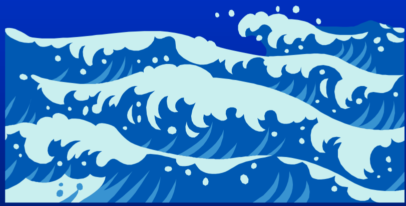
Lecture 4b

Edge waves over variable depth



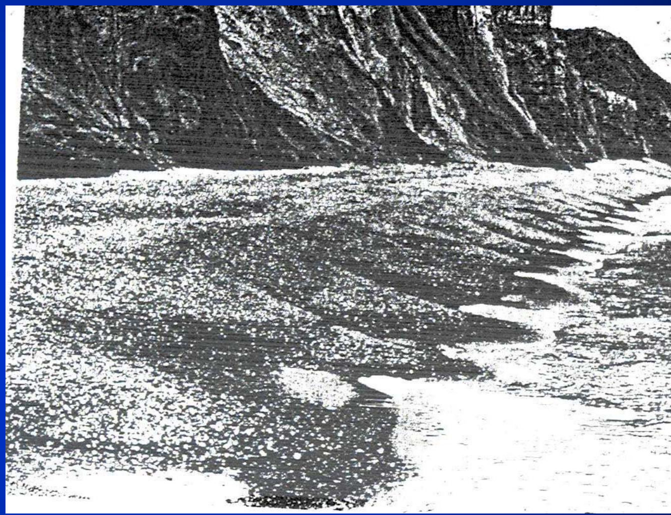
First, we show some examples of edge waves and then we briefly outline the classical edge-wave theory.

Comment: The Gerstner solution can be transformed to give an exact solution for edge waves, but for a linearly increasing depth (to infinity). Then…

this provides a scaling that can be used to develop an asymptotic theory for nonlinear edge waves over a variable depth (which therefore accommodates realistic bottom profiles away from the shoreline).

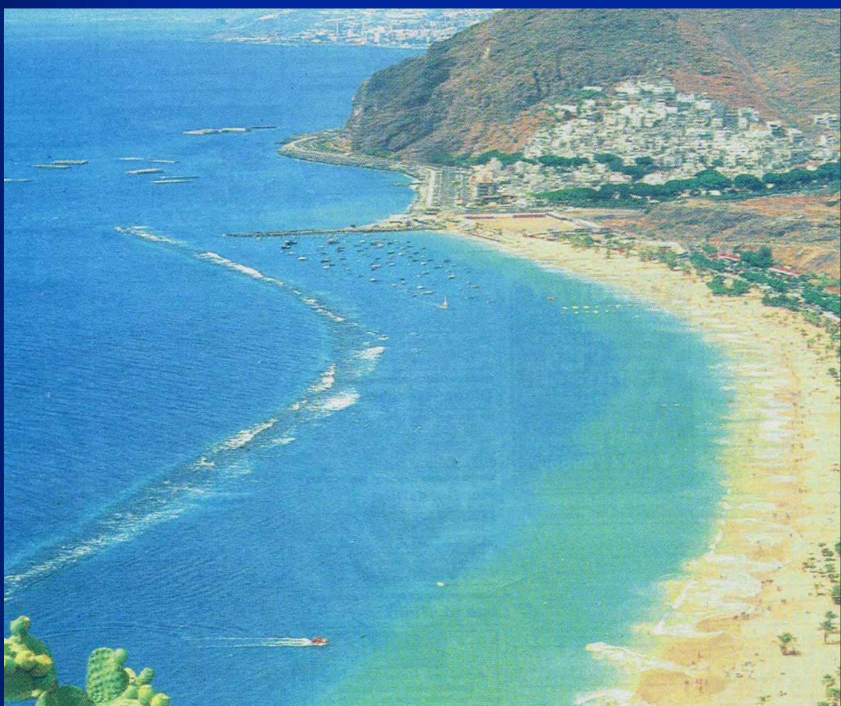
Some examples of edge waves:

A beach in Mexico



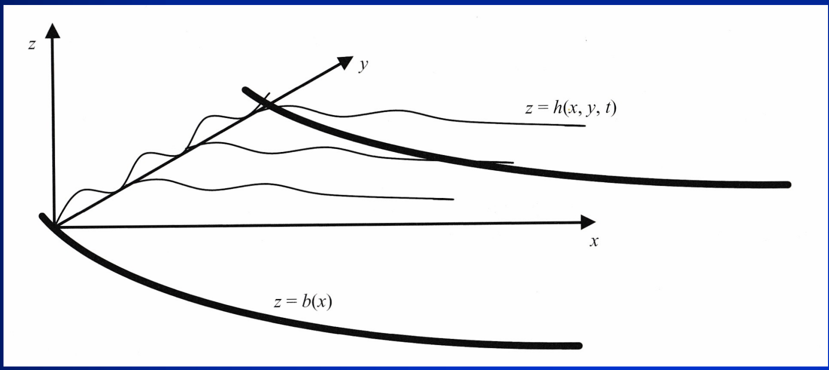
Alum Bay, Isle of Wight

Another example:



A beach in South Africa

Model and Formulation:



Surface wave:
$z = h(x, y, t)$

Variable bottom:
$z = b(x)$

Longshore coordinate: y **Seawards:** $x \to \infty$

Non-dimensionalise using a typical wave length, ℓ**, and introduce** $\sqrt{g\ell}$ **for speed and** $\ell/\sqrt{g\ell}$ **for time;**

then write pressure as $P_a - \rho g z + \rho g \ell p.$

The governing equations take the familiar form, but now with the longshore coordinate (*y*) included. We have

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u, v, w) = -\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)p\,;$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

with boundary conditions:

$$p = h \quad \& \quad w = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)h \quad \textbf{on} \quad z = h$$

and

$$w = u\frac{\mathrm{d}b}{\mathrm{d}x} \quad \textbf{on} \quad z = b(x)$$

(together with suitable initial data, chosen to be consistent with the solution that we develop).

Classical (Stokes) theory (1846):

Invoke the approximation:

- small amplitude
- long waves (but not essential)

then we obtain

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}\ ;\ \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y};\ \frac{\partial p}{\partial z} = 0\ ;\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

with $p = h$ **&** $w = \frac{\partial h}{\partial t}$ **on** $z = 0$ **and** $w = u\frac{\mathrm{d}b}{\mathrm{d}x}$ **on** $z = b$,

where we set $b(x) = -\alpha x \quad (\alpha > 0, \text{constant}).$

Seek a solution

$h = A(x)E\ ;\ p = P(x,z)E\ ;\ (u,v,w) = (U,V,W)(x,z)E$

$\left(E = \exp\{\mathrm{i}(\ell y - \omega t)\}\right)$

then the equation for ***A*(*x*)** **becomes**

$$\alpha x\left(\frac{\mathrm{d}^2 A}{\mathrm{d}x^2} - \ell^2 A\right) + \alpha\frac{\mathrm{d}A}{\mathrm{d}x} + \omega^2 A = 0.$$

Write in terms of ***L*(*Y*):** $A(x) = \mathrm{e}^{-\ell x}L(Y),\ Y = 2\ell x$ **to give**

$$YL'' + (1 - Y)L' + \gamma L = 0,\ \ \gamma = \tfrac{1}{2}\left(\omega^2/\alpha\ell - 1\right),$$

which has bounded solutions for ***A*(*x*)**, **as** $x \to +\infty$, **only if**

$$\gamma = n \quad (n = 0, 1, 2, \ldots);$$

these are the ***Laguerre*** **polynomials,** $L_n(Y)$.

The dispersion relation is then: $\omega^2 = \alpha\ell(1 + 2n)$,

and so there are no waves if $\alpha = 0$.

We now introduce a **slowly varying depth**, as follows:

Set $\varepsilon = -\dfrac{\mathrm{d}b}{\mathrm{d}x}(0)$ and consider the problem with

$$b = -B(X) \quad \text{with} \quad X = \varepsilon x.$$

From the transformation of the **Gerstner solution**, we see that the relevant scaling is

$$(u, v, w) \to \sqrt{\varepsilon}(u, v, \varepsilon w)\,;(p, h) \to \varepsilon(p, h),$$

with
$$\xi = \ell y - \omega\sqrt{\varepsilon}\,t\ ;\ \theta = \varepsilon^{-1}\int_0^X \alpha(X';\varepsilon)\,\mathrm{d}X',$$

together with X, of course.

Note that we describe a *travelling* wave here.

The equations then become the following:

Euler's equation (components)

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\left(\alpha\frac{\partial p}{\partial\theta} + \varepsilon\frac{\partial p}{\partial X}\right);\ \frac{\mathrm{D}v}{\mathrm{D}t} = -\ell\frac{\partial p}{\partial\xi}\ ;\ \varepsilon\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}.$$

Mass conservation
$$\alpha\frac{\partial u}{\partial\theta} + \ell\frac{\partial v}{\partial\xi} + \varepsilon\left(\frac{\partial u}{\partial X} + \frac{\partial w}{\partial z}\right) = 0.$$

Boundary conditions

$$p = h \quad \& \quad w = -\omega\frac{\partial h}{\partial\xi} + \alpha u\frac{\partial h}{\partial\theta} + \ell v\frac{\partial h}{\partial\xi} + \varepsilon u\frac{\partial h}{\partial X} \quad \text{on} \quad z = \varepsilon h$$

and
$$w = -u\frac{\mathrm{d}B}{\mathrm{d}X} \quad \text{on} \quad z = -B(X).$$

We have written
$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv -\omega\frac{\partial}{\partial\xi} + \alpha u\frac{\partial}{\partial\theta} + \ell v\frac{\partial}{\partial\xi} + \varepsilon\left(u\frac{\partial}{\partial X} + w\frac{\partial}{\partial z}\right).$$

We seek an asymptotic solution in the usual fashion giving, at leading order,

$$-\omega\frac{\partial u_0}{\partial \xi}+\alpha_0 u_0\frac{\partial u_0}{\partial \theta}+\ell v_0\frac{\partial u_0}{\partial \xi}=-\alpha_0\frac{\partial p_0}{\partial \theta};$$

$$-\omega\frac{\partial v_0}{\partial \xi}+\alpha_0 u_0\frac{\partial v_0}{\partial \theta}+\ell v_0\frac{\partial v_0}{\partial \xi}=-\ell\frac{\partial p_0}{\partial \xi};\quad \frac{\partial p_0}{\partial z}=0\ ;\ \alpha_0\frac{\partial u_0}{\partial \theta}+\ell\frac{\partial v_0}{\partial \xi}=0,$$

with $p_0=h_0$ **on** $z=0$.

N.B. We have expanded the surface boundary conditions about $z = 0$; also note that w does not appear at this order.

This nonlinear system has the exact solution

$$u_0=-\frac{\ell}{\omega}A_0\mathrm{e}^{\theta}\sin\xi\ ;\ v_0=\frac{\ell}{\omega}A_0\mathrm{e}^{\theta}\cos\xi\ ;$$

$$p_0=h_0=A_0\mathrm{e}^{\theta}\cos\xi-\frac{1}{2}\frac{\ell^2}{\omega^2}A_0^2\mathrm{e}^{2\theta},$$

with $\alpha_0=-\ell$ and for $A_0(X)$, ω both arbitrary.

At the next order we impose the condition which ensures that the asymptotic expansion is uniformly valid as $\theta\to\infty$; this requires

$$A_0\frac{\mathrm{d}B}{\mathrm{d}X}+2B\frac{\mathrm{d}A_0}{\mathrm{d}X}=\frac{\omega^2}{\ell}A_0 \quad \text{and so} \quad A_0(X)=\frac{1}{\sqrt{B(X)}}\exp\left\{\frac{\omega^2}{\ell}\int^X\frac{\mathrm{d}X'}{B(X')}\right\}.$$

The free surface is then given by

$$h(\theta,X,\xi;\varepsilon)\sim A_0\mathrm{e}^{\theta}\cos\xi-\frac{\ell^2}{2\omega^2}A_0^2\mathrm{e}^{2\theta}+\varepsilon h_1$$

and the beach is represented by $B(X)\sim X$ **as** $X\to 0$ **which gives**

$$A_0(X)\sim kX^{\beta},\ \beta=\tfrac{1}{2}\left[\left(\omega^2/\ell\right)-1\right] \quad \text{as} \quad X\to 0.$$

If $A_0(X)$ and all its derivatives are to exist as $X\to 0$ then $\beta=n$

i.e. $\dfrac{\omega^2}{\ell}=1+2n \quad (n=0,1,2,\ldots)$ **– the classical result.**

N.B. Uniformity & convergence as $X \to 0$ require finite A_0^2/B so $n = 1, 2, 3, \ldots$. [$n = 0$ is a special case.]

We can now find the run-up pattern of the waves.

This is given by the intersection of the surface wave with the bottom profile i.e. $z = -B(X) = \varepsilon h$.

At the beach $B(X) \sim X$**, so we have**

$$-x \sim A_0 e^{\theta} \cos\xi - \frac{\ell^2}{2\omega^2} A_0^2 e^{2\theta} .$$

With $\theta = -\ell X/\varepsilon = -\ell x$ **and** $\beta = n$**, we take the run-up pattern to be described by the normalised form**

$$1 + \mu Z^{n-1} e^{-Z} \cos\xi - \frac{\mu^2}{2(1+2n)} Z^{2n-1} e^{-2Z} = 0$$

where $Z = \ell x, \mu = k\varepsilon^n / \ell^{n-1}\ (n = 1, 2, \ldots)$ **and** $Z \neq 0.$

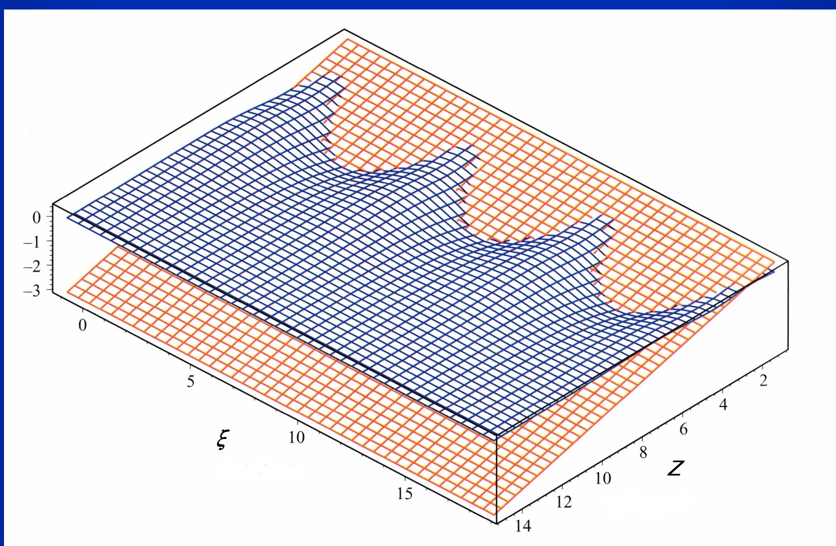
N.B. Continuous, bounded, periodic solutions exist for

$$\mu \geq \mu_n > 0 \quad [\mu_1 \approx 7\cdot 27; \mu_2 \approx 5\cdot 87; \mu_3 \approx 2\cdot 67].$$

Example:

blue = sea

orange = sand/beach



seawards

longshore direction

Example plotted for $n = 4, \mu = 4$.

End of

Lecture 4b