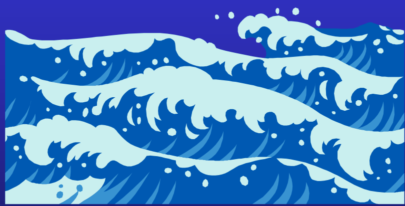
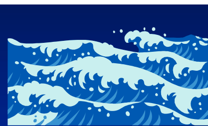
# Lecture 5a

# Periodic water waves in the presence of constant vorticity



## Background



Existence of inviscid, periodic water waves with vorticity over finite depth has been proved. In particular

waves are symmetric;

in some cases, the waves take extreme forms, i.e. included angle at the crest;

for negative vorticity, any stagnation point is at the crest;

for positive vorticity, any stagnation point is on the bottom, directly below the crest;

horizontal velocity component below the surface is strictly monotone.

**All these are important, general observations which have been confirmed by a number of numerical solutions.**

**We will use asymptotic methods to extract some *analytical detail* about the solutions – an example of the power of these methods.**

**To use this approach we need a suitable small/large parameter; various possibilities are available, e.g. very large of very small vorticity, which may be associated with the size of the wave.**

**Here we introduce an amplitude parameter, for fixed constant vorticity.**



## Governing equations

**Model (the classical one) based on inviscid, incompressible fluid, in the absence of surface tension and with constant pressure at the free surface:**

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \equiv \left( \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right)\mathbf{u} = -\nabla p, \;\; \nabla \cdot \mathbf{u} = 0,$$

**with**

$$p = h \quad \& \quad w = \frac{\mathrm{D}h}{\mathrm{D}t} \equiv \left( \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right)h \quad \text{on} \quad z = h(x, y, t)$$

**and**

$$w = \frac{\mathrm{D}b}{\mathrm{D}t} \equiv \left( u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \right)b \quad \text{on} \quad z = b(x, y),$$

**written in nondimensional variables, where *p* is the deviation from the hydrostatic pressure distribution.**

**We consider one-dimensional, periodic waves over constant vorticity (in a formulation which follows Constantin & Strauss).**

**Let the speed of propagation of steady waves be *c*, and introduce *X* = *x* – *ct* and *U* = *u* – *c*; then we have**

$$U\frac{\partial U}{\partial X}+w\frac{\partial U}{\partial z}=-\frac{\partial p}{\partial X};\, U\frac{\partial w}{\partial X}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z};\, \frac{\partial U}{\partial X}+\frac{\partial w}{\partial z}=0,$$

with $p=h$ & $w=U\dfrac{\mathrm{d}h}{\mathrm{d}X}$ on $z=h(X)$; $w=0$ on $z=-d_0$.

**Equivalently, at the surface, we have Bernoulli's equation:**

$$U^2+w^2+2(h+d_0)=Q=\text{constant} \ \text{ on } \ z=h(X).$$

**The constant vorticity is** $\omega=\dfrac{\partial w}{\partial X}-\dfrac{\partial U}{\partial z}$ **.**



**Introduce the stream function**

$$\psi(X,z):\quad U=\frac{\partial \psi}{\partial z} \ \text{ and } \ w=-\frac{\partial \psi}{\partial X},$$

**and write the total mass flux, in this moving frame, as**

$$\int_{-d_0}^{h(X)} U(X,z)\,\mathrm{d}z=-p_0=\text{constant};\quad \begin{cases}\psi=0 \ \text{ on } \ z=h(X)\\ \psi=p_0(>0) \ \text{ on } \ z=-d_0\end{cases}.$$

**Now use the Dubreil-Jacotin transformation, by writing**

$$D(X,\psi)=z+d_0$$

**to give**

$$\frac{\partial^2 D}{\partial \psi^2}+\left(\frac{\partial D}{\partial \psi}\right)^2\frac{\partial^2 D}{\partial X^2}+\left(\frac{\partial D}{\partial X}\right)^2\frac{\partial^2 D}{\partial \psi^2}-2\frac{\partial D}{\partial \psi}\frac{\partial D}{\partial X}\frac{\partial^2 D}{\partial \psi \partial X}=\omega\left(\frac{\partial D}{\partial \psi}\right)^3$$

with $1+(2D-Q)\left(\dfrac{\partial D}{\partial \psi}\right)^2+\left(\dfrac{\partial D}{\partial X}\right)^2=0$ on $\psi=0$; $D=0$ on $\psi=p_0$,

and $\omega=$ constant.

**Seek a solution in** $0 \le \psi \le p_0, -\pi \le X \le \pi$

**(periodic, where the actual period is determined by the length scale used in the non-dimensionalisation).**

**Convenient to rescale:** $D = d/\sqrt{\omega}\,, Q = \omega q$

**to give** $$\frac{\partial^2 d}{\partial \psi^2} - \left(\frac{\partial d}{\partial \psi}\right)^3 = \frac{1}{\omega}\left(2\frac{\partial d}{\partial \psi}\frac{\partial d}{\partial X}\frac{\partial^2 d}{\partial \psi \partial X} - \left(\frac{\partial d}{\partial \psi}\right)^2 \frac{\partial^2 d}{\partial X^2} - \left(\frac{\partial d}{\partial X}\right)^2 \frac{\partial^2 d}{\partial \psi^2}\right)$$

with $1 - \left(q - \frac{2}{\omega^{3/2}} d\right)\left(\frac{\partial d}{\partial \psi}\right)^2 + \frac{1}{\omega}\left(\frac{\partial d}{\partial X}\right)^2 = 0$ on $\psi = 0$; $d = 0$ on $\psi = p_0$.

**Can examine various problem: e.g. fixed $\omega$, with a small-wave perturbation; or let $\omega \to \pm\infty$.**



**We choose to examine small-amplitude (i.e. linear) waves with fixed, positive vorticity.**

**So we start with the exact solution representing a uniform flow:**

$$d = d_0(\psi) = \sqrt{b - 2\psi} - \sqrt{b - 2p_0}\,,$$

**with the Bernoulli condition which gives**

$$q = b + 2\omega^{-3/2}\left(\sqrt{b} - \sqrt{b - 2p_0}\right), \text{ for arb. real const. } b \ge 2p_0.$$

**Introduce a (wave) perturbation of this background flow, with amplitude measured by $\varepsilon$:**

$$d \sim d_0(\psi) + \varepsilon d_1(\psi, X), \quad \varepsilon \to 0.$$

**The equation for $d_1$ is then**

$$\frac{\partial^2 d_1}{\partial \psi^2} - 3\left(\frac{\mathrm{d}d_0}{\mathrm{d}\psi}\right)^2 \frac{\partial d_1}{\partial \psi} = -\frac{1}{\omega}\left(\frac{\mathrm{d}d_0}{\mathrm{d}\psi}\right)^2 \frac{\partial^2 d_1}{\partial X^2}$$

**with**

$$-2q\frac{\mathrm{d}d_0}{\mathrm{d}\psi}\frac{\partial d_1}{\partial \psi} + \frac{2}{\omega^{3/2}}\left(2d_0\frac{\mathrm{d}d_0}{\mathrm{d}\psi}\frac{\partial d_1}{\partial \psi} + \left(\frac{\mathrm{d}d_0}{\mathrm{d}\psi}\right)^2 d_1\right) = 0 \text{ on } \psi = 0$$

**(surface)**

**and** $d_1 = 0$ on $\psi = p_0$. **(bottom)**

**A solution for $d_1$ which satisfies the bottom boundary condition is**

$$d_1(\psi, X) = \frac{\cos X}{\sqrt{b-2p_0}}\sinh\left[\omega^{-1/2}\left(\sqrt{b-2\psi} - \sqrt{b-2p_0}\right)\right],$$

**and the top condition is satisfied if**

$$\tanh\left[\omega^{-1/2}\left(\sqrt{b} - \sqrt{b-2p_0}\right)\right] = \frac{b}{\sqrt{\omega b} + \omega^{-1}}.$$

**Real solutions for *b* only if** $0 < \omega/p_0 \le \kappa \approx 7\cdot 009$



**In summary, we have the asymptotic solution**

$$d \sim \sqrt{b-2\psi} - \sqrt{b-2p_0} + \varepsilon\frac{\cos X}{\sqrt{b-2p_0}}\sinh\left[\omega^{-1/2}\left(\sqrt{b-2\psi} - \sqrt{b-2p_0}\right)\right]$$

**with** $$\tanh\left[\omega^{-1/2}\left(\sqrt{b} - \sqrt{b-2p_0}\right)\right] = \frac{b}{\sqrt{\omega b} + \omega^{-1}}$$

**and** $$q = b + 2\omega^{-3/2}\left[\sqrt{b} - \sqrt{b-2p_0}\right].$$

**This asymptotic expansion for *d* breaks down near the bottom where $b - 2p_0 = \mathrm{O}(\varepsilon^2)$ and then $\psi - p_0 = \mathrm{O}(\varepsilon^2)$.**

**Introduce**

$$b - 2p_0 = \varepsilon^2\lambda^2 \ (\lambda > 0), \quad \psi = p_0 - \varepsilon^2\Psi \ (\Psi \ge 0)$$

**and then *d* = O(*ε*), so we write $d = \varepsilon\bar{d}(\Psi, X; \varepsilon)$ to describe the problem valid near the bottom of the flow.**

**Thus we obtain**

$$\frac{\partial^2 \bar{d}}{\partial \Psi^2} + \left(\frac{\partial \bar{d}}{\partial \Psi}\right)^3 = \frac{\varepsilon^2}{\omega}\left(2\frac{\partial \bar{d}}{\partial \Psi}\frac{\partial \bar{d}}{\partial X}\frac{\partial^2 \bar{d}}{\partial \Psi \partial X} - \left(\frac{\partial \bar{d}}{\partial \Psi}\right)^2 \frac{\partial^2 \bar{d}}{\partial X^2} - \left(\frac{\partial \bar{d}}{\partial X}\right)^2 \frac{\partial^2 \bar{d}}{\partial \Psi^2}\right)$$

**with** $\bar{d} = 0 \quad \text{on} \quad \Psi = 0,$

**plus matching condition for** $\Psi \to \infty$**, i.e. away from the bottom.**

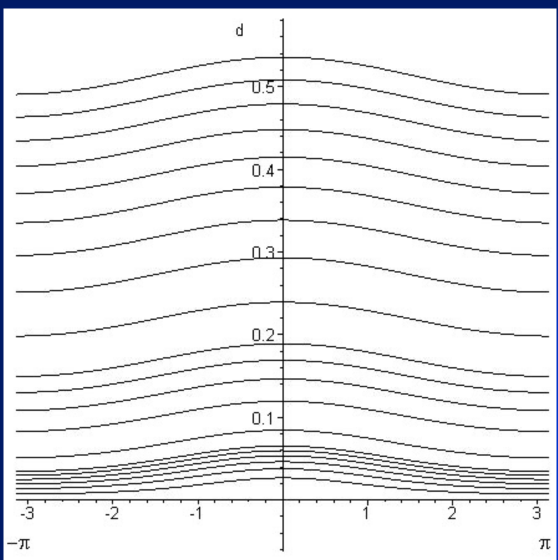
**The resulting leading-order solution, satisfying the matching condition, is**

$$\bar{d} \sim \sqrt{2\Psi + \left(\lambda - \omega^{-1/2}\cos X\right)^2} - \left(\lambda - \omega^{-1/2}\cos X\right), \ \varepsilon \to 0,$$

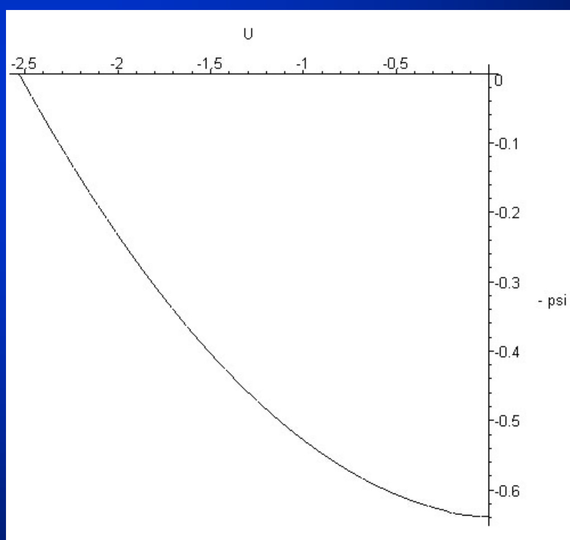
**with** $q = 2p_0 + 2\omega^{-3/2}\left(\sqrt{2p_0} - \varepsilon\lambda\right) + \mathrm{O}(\varepsilon^2).$

**This admits solution with a stagnation point (on the bed, under the crest) if** $\lambda = 1/\sqrt{\omega}$.



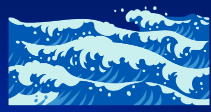



**An example of the stream lines for small-amplitude waves based on the asymptotic solution.**




**The corresponding horizontal velocity component below the crest (in the moving frame).**

**This example shows how the use of (formal) parameter expansions can extract analytical details of a complicated flow. In particular:**



**the asymptotic structure is fairly simple, but contains a non-uniformity leading to a breakdown (near the bottom of the flow field) and consequent rescaling;**

**the solution near the bottom has been found, requiring matching to be invoked;**

**the solution confirms the general structure implied by the proven results, with additional fine detail evident.**

**N.B. Other cases can also be examined, e.g. large (constant) vorticity and variable vorticity.**



# *End of*

# *Lecture 5a*