\left(\nabla'\cdot\mathbf{u}'\right)$$ (= 0 because the fluid is incompressible).

Also add $\dfrac{\partial}{\partial t'}(g'z')$, which is zero, and multiply by ρ' (= constant) to give

$$\frac{\partial}{\partial t'}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+\rho'g'z'\right)+\nabla'\cdot\left\{\mathbf{u}'\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+p'+\rho'g'z'\right)\right\}=0;$$

then separate by writing $\nabla'\equiv\left(\nabla'_{\perp},\dfrac{\partial}{\partial z'}\right)$ and integrate from $z'=b'(\mathbf{x}'_{\perp})$ to $z'=h'(\mathbf{x}'_{\perp},t')$:

$$\int_{b'}^{h'}\left\{\frac{\partial}{\partial t'}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+\rho'g'z\right)+\nabla'_{\perp}\cdot\left[\mathbf{u}'_{\perp}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+p'+\rho'g'z\right)\right]\right\}\mathrm{d}z$$

$$+\left[w\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+p'+\rho'g'z\right)\right]_{z=b'}^{z=h'}=0.$$

Introducing the boundary conditions on the bottom and at the surface, and interchanging the integral and differential operators (by invoking the technique of 'differentiating under the integral sign', sometimes referred to as Leibniz's integral rule), gives, in general,

$$\frac{\partial}{\partial t'}\int_{b'}^{h'}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+\rho'g'z\right)\mathrm{d}z+\nabla'_{\perp}\cdot\int_{b'}^{h'}\mathbf{u}'_{\perp}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}'+p'+\rho'g'z\right)\mathrm{d}z$$

$$+P'_s\frac{\partial h'}{\partial t'}-P'_b\frac{\partial b'}{\partial t'}=0,$$

where $p'=P'_s,P'_b$ are the pressures at the surface and on the bottom, respectively. (We assume, throughout this process, that all the integrals exist.) But, for our water-wave model, we have $\dfrac{\partial b'}{\partial t'}=0$ and, for constant atmospheric pressure at the free surface, we may define the surface pressure as relative to this pressure, so then $P'_s=0$.

Classical linear problem: The equations for (U,W,P) are

$$\frac{\omega}{k}U=P,\quad \mathrm{i}\omega\delta^2W=\frac{\mathrm{d}P}{\mathrm{d}z},\quad \mathrm{i}kU+\frac{\mathrm{d}W}{\mathrm{d}z}=0,$$

with

$$P(1)=\left(1+\delta^2k^2W_e\right)A,\quad W(1)=-\mathrm{i}\omega A,\quad W(0)=0.$$

Further, we see that $$P(1)=\frac{\omega}{k}U(1)=\frac{\mathrm{i}\omega}{k^2}\frac{\mathrm{d}W}{\mathrm{d}z}(1).$$

The resulting dispersion relation, without surface tension, is $c_p^2=\dfrac{\tanh(\delta k)}{\delta k}$; this, in general, describes a dispersive wave: the phase speed depends on the wave number. However, for long waves/shallow water ($\delta k\to 0$), we obtain $c_p\sim\pm 1$ (non-dispersive); for short waves/deep water ($\delta k\to\infty$), we have $c_p\sim\pm 1\big/\sqrt{\delta k}$ (dispersive). The energy in the wave, which is proportional to (amplitude)^2, propagates at the group speed $(c_g=\mathrm{d}\omega/\mathrm{d}k)$; for gravity waves, this is

$$c_g=\frac{1}{2}c_p\left(1+2\delta k\operatorname{cosech}(2\delta k)\right)\ \text{ and so }\ \frac{1}{2}<\frac{c_g}{c_p}<1.$$

We can also find the leading approximation to the particle paths. Using our non-dimensionalisation and scaling, the particle paths are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\varepsilon u,\quad \frac{\mathrm{d}z}{\mathrm{d}t}=\varepsilon w;$$

Relative to a fixed point, (x_0,z_0), we write $(x,z)=(x_0+\varepsilon X, z_0+\varepsilon Z_0)$ and so obtain

$$\frac{\mathrm{d}X}{\mathrm{d}t}\sim\delta\omega A\frac{\cosh(\delta k z_0)}{\sinh(\delta k)}\mathrm{e}^{\mathrm{i}(kx_0-\omega t)}(+c.c),\quad \frac{\mathrm{d}Z}{\mathrm{d}t}\sim-\mathrm{i}\omega A\frac{\sinh(\delta k z_0)}{\sinh(\delta k)}\mathrm{e}^{\mathrm{i}(kx_0-\omega t)}(+c.c);$$

This produces the particle paths

$$\left(\frac{X}{\delta\cosh(\delta k z_0)}\right)^2+\left(\frac{Z}{\sinh(\delta k z_0)}\right)^2=\frac{4|A|^2}{(\sinh(\delta k))^2},\quad 0<z_0\le 1,$$

describing different ellipses at different depths.

Classical nonlinear system: We have

$$u+2c=\text{constant on characteristic lines } C^+:\frac{\mathrm{d}x}{\mathrm{d}t}=u+c;$$
$$u-2c=\text{constant on characteristic lines } C^-:\frac{\mathrm{d}x}{\mathrm{d}t}=u-c.$$

Example 1 ‘simple’ wave

Consider propagation to the right into stationary water of constant depth, $h=h_0$; all the C^- characteristics emanate from the undisturbed region, so $u-2c=-2c_0$ where $u_0=0$ and $c_0=\sqrt{h_0}$. Thus $u-2c=-2c_0$ everywhere (so we have ‘simple waves’) and $u+2c=$ constant on the C^+ characteristics, so u and c are each constant on C^+, which are therefore the lines $x-(u+c)t=\alpha$ with general solution

$$u+2c=f(\alpha)=f\left[x-(u+c)t\right].$$

Given the wave profile $h(x,0)=H(x)$, then

$$f(x)=(u+2c)\big|_{t=0}=4c\big|_{t=0}-2c_0=4\sqrt{H(x)}-2\sqrt{h_0}\,.$$

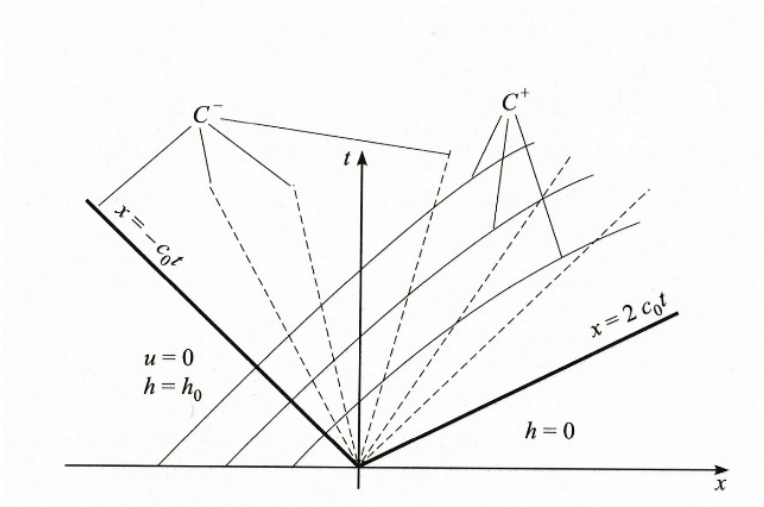
Thus

$$u+2c\left(=4\sqrt{h(x,t)}-2\sqrt{h_0}\right)=4\sqrt{H\left[x-(u+c)t\right]}-2\sqrt{h_0}\,,$$

and so $h(x,t)=H\left[x-(u+c)t\right]$ where $u(x,t)=2c(x,t)-2c_0=2\left(\sqrt{h(x,t)}-\sqrt{h_0}\right)$,

i.e. $h(x,t)=H\left[x-\left(3\sqrt{h}-2\sqrt{h_0}\right)t\right]$, $t\geq 0$.

This is an implicit equation for $h(x,t)$, given $H(x)$ and h_0; if $H(x)>0$ for some *x*, then this solution produces a wave which 'breaks', i.e. the characteristic lines cross after a certain finite time: the solution has become multi-valued.



Example 2 'dam break' problem

At time $t = 0$, the dam is broken; we describe this by writing

$$u=0 \text{ everywhere and } h(x)=\begin{cases} h_0, & x<0 \\ 0, & x>0 \end{cases} \quad \text{all at } t=0,$$

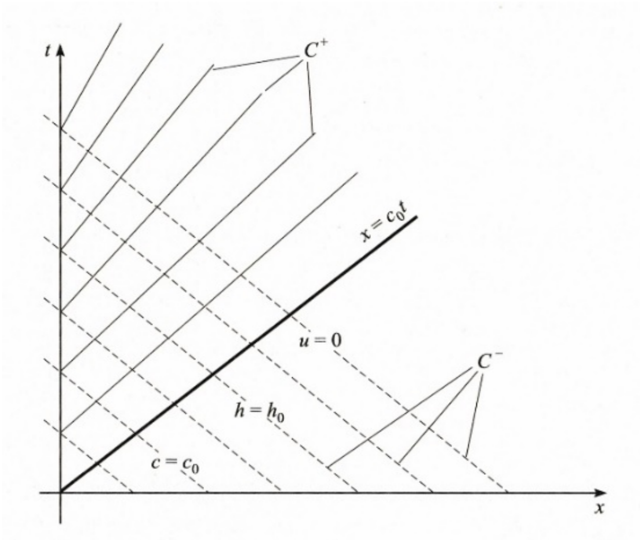
where $h_0>0$ is a constant. Now the C^+ characteristics emanate from $x<0$ where $u=0$ and $c=c_0=\sqrt{h_0}$, so $u+2c=2\sqrt{h_0}$ everywhere in the flow. But on C^- characteristics, we have $u-2c=\text{constant}$; thus *u*, *c* and $u-2c$ are constant on C^- and so these are the lines $x-(u-c)t=\text{constant}=0$ (since all pass through the (*x*, *t*)-origin). Thus we have $u+2c=2\sqrt{h_0}$ and $u-c=x/t$ ($t>0$); so

$$\sqrt{h}=\frac{1}{3}\left(2\sqrt{h_0}-\frac{x}{t}\right) \text{ and } u=\frac{2}{3}\left(\sqrt{h_0}+\frac{x}{t}\right),$$

and then $\dfrac{x}{t}=u-\sqrt{h}=2\sqrt{h_0}-3\sqrt{h}$ which shows that the solution is defined in the wedge $-\sqrt{h_0}\leq\dfrac{x}{t}\leq 2\sqrt{h_0}$. The surface profile is therefore

$$h(x,t) = \frac{1}{9}\left(2\sqrt{h_0} - \frac{x}{t}\right)^2, \quad -\sqrt{h_0} \le \frac{x}{t} \le 2\sqrt{h_0},$$

and then at the front, where $h = 0$, we have $x = 2\sqrt{h_0}\,t$: the front moves into the zero conditions ahead at a speed $2\sqrt{h_0}$. At the rear, where $h = h_0$, we have $x = -\sqrt{h_0}\,t$: the top of the collapsing wall of water propagates *backwards* into the undisturbed conditions at the speed $\sqrt{h_0}$.



Lecture 2a: The solitary wave and Gerstner's exact solution

The solitary wave dynamic boundary condition: We use the pressure equation to provide a suitable version of the surface dynamic (pressure) boundary condition. We evaluate

$$\frac{\partial \phi'}{\partial t'} + \frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + g'z' = f(t')$$

on the free surface where $p' = P_a'$ $(=\text{constant})$ with $z' = h'$:

$$\frac{\partial \phi'}{\partial t'} + \frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{P_a'}{\rho'} + g'h' = f(t') \quad \text{on} \quad z' = h'(\mathbf{x}_\perp', t').$$

Further, the flow is to be at rest at infinity $(|\mathbf{x}_\perp'| \to \infty)$, so $\phi' \to \text{constant}$ and $h' \to h_0'$ $(=\text{constant})$, which gives

$$\frac{P_a'}{\rho'} + g'h_0' = f(t');$$

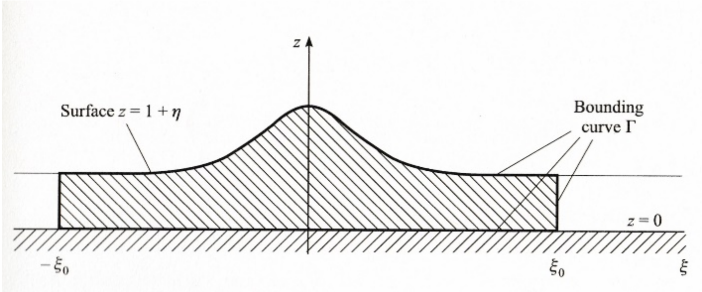
thus the dynamic boundary condition becomes

$$\frac{\partial \phi'}{\partial t'} + \frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + g'\left(h' - h_0'\right) = 0 \quad \text{on} \quad z' = h'(\mathbf{x}_\perp', t').$$

A second integral relation: We start with Green's theorem in the form

$$\int_V \left[(\nabla f)\cdot(\nabla g)+f\nabla^2 g\right]\mathrm{d}V=\int_S f(\nabla g)\cdot\mathrm{d}\mathbf{S},$$

applied per unit length in the y-direction, so we have $\mathrm{d}V=1\times\mathrm{d}s$ and $\mathrm{d}\mathbf{S}=\mathbf{n}(1\times\mathrm{d}l)$. Choose $f=g=\phi-c\xi$ and $\nabla\equiv\left(\dfrac{\partial}{\partial\xi},\dfrac{1}{\delta}\dfrac{\partial}{\partial z}\right)$, together with the plane region which is bounded by the curve Γ:



$$\begin{cases} z=1+\eta(\xi) \quad \text{and} \quad z=0 \quad \text{for} \quad -\xi_0\le\xi\le\xi_0 \\ \xi=\pm\xi_0 \quad (\text{with} \quad \xi_0>0) \end{cases},$$

which is shown in the figure. Now

$$\nabla^2(\phi-c\xi)=\nabla^2\phi=\frac{\partial^2\phi}{\partial\xi^2}+\frac{1}{\delta^2}\frac{\partial^2\phi}{\partial z^2}=0,$$

and so Green's theorem becomes

$$\int_{-\xi_0}^{\xi_0}\int_0^{1+\eta}\left[\frac{1}{\delta^2}\frac{\partial^2\phi}{\partial z^2}+\left(\frac{\partial\phi}{\partial\xi}-c\right)^2\right]\mathrm{d}z\,\mathrm{d}\xi=\int_\Gamma(\phi-c\xi)\left[m\left(\frac{\partial\phi}{\partial\xi}-c\right)+\frac{n}{\delta}\frac{\partial\phi}{\partial z}\right]\mathrm{d}l, \qquad \text{(A)}$$

where $\mathbf{n}\equiv(m,n)$ is the outward unit normal on Γ. The left-hand side of (A) can be written

$$2\hat{T}-2c\int_{-\xi_0}^{\xi_0}\int_0^{1+\eta}\frac{\partial\phi}{\partial\xi}\mathrm{d}z\,\mathrm{d}\xi+c^2\int_{-\xi_0}^{\xi_0}\int_0^{1+\eta}1\,\mathrm{d}z\,\mathrm{d}\xi=2\hat{T}-2c\hat{I}+2c^2\xi_0+c^2\hat{M},$$

where $\hat{T}\to T, \hat{I}\to I, \hat{M}\to M$ as $\xi_0\to\infty$; the right-hand side uses

$$\begin{cases} \text{on} \quad z=0: \quad m=0, n=-1; \partial\phi/\partial z=0, \\ \text{on} \quad \xi=\xi_0: \quad m=1, n=0; \partial\phi/\partial\xi-c=-\hat{c}, \\ \text{on} \quad \xi=-\xi_0: \quad m=-1, n=0; \partial\phi/\partial\xi-c=-\hat{c}, \\ \text{on} \quad z=1+\eta(\xi): \quad \mathbf{n}\text{ is normal to }\nabla(\phi-c\xi), \end{cases}$$

where we have introduced $\hat{c}$ such that $\hat{c}\to c$ as $\xi_0\to\infty$. The right-hand side then becomes

$$\int_0^{1+\eta} \left[-(\phi - c\xi)_+ + (\phi - c\xi)_- \right] \hat{c}\,\mathrm{d}z ,$$

where $\pm$ denotes evaluation on $\xi = \pm\xi_0$; for $\xi_0 \to \infty$, this is readily estimated to produce

$$\int_0^{1+\eta} \left[-(\phi - c\xi)_+ + (\phi - c\xi)_- \right] \hat{c}\,\mathrm{d}z \sim -c(\phi - c\xi)_+ + c(\phi - c\xi)_- = -c[\phi]_{-\infty}^{\infty} + 2c^2\xi_0 ,$$

and so (A) can be written (for finite ξ_0 and then imposing $\xi_0 \to \infty$) as

$$2T - 2cI + c^2 M = -cC \quad \text{or} \quad 2T = c(I - C) \text{ (McCowan, 1891).}$$

There is one other important relation, due to Longuet-Higgins (1974):

$$3V = (c^2 - 1)M .$$

The Gerstner wave: The velocity components are

$$(u', w') = \left(\sqrt{\frac{g'}{k}} \mathrm{e}^{kb} \cos\left(ka - t'\sqrt{g'k} \right), \sqrt{\frac{g'}{k}} \mathrm{e}^{kb} \sin\left(ka - t'\sqrt{g'k} \right) \right),$$

which confirms that the motion decays with depth (corresponding to $b \to -\infty$). The wave speed is $c = \sqrt{g'/k}$, and with wavelength $\lambda = 2\pi/k$ this gives $c = \sqrt{\frac{g'\lambda}{2\pi}}$ which is the wave speed for deep-water waves in *irrotational* flow.

The pressure is obtained by direct integration, to produce

$$p'(t'; a, b) = p'_a - \rho' g'(b - b_0) + \frac{\rho' g'}{2k}\left(\mathrm{e}^{2kb} - \mathrm{e}^{2kb_0} \right),$$

which is constant on particles (i.e. fixed b) as they move. The surface kinematic condition, requiring that points of the free surface remain in the surface, corresponds to a specific value of b, namely $b = b_0$.

The Jacobian becomes $J = \frac{\partial X'}{\partial a}\frac{\partial Z'}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial Z'}{\partial a} = 1 - \mathrm{e}^{kb}$, and for $-\infty < b \le b_0$ we require $b_0 \le 0$ for otherwise J will pass through zero. The special case $b_0 = 0$ corresponds to a degenerate form of the surface wave – a cusped wave.

The vorticity is

$$\frac{\partial u'}{\partial z'}-\frac{\partial w'}{\partial x'}=\frac{1}{J}\left(\frac{\partial X'}{\partial a}\frac{\partial^2 X'}{\partial t'\partial b}-\frac{\partial X'}{\partial b}\frac{\partial^2 X'}{\partial t'\partial a}+\frac{\partial Z'}{\partial a}\frac{\partial^2 Z'}{\partial t'\partial b}-\frac{\partial Z'}{\partial b}\frac{\partial^2 Z'}{\partial t'\partial a}\right)=-\frac{2\sqrt{g'k}}{\mathrm{e}^{-2kb}-1},$$

which shows that the Gerstner-wave flow is rotational, with a vorticity which decays with depth ($b\to-\infty$). For the cusped wave ($b_0=0$), the vorticity is singular at the surface (which is where $b=b_0$).

Lecture 2b: <u>Introduction to parameter asymptotics: ideas and method</u>

Here is another example which contains the important ingredient often encountered in the solution of some differential equations: an exponentially small term.

We consider $f(x;\varepsilon)=\dfrac{1}{1+\varepsilon x+\mathrm{e}^{-x/\varepsilon}}$ for $x\ge 0$ and $\varepsilon>0$ then, for arbitrary fixed $x > 0$, with $\varepsilon\to 0$, we see that

$$f(x;\varepsilon)\sim(1+\varepsilon x)^{-1}\sim 1-\varepsilon x,$$

and any number of terms could be included here (based on the sequence $\left\{\varepsilon^n\right\}$), although there will still be an overall error generated by the exponentially small term. However, we observe that

$$f(0;\varepsilon)=\frac{1}{2} \text{ and that } f\to 0 \text{ as } x\to\infty,$$

neither of which can be recovered from our original asymptotic expansion. This expansion is therefore not uniformly valid: there is a *breakdown* for sufficiently small *x*, and also for sufficiently large *x*. To proceed, we must rescale *x*: for small *x* we set $x=\varepsilon X$ to give

$$f(x;\varepsilon)=f(\varepsilon X;\varepsilon)\equiv F(X;\varepsilon)=\frac{1}{1+\varepsilon^2 X+\mathrm{e}^{-X}}$$

and then, for fixed X as $\varepsilon\to 0$, we obtain

$$F(X;\varepsilon)\sim\frac{1}{1+\mathrm{e}^{-X}}\left(1-\frac{\varepsilon^2 X}{1+\mathrm{e}^{-X}}\right),$$

which does recover the correct behaviour on $X=0$, i.e. on $x=0$.

Correspondingly, for large *x*, we introduce $x=\chi/\varepsilon$:

$$f(x;\varepsilon)=f\left(\chi/\varepsilon\,;\varepsilon\right)\equiv\Im(\chi;\varepsilon)=\frac{1}{1+\chi+\mathrm{e}^{-\chi/\varepsilon^2}},$$

and so, for fixed χ (> 0 because *x* is large) as $\varepsilon\to 0$, this gives

$$\Im(\chi;\varepsilon) \sim \frac{1}{1+\chi},$$

which produces the correct behaviour as $\chi \to \infty$, i.e. as $x \to \infty$.

In this example, the given function (in the specified domain) requires three different asymptotic expansions in order to cover the whole domain, associated with three different sizes of x: $x = \mathrm{O}(1)$, $x = \mathrm{O}(\varepsilon)$ and $x = \mathrm{O}(\varepsilon^{-1})$. Furthermore, we see that the original expansion includes its own region of validity (if we think of including the exponentially small terms): these become O(1) where $x = \mathrm{O}(\varepsilon)$ and, furthermore, the correction term explicitly written down here (namely εx) becomes O(1) where $x = \mathrm{O}(\varepsilon^{-1})$, the same size as the first term.

Usually, it is altogether straightforward to find asymptotic expansions of given functions which contain a parameter. However, their most powerful use is in the construction of solutions of differential equations. Obviously we do not know the function – that is what we are seeking – and we do not even have an asymptotic sequence. The procedure, guided by the appearance of the (small) parameter in the equation(s), is to assume a suitable asymptotic sequence, represent the solution on this basis and then solve the set of problems (presumably reduced versions of the original equations) that arise at each order. If the solution can be found, to all orders if possible, it may be examined for validity in the given domain of the solution, rescaling as necessary and repeating the process. On occasions, the scalings associated with different regions can be deduced directly from the differential equation(s).

This technique is much used in the investigation of problems in fluid dynamics – indeed, it was first developed to deal with complex fluid flows, boundary layers in particular – and we will spend the rest of the course looking at some specific, but important, flows. These will demonstrate how this approach can be used to extract fine detail (and aid understanding) when we do not have exact solutions and, at best, we may have only a proof of existence (which is always most gratifying!). Essentially, the procedure of non-dimensionalisation and scaling, using a suitable parameter, identifies specific properties in the system of equations and then extracts them at each order.