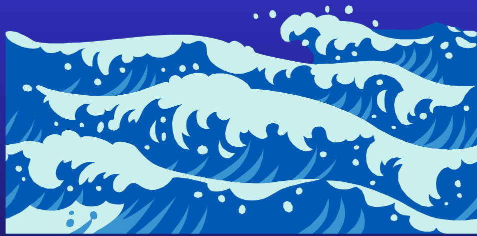
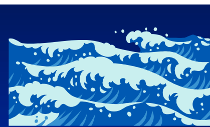
Lecture 4a

Asymptotic theory for the appearance of ‘cat’s-eyes’ and of critical levels



Preamble:



These last few lectures present asymptotic results for a few important properties associated with the propagation of water-waves. The aim is to show how this approach can extract the detailed structure of complicated flows, as well as showing how diverse can be the motion under the umbrella of ‘water waves’.

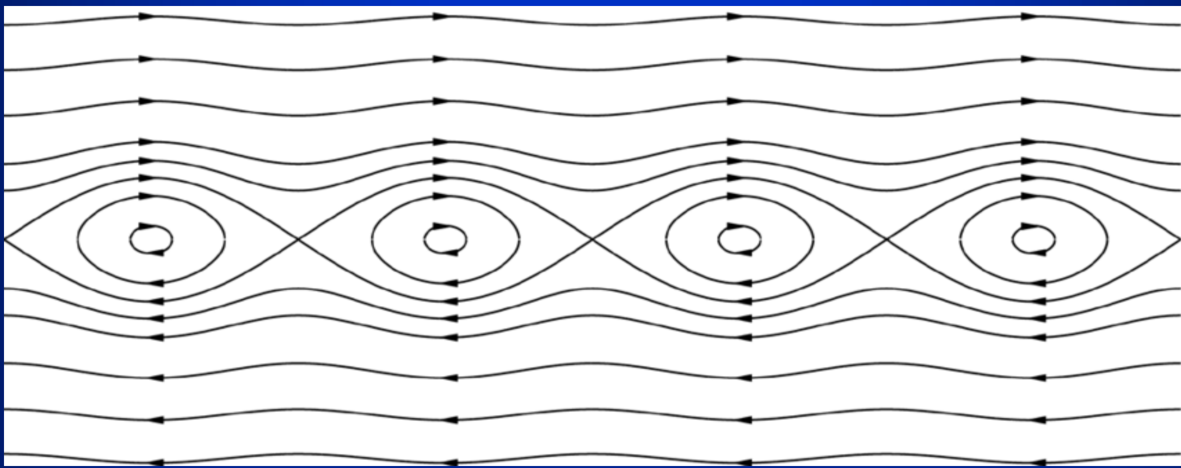
We start by picking up one of the flows described in the previous lecture: water waves in the presence of vorticity (i.e. waves over a ‘shear’ flow). In particular, we will address two important issues:

1. Can we have flows which initially have no cat’s-eyes, which then appear at later times?

2. Can we have flows which initially have no critical level, which then appears at a later time?

Kelvin's cat's-eyes (1880)

Lord Kelvin (W. Thomson) was the first to describe the closed streamlines that appear under a surface wave which moves at the same speed as the flow below it, at a particular level:



An example of classical cat's-eyes, drawn in the frame moving with the periodic surface wave.

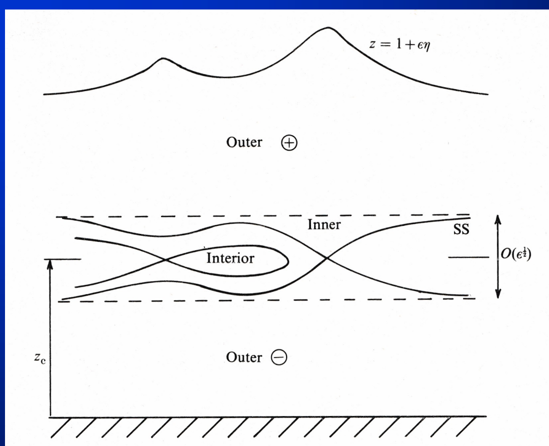
Formulation:

For inviscid, but rotational, flow, the thickness of the critical layer is $O\left(\sqrt{\varepsilon}\right)$ with the surface wave $z = 1 + \varepsilon\eta$;

introduce $z - z_c = \sqrt{\varepsilon}\, Z$, where the Burns condition gives the critical level at $z = z_c$ $(0 < z_c < 1)$.

The solution requires a number of matched, asymptotic regions:

outside the critical layer, above and below; inside the critical layer, either side of the separating streamline SS (across which a jump in vorticity is required at $O\left(\sqrt{\varepsilon}\right)$).



In the outer regions we have

$$[U(z)-c]\frac{\partial u}{\partial \xi}+\frac{\mathrm{d}U}{\mathrm{d}z}w+\varepsilon\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial \xi};$$

$$\varepsilon\left[\{U(z)-c\}\frac{\partial w}{\partial \xi}+\varepsilon\left(\frac{\partial w}{\partial \tau}+u\frac{\partial w}{\partial \xi}+w\frac{\partial w}{\partial z}\right)\right]=-\frac{\partial p}{\partial z};$$

$$\frac{\partial u}{\partial \xi}+\frac{\partial w}{\partial z}=0,$$

with $p=\eta$ **&** $w=(U-c)\dfrac{\partial \eta}{\partial \xi}+\varepsilon\left(\dfrac{\partial \eta}{\partial \tau}+u\dfrac{\partial \eta}{\partial \xi}\right)$

on $z=1+\varepsilon\eta$

and $w=0$ **on** $z=0$,

exactly as before, with $\xi=x-ct,\ \tau=\varepsilon t.$

For the critical layer, we define

$$z-z_c=\sqrt{\varepsilon}\,Z,\ u=V(\xi,\tau,Z)\big/\sqrt{\varepsilon},\ w=W(\xi,\tau,Z),\ p=P(\xi,\tau,Z)$$

$$\left(\hat{U}-c\right)\frac{\partial V}{\partial \xi}+\sqrt{\varepsilon}\left(\hat{U}'W+V\frac{\partial V}{\partial \xi}+W\frac{\partial V}{\partial Z}\right)+\varepsilon\frac{\partial V}{\partial \tau}=-\sqrt{\varepsilon}\frac{\partial P}{\partial \xi};$$

$$\varepsilon^{3/2}\left(\hat{U}-c\right)\frac{\partial W}{\partial \xi}+\varepsilon^{2}\left(V\frac{\partial W}{\partial \xi}+W\frac{\partial W}{\partial Z}\right)+\varepsilon^{5/2}\frac{\partial W}{\partial \tau}=-\frac{\partial P}{\partial Z};$$

$$\frac{\partial V}{\partial \xi}+\frac{\partial W}{\partial Z}=0,$$

where $\hat{U}=U\left(z_c+\sqrt{\varepsilon}\,Z\right)$ **and the prime denotes the derivative.**

Note that there are no boundary conditions here; these are replaced by matching conditions to the outer flow.

The development is lengthy and tiresome; we provide an outline of the main results.

The outer solution, at leading order, gives the Burns condition in the form

$$-\!\!\!\!\!\!\int_0^1 \frac{dz}{[U(z)-c]^2} = 1$$

where $-\!\!\!\!\!\!\int_0^1$ **denotes the finite part of the integral, defined as**

the finite part as $\varsigma \to 0^+$ **of** $\displaystyle \int_0^{z_c-\varsigma} \frac{\mathrm{d}z}{[U(z)-c]^2} + \int_{z_c+\varsigma}^{1} \frac{\mathrm{d}z}{[U(z)-c]^2},$

where $U(z_c) = c, 0 < z_c < 1.$

For monotonic profiles, $U(0) \le U(z) \le U(1)$**, there is always one solution** $c < U(0)$ **and one solution** $c > U(1).$

If, in addition, we have

$$\frac{\mathrm{d}U}{\mathrm{d}z} > 0, \ \frac{\mathrm{d}^2U}{\mathrm{d}z^2} < 0 \ \text{ both for } \ 0 \le z < 1$$

and

$$\frac{\mathrm{d}U}{\mathrm{d}z} = 0, \ \frac{\mathrm{d}^2U}{\mathrm{d}z^2} \le 0, \ \frac{\mathrm{d}^3U}{\mathrm{d}z^3} > 0 \ \text{ all on } \ z = 1$$

(choices that model realistic velocity profiles), then there is just one critical level: $U(z_c) = c, 0 < z_c < 1.$

At O(ε), the solution which matches across the critical layer gives

$$-2\not{\mathcal{J}}_{31} \frac{\partial \eta_0}{\partial \tau} + 3\not{\mathcal{J}}_{41} \eta_0 \frac{\partial \eta_0}{\partial \xi} + \not{\mathcal{J}} \frac{\partial^3 \eta_0}{\partial \xi^3} = 0,$$

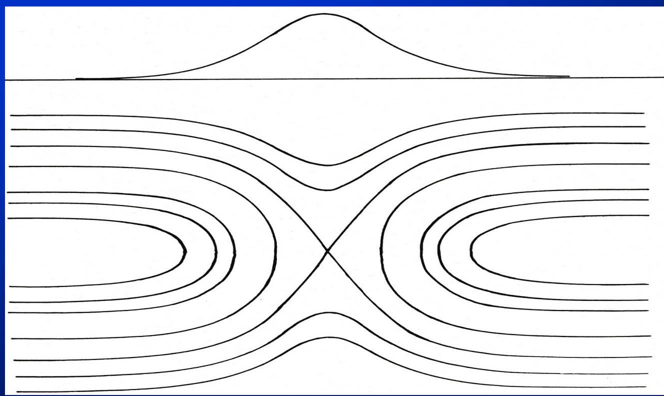
KdV as before, but the coefficients now evaluated as finite parts of the corresponding integrals.

The streamlines in the critical layer are given by

$$Z^2 + 2\frac{\eta_0}{\left[\frac{dU}{dz}(z_c)\right]^2} = \text{constant}$$

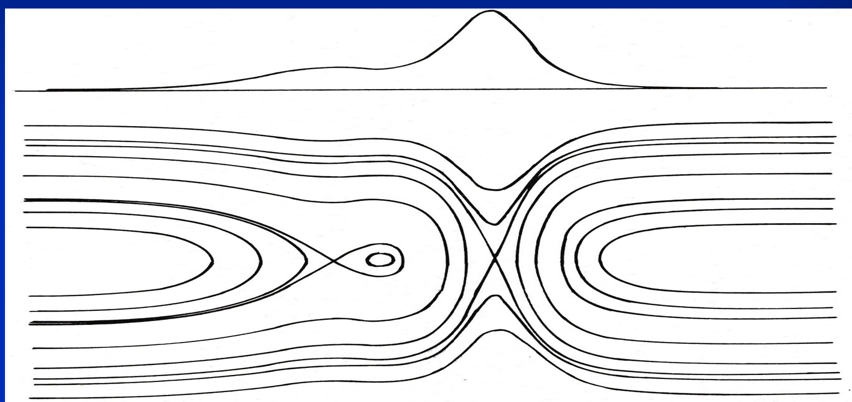
and then the evolution of the surface wave from one with a single peak to n-solitons produces $(n-1)$ cat's-eyes within the critical layer.

So we start with this scenario

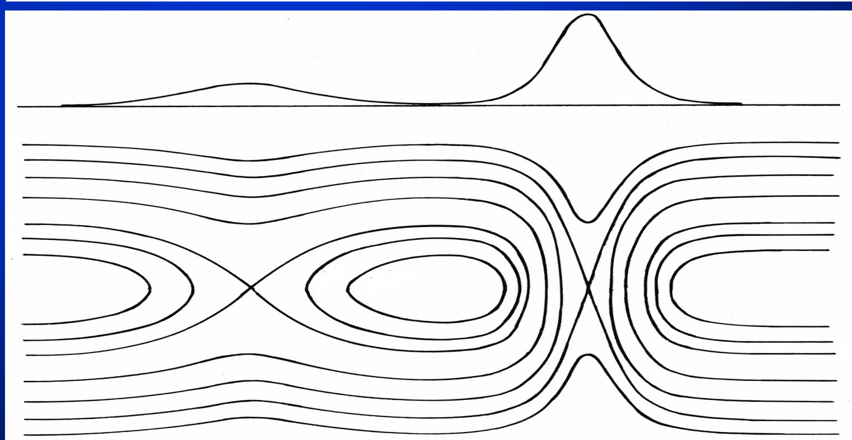


The two-soliton solution of the KdV equation then gives

A cat's-eye appears



and evolves



Appearance of critical levels

The existence of steady, periodic waves (in 2D), with given vorticity, mass flow rate and the total energy, has been proved, and some properties determined.

With suitable choices of the parameters measuring the strength of the vorticity, the mass flow rate and the total energy, critical layers can be present.

For fixed parameters then either there *is*, or there *is not*, a critical layer in the flow.

If we can change one (or more) of these physical properties of the flow, then a critical layer may appear, where previously one did not exist.

We cannot, in a realistic model of the (inviscid) flow, change the vorticity or the mass flow rate, but we can (reasonably) change the total energy .

So we choose to do this, keeping the mass flow rate and vorticity fixed; this is accomplished by

adjusting the conditions at the surface.

In particular we choose to control the atmospheric pressure at the free surface, which mimics the passage of a low-pressure region (e.g. typical of storms at sea). This requires the choice of the pressure variation at the surface and its movement (i.e speed in 1D).

We start with our familiar equations for 1D wave propagation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x};$$

$$\varepsilon\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z};\qquad \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

with $p=\varepsilon\eta+P(x,t;\varepsilon)$ **&** $w=\varepsilon\left(\frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}\right)$ **on** $z=1+\varepsilon\eta$

and $w=0$ **on** $z=0.$

The surface is $z = 1 + \varepsilon\eta(x, t; \varepsilon)$ and p is the deviation away from the hydrostatic pressure distribution.

Then $P(x, t; \varepsilon)$ is the chosen pressure variation.

We move directly to the far-field, by introducing a suitable frame moving with the wave, and a corresponding timescale:

$$\xi=x-\varepsilon^{-1}\int_0^{\tau}c(\tau';\varepsilon)\,\mathrm{d}\tau',\qquad \tau=\varepsilon t$$

where $c(\tau; \varepsilon)$ is the (variable) speed of the wave.

In this far-field, based on ε, the relevant solution is obtained by transforming according to

$$u\to-\gamma z+\varepsilon u(\xi,\tau;\varepsilon),\ w\to\varepsilon w,\ p\to\varepsilon p,\ P\to\varepsilon P.$$

We have chosen the vorticity, $\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=-\gamma$, to be a constant.

The problem is now described by

$$-[\gamma z + c(\tau)]\frac{\partial u}{\partial \xi} - \gamma w + \varepsilon\left(\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial \xi};$$

$$\varepsilon\left[-\{\gamma z + c(\tau)\}\frac{\partial w}{\partial \xi} + \varepsilon\left(\frac{\partial w}{\partial \tau} + u\frac{\partial w}{\partial \xi} + w\frac{\partial w}{\partial z}\right)\right] = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial z} = 0,$$

with $p = \eta + P$ **&** $w = -[\gamma + c(\tau)]\dfrac{\partial \eta}{\partial \xi} + \varepsilon\left(\dfrac{\partial \eta}{\partial \tau} - \eta\dfrac{\partial \eta}{\partial \xi} + u\dfrac{\partial \eta}{\partial \xi}\right)$

on $z = 1 + \varepsilon\eta$

and $w = 0$ **on** $z = 0.$

We seek a solution in

$$0 \le z \le 1 + \varepsilon\eta(\xi, \tau; \varepsilon), -\infty < \xi < \infty, \tau \ge 0.$$

An observation: there are three wave components, with speeds $c_1 > (-\gamma z)_{max}$ and $c_2 < (-\gamma z)_{min}$, and c; we select c so that, initially, it is between c_1, c_2, and in particular between the corresponding max/min values. (We allow both $\gamma > 0$ and $\gamma < 0$.)

In detail, our model is as follows:

We assume initial data, in the near-field, that is on compact support (for $x = O(1)$), then in the far-field there will be three well-separated components – and we follow only the one moving at speed c.

Consistent with the equations (and the observations above), we consider the two problems given by: $c = \varepsilon C(\tau)$, $c = -\gamma + \varepsilon C(\tau)$, but give the details only for the former.

Model with $c = \varepsilon C(\tau)$

This choice requires $p = O(\varepsilon)$, so we further transform

$$p \to \varepsilon p$$

and then seek an asymptotic solution in the usual form

$$q(\xi,\tau,z;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(\xi,\tau,z) \quad (q \equiv u, w, p)$$

$$\eta(\xi,\tau;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n \eta_n(\xi,\tau)$$

given $P(\xi,\tau) \in C^\infty$ and $c \in C^0$; valid for $\tau \in [0,\tau_0]$.

Solution at leading order is

$$\eta_0 = -P;\, u_0 = -\gamma P;\, w_0 = \gamma z P_\xi, \text{ with } (-P > 0);\; p_0 = \eta_1 - \tfrac{1}{3}\gamma^2(1 - z^3)P_{\xi\xi}$$

and then $w_1 = Az - \frac{1}{6}\gamma z^3 P_{\xi\xi\xi}$ where

$$A(\xi,\tau) = (1+\gamma^2)(CP_\xi - P_\tau) + \tfrac{1}{3}\gamma(9 + 2\gamma^2)PP_\xi + \tfrac{1}{6}\gamma(1 - 2\gamma^2)P_{\xi\xi\xi}$$

and correspondingly for the other functions at this order.

N.B. We do not require these higher-order terms in order to give the dominant description of the solution.

Thus we have $u - c \sim -\varepsilon C - \gamma z - \varepsilon\gamma P$

and so the critical level ($u - c = 0$) is given by

$$z \sim \varepsilon(-P - C/\gamma).$$

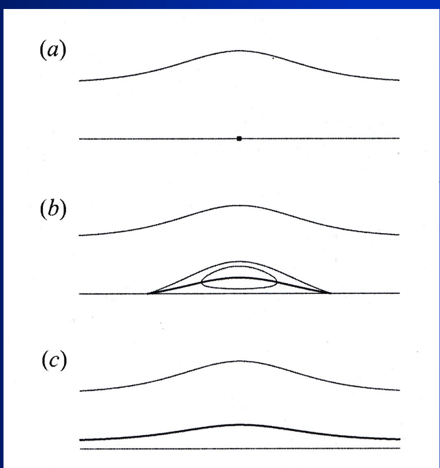
Consider $(-P)$ with a single peak $(-P)_{max}$ at $\xi = 0$ – so a region of low pressure, which otherwise decays as $|\xi| \to \infty$; we take $\gamma > 0$ and allow the pressure distribution to move with decreasing speed, $C(\tau)$. We see that

$C(\tau) > \gamma(-P)_{max}$	no critical level
$C(\tau) = \gamma(-P)_{max}$	stagnation point on bottom directly below the peak
$0 < C(\tau) < \gamma(-P)_{max}$	critical level in the flow
$C(\tau) < 0$	critical level extends to infinity

Furthermore, at leading order, the streamlines are given by

$$\psi_z \sim -\gamma z - \varepsilon\gamma P \quad \text{and so} \quad Z(Z+2P) = \text{constant}\ (z = \varepsilon Z),$$

which we include in the description of the flow field.



(a) Stagnation point appears
(b) Critical level appears (and a streamline is included)
(c) Critical level extends to infinity

There is a similar result for $\gamma < 0$, and also for the appearance of a critical level near the surface with $c = -\gamma + \varepsilon C(\tau)$.

End of

Lecture 4a