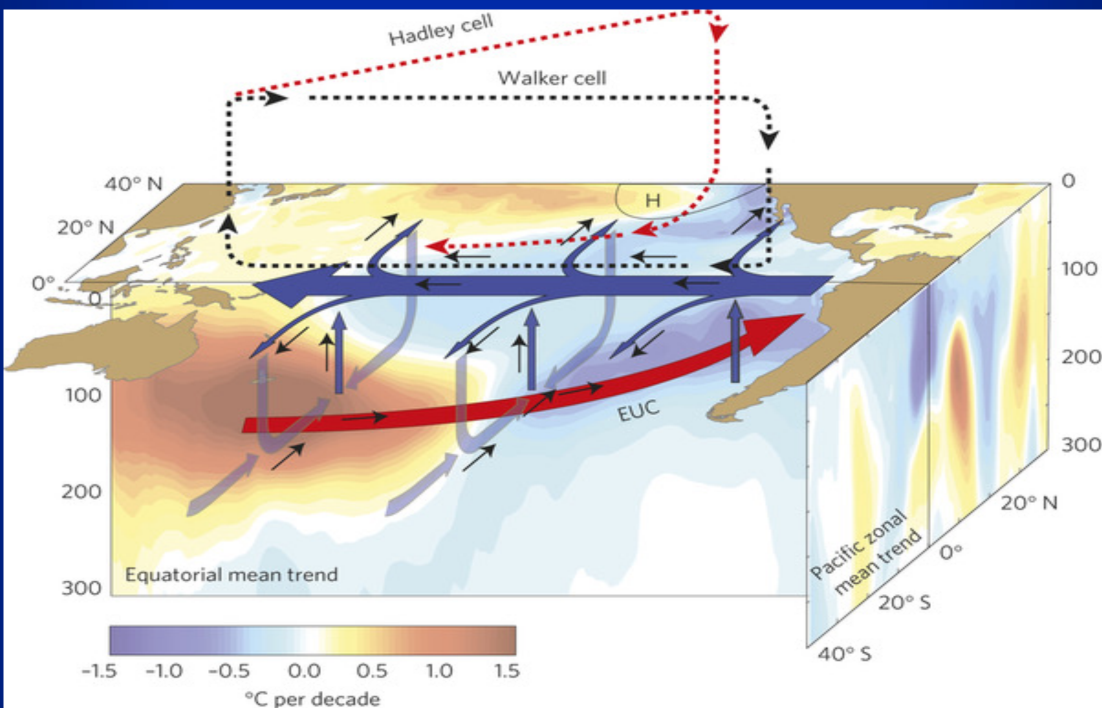
Lecture 5b

The Pacific Equatorial Undercurrent (EUC) and associated waves

Pacific Ocean in the neighbourhood of the Equator: schematic



Background

Oceanic flows involve

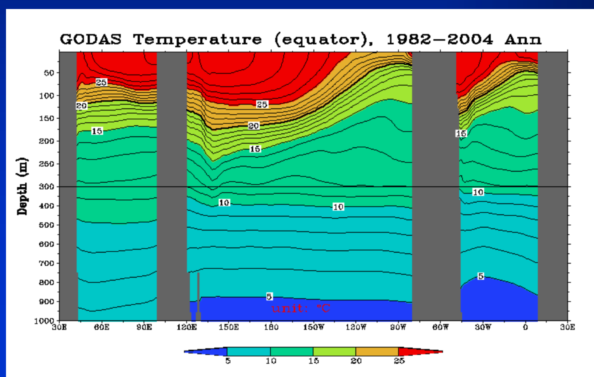
wind-driven surface waves, underwater currents,
upwelling and downwelling,
and thermal, density and salinity stratification
…….and much more; so, for example,

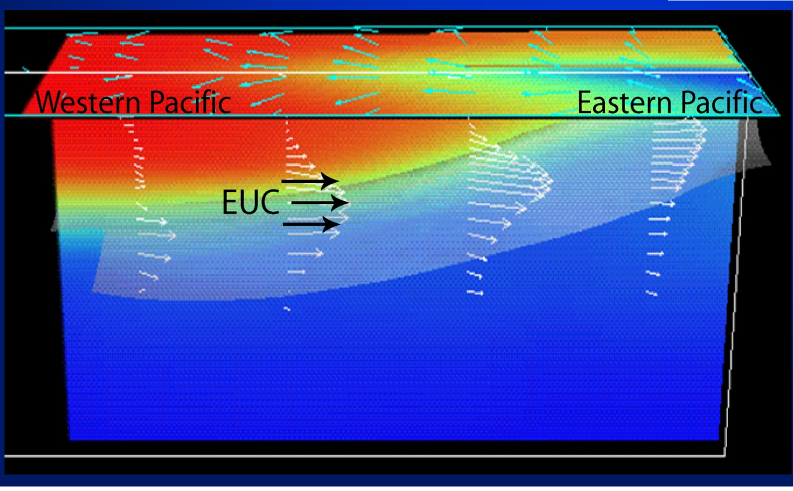
all these can change on both long and short timescales, vary with the seasons….and they are affected by the climate and by climate change.

Many *ad hoc* models have been used to describe specific phenomena; we approach the analysis of one particular flow by treating it as a problem in classical mathematical fluid dynamics (some details omitted).

Some information about the EUC:

Data for the temperature variation along the Pacific Equator.





Schematic of the flow in the neighbourhood of the Pacific Equator.

We consider flow in the vertical plane at the Pacific Equator, written in rotating, spherical coordinates (but approximated locally); in nondimensional variables we have

$$\frac{\partial u}{\partial t}+[U(z)+\varepsilon u]\frac{\partial u}{\partial x}+w\left(\frac{\mathrm{d}U}{\mathrm{d}z}+\varepsilon\frac{\partial u}{\partial z}\right)+2\Omega w=-\frac{1}{\rho}\frac{\partial p}{\partial x};$$

$$\frac{\partial w}{\partial t}+[U(z)+\varepsilon u]\frac{\partial w}{\partial x}+\varepsilon w\frac{\partial w}{\partial z}-2\Omega u=-\frac{1}{\rho}\frac{\partial p}{\partial z};$$

(where $\rho = 1$ above the thermocline, and $\rho = 1 + r$ below: $r \approx 0{\cdot}005$)

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

with $P(z)+p=P_a$ & $w=\frac{\partial \eta}{\partial t}+[U(z)+\varepsilon u]\frac{\partial \eta}{\partial x}$ on $z=\varepsilon\eta$ **(surface)**

and $w=\frac{\partial H}{\partial t}+[U(z)+\varepsilon u]\frac{\partial H}{\partial x}$ on $z=-1+\varepsilon H$; **(thermocline)**

$[p]_{below}^{above}=-r[1-2\Omega U(z)]H$ on $z=-1+\varepsilon H$, **(pressure continuous across the thermocline)**

also $w=0$ on $z=-d$. **(bottom)**

The parameter ε measures the amplitude of the wave (relative to the average depth of the undisturbed thermocline, $h' \approx 120\text{m}$); the background flow (with $w = 0$) satisfies

$$-2\Omega U(z)=-\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z}-1 \quad \text{with} \quad P=P_a=\text{constant on } z=0$$

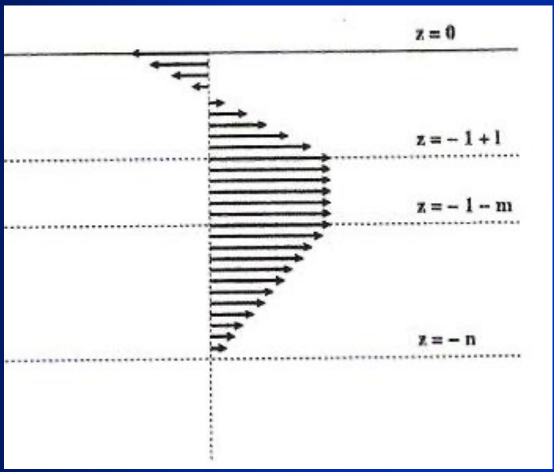
where $\rho = 1$ above the thermocline, and $\rho = 1 + r$ below.

The parameter $\Omega=\Omega' h'/\sqrt{g'h'}$ provides the contribution from the rotation of the Earth: the Coriolis term.

We choose a background flow to represent the EUC.

Plan: expand for small ε, using Taylor expansions for the boundary conditions, and find the dispersion relation for the waves. THEN consider nonlinearity.

The background flow which models the observed EUC (simply) is



$$U(z)=\begin{cases}-V-z\left(\dfrac{V+W}{1-\ell}\right), & 0\geq z>-1+\ell\\ W, & -1+\ell\geq z>-1-m\\ W\left(\dfrac{z+n}{n-1-m}\right), & -1-m\geq z>-n\\ 0, & -n\geq z\geq -d\end{cases}$$

Typical values are

$$V=0\cdot 014,\ W=0\cdot 029,\ m=\ell=0\cdot 33,\ n=2,\ d=33,$$

based on the speed scale $\sqrt{g'h'}$ and the average depth of the undisturbed thermocline, h'.

The linearised problem becomes

$$\frac{\partial u}{\partial t}+U(z)\frac{\partial u}{\partial x}+w\frac{\mathrm{d}U}{\mathrm{d}z}+2\Omega w=-\frac{1}{\rho}\frac{\partial p}{\partial x};$$

$$\frac{\partial w}{\partial t}+U(z)\frac{\partial w}{\partial x}-2\Omega u=-\frac{1}{\rho}\frac{\partial p}{\partial z}-1;$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

($\rho = 1$, $\rho = 1 + r$)

with $\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}=0$ **: for an irrotational perturbation.**

Taylor expansions give

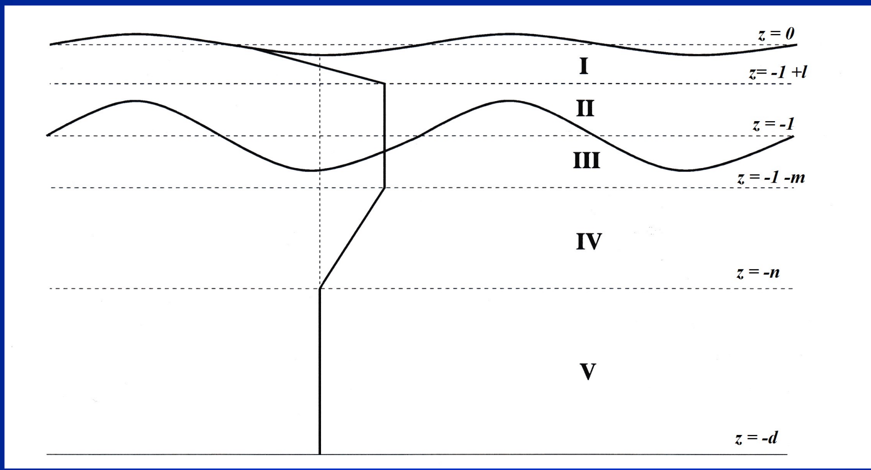
$$\left.\begin{aligned}p&=(1+2\Omega V)\eta\\ w&=\frac{\partial\eta}{\partial t}-V\frac{\partial\eta}{\partial x}\end{aligned}\right\}\quad \text{on}\quad z=0;$$

$$w=\frac{\partial H}{\partial t}+W\frac{\partial H}{\partial x}\quad \text{on}\quad z=-1;$$

with $w=0$ on $z=-d$

and $[p]_{below}^{above}=-r(1-2\Omega W)H$ across $z=-1.$

The linear problem involves finding the solution in 5 regions:



A lengthy and tedious calculation, but quite straightforward, seeking a harmonic-wave solution with wave number *k*.

We do not reproduce the calculation here – details in the reading list – we will simply quote a few results.

The dispersion relation, for arbitrary *k*, can be found; we are interested only in long waves: $k \to 0$; these come in two variants.

1. **Intermediate long waves**

These are defined by $k \to 0,\ kd \to \infty$, i.e. long with respect to the depth scale h', but short relative to the depth of the ocean. We find that

$$c \sim \pm\sqrt{r} + W - \frac{1}{2}(1-\ell)(V+W),\ c \sim W - \ell(V+W) + \left(\frac{1-\ell}{r}\right)(V+W)^3.$$

Typical values are

$$c \approx 2\cdot 7\text{ms}^{-1}\ \text{(eastwards)},\ 1\cdot 7\text{ms}^{-1}\ \text{(westwards)},\ 0\cdot 55\text{ms}^{-1}\ \text{(eastwards)};$$

(respectively)

corresponding observed values are

$$c \approx 2\cdot 5\text{ms}^{-1},\ 0\cdot 5\text{ms}^{-1},\ 0\cdot 55\text{ms}^{-1},$$

but the second is for very long waves – see later – and the third turns out to be a critical speed.

In addition, the ratio of amplitudes gives

$$\frac{\text{surface wave}}{\text{thermocline wave}} \sim -r$$

so a large wave on the thermocline is virtually invisible at the surface. N.B. Out of phase.

2. Very long waves

These are defined by $k \to 0,\ kd \to 0$, i.e. long with respect to the depth of the ocean. We find that

$$c \sim W \pm \sqrt{\frac{\ell m r}{\ell + m}(1 - 2\Omega W)},\ c \sim -V + \frac{(1-\ell)V^2}{d(V+W)},\ c \sim \pm\sqrt{d}.$$

last two: $c \approx 200\text{ms}^{-1}$ – tsunami waves

Typical/observed values are

$c \approx 1{\cdot}9/1{\cdot}7\text{ms}^{-1},\ 0{\cdot}11/0{\cdot}12\text{ms}^{-1},\ 0{\cdot}51/0{\cdot}5\text{ms}^{-1}$, respectively, for the first three;

and $$\frac{\text{surface wave}}{\text{thermocline wave}} \sim -\frac{rm^2}{(\ell+m)(1+m)},\ \frac{V^2}{V^2+d}$$ (ditto).

Nonlinearity: critical levels

Critical levels can appear in Region I and/or Region IV. Observations confirm that large eddies, with closed streamlines, can occur in the neighbourhood of the thermocline (most often in the western Pacific).

We use our equations to obtain the structure of the flow in these critical layers; we consider Region I only, i.e. for the constant wave speed *c*: $-V < c < W$.

For steady waves travelling at the speed *c* (satisfying the above), we introduce $\xi = x - ct$ and use the explicit form of the background velocity profile appropriate for this region; the equations are therefore

$$(-c-V-\beta z+\varepsilon u)\frac{\partial u}{\partial \xi}+w\left(-\beta+\varepsilon\frac{\partial u}{\partial z}\right)+2\Omega w=-\frac{\partial p}{\partial \xi};$$

$$(-c-V-\beta z+\varepsilon u)\frac{\partial w}{\partial \xi}+\varepsilon w\frac{\partial w}{\partial z}-2\Omega u=-\frac{\partial p}{\partial z};\quad \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

with $\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=0$ **for irrotational wave perturbations,**

where $\beta=\frac{V+W}{1-\ell}.$

For the flow outside the critical layer, and a wave with wave number *k*, the stream function can be written as

$$\psi \sim -Vz-\frac{1}{2}\beta z^2+\varepsilon\left(Ae^{kz}+Be^{-kz}\right)\frac{1}{k}\cos(k\xi)$$

where *A*, *B* are arbitrary constants.

The critical level, $z=z_c$, **is defined by**

$$c+V+\beta z_c=0$$

and then a rescaled *z*: $z=z_c+\sqrt{\varepsilon}\,Z$, **with**

$$u=\frac{1}{\sqrt{\varepsilon}}\hat{U}(Z,\xi;\varepsilon),\ w=\hat{W}(Z,\xi;\varepsilon),\ p=\hat{P}(Z,\xi;\varepsilon),$$

gives $$(-\beta Z+\hat{U})\frac{\partial \hat{U}}{\partial \xi}+\hat{W}\left(-\beta+\frac{\partial \hat{U}}{\partial Z}\right)+2\Omega\hat{W}=-\frac{\partial \hat{P}}{\partial \xi};$$

$$\varepsilon\left\{(-\beta Z+\hat{U})\frac{\partial \hat{W}}{\partial \xi}+\hat{W}\frac{\partial \hat{W}}{\partial Z}\right\}-2\Omega\hat{U}=-\frac{\partial \hat{P}}{\partial Z};\qquad \frac{\partial \hat{U}}{\partial \xi}+\frac{\partial \hat{W}}{\partial Z}=0.$$

Introduce the stream function here: $\hat{U}=\frac{\partial \hat{\Psi}}{\partial Z},\ \hat{W}=-\frac{\partial \hat{\Psi}}{\partial \xi}$

then at leading order we get $\frac{\partial^2 \hat{\Psi}}{\partial Z^2}=F\left(\hat{\Psi}-\frac{1}{2}\beta Z^2\right).$ **(arb. F)**

This must match to the solution valid outside the critical layer; this gives

$$\hat{\Psi} = -\frac{1}{2}\beta Z^2 + \left(A\mathrm{e}^{kz_c} + B\mathrm{e}^{-kz_c}\right)\frac{1}{k}\cos(k\xi),$$

and then the streamlines in the critical layer are described by

$$\left(A\mathrm{e}^{kz_c} + B\mathrm{e}^{-kz}\right)\frac{1}{k}\cos(k\xi) - \frac{1}{2}\beta Z^2 = \text{constant}$$

which recovers the classical Kelvin cats'-eyes pattern; see Lecture 4a.

Comment: another approach enables the scales to be identified which give rise to the Korteweg-de Vries equation for nonlinear waves, but the resulting analysis – because of the detailed background state – is very involved.

We have seen that our asymptotic approach can be extended to encompass very complicated flows; here, in particular, we have

a well-defined, i.e. based on a precise asymptotic limit, linear problem which can be solved in detail and for a general wave number (with gratifyingly good agreement with the observed wave speeds);

the asymptotic approach allows the extension, precisely defined, to incorporate nonlinearity (to describe the critical layer);

higher-order terms are accessible, and application to nonlinear wave propagation is possible (but messy!).

End of Lecture 5b