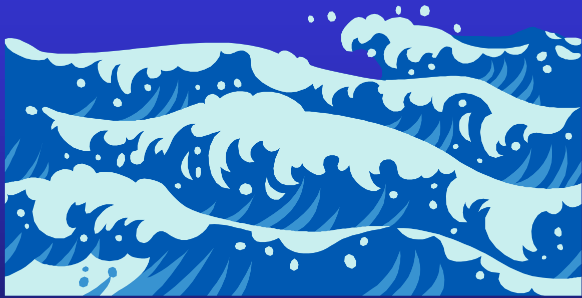
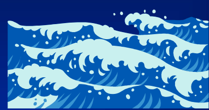
Lecture 3b

The KdV equation for variable depth and in a flow with background vorticity





In this lecture, we show how standard asymptotic methods can be used to develop a theory for the KdV equation within a more realistic scenario. We examine two cases.

1. Variable depth

In general, this is an important type of problem in water-wave theory, both for river flows and for ocean waves (in the neighbourhood of shorelines).

The simplest problem that is associated with our weakly nonlinear, dispersive waves, is that generated by a KdV-type propagation over variable depth.

The scale on which the depth varies gives rise to different asymptotic approximations – and this usually produces a very involved structure, often requiring the consideration of many terms in the asymptotic expansions (and, typically, the resulting equation is NOT completely integrable). In general, both transmitted and reflected waves must be included, starting from the general governing equations.

In this example of variable depth, we start with the governing equations (as used in the previous lecture for one-dimensional wave propagation with δ scaled out), but now rewritten to accommodate a variable bottom profile, and develop the relevant KdV equation.

Thus we have

$$\frac{\partial u}{\partial t} + \varepsilon\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x};$$

$$\varepsilon\left\{\frac{\partial w}{\partial t} + \varepsilon\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)\right\} = -\frac{\partial p}{\partial z};\quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

with $p = \eta$ **&** $w = \dfrac{\partial \eta}{\partial t} + \varepsilon u\dfrac{\partial \eta}{\partial x}$ **on** $z = 1 + \varepsilon\eta$

and $w = u\dfrac{\mathrm{d}b}{\mathrm{d}x}$ **on** $z = b(x)$.

Write $b(x) = B(\alpha x)$ and then we consider the different scales on which the depth might vary.

The various cases are summarised by

$$\alpha = \mathrm{o}(\varepsilon); \quad \alpha = \mathrm{O}(\varepsilon); \quad \varepsilon = \mathrm{o}(\alpha),$$

and $\alpha = \mathrm{O}(1)$ i.e. α fixed as ε varies – the mathematically *most difficult case*.

The mathematically *most interesting case*, in the context that we are exploring here (KdV-type problems), is

$$\alpha = \mathrm{O}(\varepsilon) \text{ so we write } b = B(\varepsilon x).$$

We introduce the variables $\xi = \varepsilon^{-1} k(X) - t,\ X = \varepsilon x$ where ***k(X)*** accommodates the expected variable speed over the varying depth. (We consider right-running waves.)

N.B. The choice $k(X) = X$ gives $\xi = X/\varepsilon - t = x - t.$

Written in our new variables, we have

$$-\frac{\partial u}{\partial \xi} + \varepsilon\left\{u\left(\frac{\mathrm{d}k}{\mathrm{d}X}\frac{\partial u}{\partial \xi} + \varepsilon\frac{\partial u}{\partial X}\right) + w\frac{\partial u}{\partial z}\right\} = -\left(\frac{\mathrm{d}k}{\mathrm{d}X}\frac{\partial p}{\partial \xi} + \varepsilon\frac{\partial p}{\partial X}\right);$$

$$\varepsilon\left\{-\frac{\partial w}{\partial \xi} + \varepsilon\left[u\left(\frac{\mathrm{d}k}{\mathrm{d}X}\frac{\partial w}{\partial \xi} + \varepsilon\frac{\partial w}{\partial X}\right) + w\frac{\partial w}{\partial z}\right]\right\} = -\frac{\partial p}{\partial z};$$

$$\frac{\mathrm{d}k}{\mathrm{d}X}\frac{\partial u}{\partial \xi} + \varepsilon\frac{\partial u}{\partial X} + \frac{\partial w}{\partial z} = 0,$$

with $p = \eta$ & $w = -\dfrac{\partial \eta}{\partial \xi} + \varepsilon u\left(\dfrac{\mathrm{d}k}{\mathrm{d}X}\dfrac{\partial \eta}{\partial \xi} + \varepsilon\dfrac{\partial \eta}{\partial X}\right)$ on $z = 1 + \varepsilon\eta$

and $w = \varepsilon u\dfrac{\mathrm{d}B}{\mathrm{d}X}$ on $z = B(X)$.

We seek an asymptotic solution in ε, following the same pattern as before; at leading order we obtain

$$p_0 = \eta_0; \; u_0 = \eta_0 \frac{\mathrm{d}k}{\mathrm{d}X}; \; w_0 = [B(X) - z]\left(\frac{\mathrm{d}k}{\mathrm{d}X}\right)^2 \frac{\partial \eta_0}{\partial \xi},$$

(all for $B(X) \le z \le 1$)

and then the kinematic condition is satisfied for arbitrary $\eta_0(\xi, X)$ if

$$k(X) = \int_0^X \frac{\mathrm{d}X'}{\sqrt{D(X')}} \quad \text{where} \quad D(X) = 1 - B(X) \; (> 0).$$

***D*(*X*) is the local depth of the water; for *D* = 1, we recover $\xi = x - t$.**

At the next order we obtain the KdV equation for variable depth:

$$2\sqrt{D}\frac{\partial \eta_0}{\partial X} + \frac{1}{2\sqrt{D}}\frac{\mathrm{d}D}{\mathrm{d}X}\eta_0 + \frac{3}{D}\eta_0\frac{\partial \eta_0}{\partial \xi} + \frac{1}{3}D\frac{\partial^3 \eta_0}{\partial \xi^3} = 0.$$

With *D* = 1, so the bottom is flat and horizontal (*B* = 0), we recover our standard KdV equation.

In general, this new version of the KdV equation is not a completely integrable equation. However, it has been shown, both analytically in special cases, and numerically, that a reduction of depth leads to the appearance ('fission') of solitons.

Finally, we observe that writing $\eta_0 = D^2 H(\xi, \int \sqrt{D}\, \mathrm{d}X)$ with $D = (aX + b)^{9/4}$ leads to the cKdV equation for *H*.

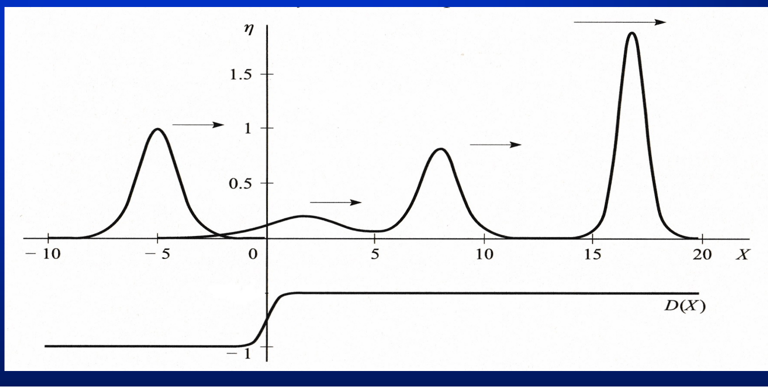
An example of 'soliton fission':

For the variable-depth KdV equation, in the case of a rapid change in depth (the details requiring a further asymptotic analysis), we find that a depth $D = 1$ which reduces to a depth

$$\left[\frac{1}{2}n(n+1)\right]^{-4/9}, \quad n = 2, 3\ldots$$

generates an n-soliton solution along the shelf.

This is a 3-soliton example:



2. <u>Waves in the presence of vorticity</u>

A more realistic model for water waves arises when we allow the water to be in motion, and the wave superimposed on that motion. This is often referred to as 'waves over a shear flow'.

In order to initiate this discussion, we therefore require the appropriate governing equations that are rewritten to include some (background) vorticity.

Thus, for one-dimensional wave propagation, we replace

$$\varepsilon \mathbf{u}_{\perp} = \varepsilon(u, 0) \text{ by } (U(z) + \varepsilon u, 0),$$

where $U(z)$ is given; this is the underlying 'shear' flow. The vorticity here is $\mathrm{d}U/\mathrm{d}z$ (perpendicular to the (x, z)-plane); this is O(1) in this formulation.

The governing equations, with background vorticity and over a flat, horizontal bed, are

$$\frac{\partial u}{\partial t}+U(z)\frac{\partial u}{\partial x}+\frac{\mathrm{d}U}{\mathrm{d}z}w+\varepsilon\left(u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x};$$

$$\varepsilon\left[\frac{\partial w}{\partial t}+U(z)\frac{\partial w}{\partial x}+\varepsilon\left(u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)\right]=-\frac{\partial p}{\partial z};$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$

with $p=\eta$ **&** $w=\dfrac{\partial \eta}{\partial t}+(U+\varepsilon u)\dfrac{\partial \eta}{\partial x}$ **on** $z=1+\varepsilon\eta$

and $w=0$ **on** $z=0$.

We now move directly to suitable far-field variables:

$$\xi=x-ct,\quad \tau=\varepsilon t.$$

This gives

$$\left[U(z)-c\right]\frac{\partial u}{\partial \xi}+\frac{\mathrm{d}U}{\mathrm{d}z}w+\varepsilon\left(\frac{\partial u}{\partial \tau}+u\frac{\partial u}{\partial \xi}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial \xi};$$

$$\varepsilon\left[\{U(z)-c\}\frac{\partial w}{\partial \xi}+\varepsilon\left(\frac{\partial w}{\partial \tau}+u\frac{\partial w}{\partial \xi}+w\frac{\partial w}{\partial z}\right)\right]=-\frac{\partial p}{\partial z};$$

$$\frac{\partial u}{\partial \xi}+\frac{\partial w}{\partial z}=0,$$

with $p=\eta$ **&** $w=(U-c)\dfrac{\partial \eta}{\partial \xi}+\varepsilon\left(\dfrac{\partial \eta}{\partial \tau}+u\dfrac{\partial \eta}{\partial \xi}\right)$ **on** $z=1+\varepsilon\eta$

and $w=0$ **on** $z=0$.

We seek an asymptotic solution, in ε, exactly as we have developed in our earlier work. At leading order we obtain

$$(U-c)\frac{\partial u_0}{\partial \xi}+\frac{\mathrm{d}U}{\mathrm{d}z}w_0=-\frac{\partial p_0}{\partial \xi};\quad \frac{\partial p_0}{\partial z}=0;\quad \frac{\partial u_0}{\partial \xi}+\frac{\partial w_0}{\partial z}=0,$$

(**all for** $z\in[0,1]$)

with $p_0=\eta_0$ **&** $w_0=(U-c)\dfrac{\partial \eta_0}{\partial \xi}$ **on** $z=1$

and $w_0=0$ **on** $z=0$.

This set has the solution, for arbitrary $\eta_0(\xi,\tau)$**:**

$$p_0=\eta_0;\ w_0=(U(z)-c)I_2\frac{\partial \eta_0}{\partial \xi};\ u_0=-\eta_0\frac{\mathrm{d}}{\mathrm{d}z}[(U-c)I_2],$$

where $I_2(z)=\displaystyle\int_0^z\frac{\mathrm{d}z'}{[U(z')-c]^2},\quad U(z)\neq c,\ z\in[0,1].$

The kinematic condition determines *c* in the form $I_2(1)=1$

which is $\displaystyle\int_0^1\frac{dz}{[U(z)-c]^2}=1.$ **(The Burns condition, 1953)**

At the next order, we obtain the KdV equation which describes propagation over an arbitrary 'shear' flow:

$$-2I_{31}\frac{\partial \eta_0}{\partial \tau}+3I_{41}\eta_0\frac{\partial \eta_0}{\partial \xi}+J\frac{\partial^3 \eta_0}{\partial \xi^3}=0$$

where $I_{n1}=I_n(1)=\displaystyle\int_0^1\frac{\mathrm{d}z}{[U(z)-c]^n}$

and $J=\displaystyle\int_0^1\int_z^1\int_0^{\zeta}\frac{[U(\zeta)-c]^2}{[U(z)-c]^2[U(Z)-c]^2}\,\mathrm{d}Z\mathrm{d}\zeta\mathrm{d}z$

in the absence of critical layers; if a critical layer is present, then closed streamlines (cat's-eyes) appear.

Example: choose the simple case $U(z) = \gamma z$

(where γ is a constant – so constant vorticity – usually called 'linear shear').

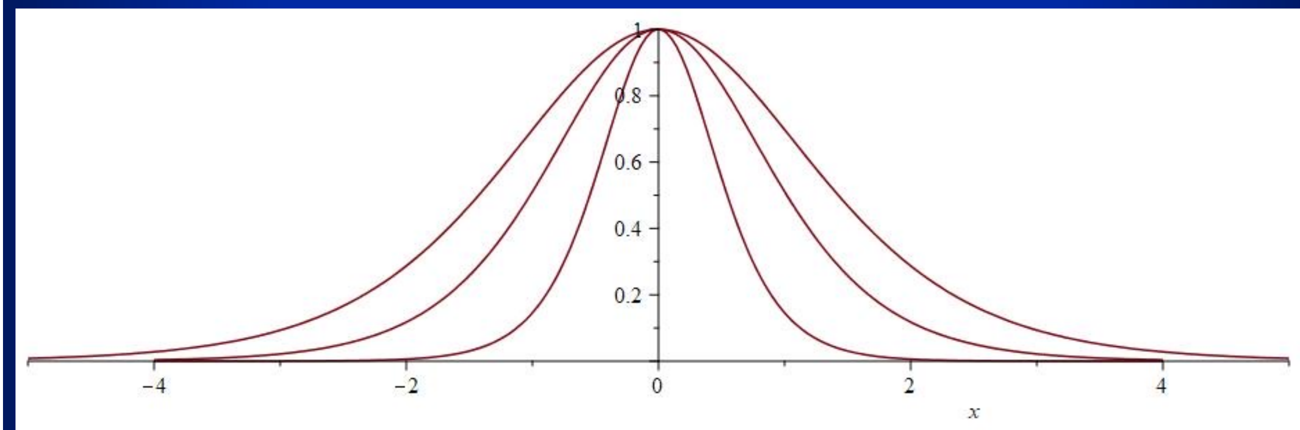
Then the *Burns condition* gives $c = \frac{1}{2}\left(\gamma \pm \sqrt{4+\gamma^2}\right)$

(and no critical layers exist for this background flow).

The KdV equation becomes

$$\pm\sqrt{4+\gamma^2}\,\eta_{0\tau} + \left(3+\gamma^2\right)\eta_0\eta_{0\xi} + \frac{1}{3}\left(1-\frac{\gamma}{c}\right)\eta_{0\xi\xi\xi} = 0.$$

Example of a solution: the solitary wave



KdV solitary wave for constant vorticity: middle profile is $\gamma = 0$ (irrotational flow); outer (broader) profile is for $\gamma = 1$, upstream propagation; inner (narrower) profile is for $\gamma = 1$, downstream propagation.

End of Lecture 3b