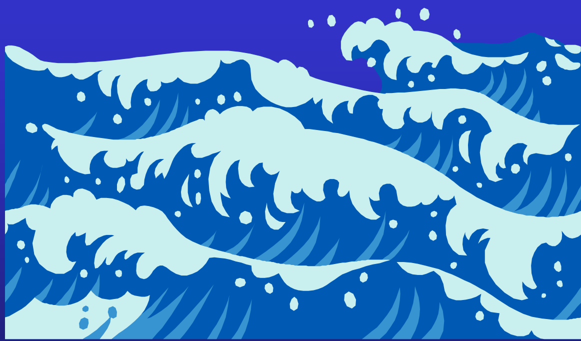
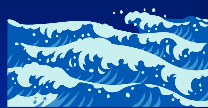
Lecture 3a

The Korteweg-de Vries (KdV) equation for water waves, and related equations



Formulation



Consider small-amplitude, one-dimensional, long waves (that decay at infinity); there is no natural horizontal scale, so we use the form of the governing equations with δ scaled out (see Lecture 1a):

$$\frac{\mathrm{D}\mathbf{u}_\perp}{\mathrm{D}t} = -\nabla_\perp p; \qquad \varepsilon\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}; \qquad \nabla\cdot\mathbf{u} = 0$$

with $$p = \eta \;\&\; w = \frac{\partial \eta}{\partial t} + \varepsilon(\mathbf{u}_\perp\cdot\nabla_\perp)\eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and $$w = 0 \;\text{ on }\; z = 0,$$

where we have made the simplest choice: the bottom is

$$z = b = 0.$$

Asymptotic solution

We seek a **formal asymptotic solution** as an **expansion in ε, written as**

$$q(x,z,t;\varepsilon) \sim \sum_{n=0}^{\infty} \varepsilon^n q_n(x,z,t), \quad \varepsilon \to 0,$$

where q (and correspondingly q_n) represent each of (u, w, p, η), although for η we omit the dependence on z.

We assume, in a reformulation of the surface boundary conditions, that the problem for $z \in [0, 1+\varepsilon\eta]$ can be mapped (using Taylor expansions) to $z \in [0,1]$.

This is readily confirmed because the underlying problem is polynomial in z.

The problem therefore can be written as

$$\frac{\partial u}{\partial t} + \varepsilon\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x};$$

$$\varepsilon\left\{\frac{\partial w}{\partial t} + \varepsilon\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)\right\} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

with

$$\left.\begin{aligned} p + \varepsilon\eta\frac{\partial p}{\partial z} + \cdots &= \eta \\ w + \varepsilon\eta\frac{\partial w}{\partial z} + \cdots &= \frac{\partial \eta}{\partial t} + \varepsilon\eta\frac{\partial \eta}{\partial x} + \cdots \end{aligned}\right\} \quad \text{on} \quad z = 1$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

Note: we have retained as many terms as appropriate in the Taylor expansions that are used in the boundary conditions.

At leading order (in ε) **we find that** $\eta_0(x,t)$ **satisfies**

$$\frac{\partial^2 \eta_0}{\partial t^2} - \frac{\partial^2 \eta_0}{\partial x^2} = 0,$$

and at the next order we have $\eta_1 = \mathcal{F}(\eta_0) + t\mathcal{G}(\eta_0)$ **where** $\mathfrak{F}$ **and** $\mathfrak{G}$ **are functionals (involving derivatives), provided that** $\eta_0 \in C^4$.

Thus the asymptotic expansion for η **takes the form**

$$\eta \sim \eta_0(x,t) + \varepsilon\{\mathcal{F}(\eta_0) + t\,\mathcal{G}(\eta_0)\}$$

which is not uniformly valid as t **increases; provided that we have sufficiently smooth initial data, there is no other non-uniformity. However, this solution is valid only for** $t = O(1)$ **– the near field.**

The far field is defined by times $t = O(\varepsilon^{-1})$**, so we introduce** $\tau = \varepsilon t$ **and consider right-running waves by using** $\xi = x - t$.

Thus we transform to far-field coordinates, to give

$$-\frac{\partial u}{\partial \xi} + \varepsilon\frac{\partial u}{\partial \tau} + \varepsilon\left(u\frac{\partial u}{\partial \xi} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial \xi};$$

$$\varepsilon\left\{-\frac{\partial w}{\partial \xi} + \varepsilon\frac{\partial w}{\partial \tau} + \varepsilon\left(u\frac{\partial w}{\partial \xi} + w\frac{\partial w}{\partial z}\right)\right\} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial \xi} + \frac{\partial w}{\partial z} = 0,$$

with

$$\left.\begin{aligned} p + \varepsilon\eta\frac{\partial p}{\partial z} + \cdots &= \eta \\ w + \varepsilon\eta\frac{\partial w}{\partial z} + \cdots &= -\frac{\partial \eta}{\partial \xi} + \varepsilon\frac{\partial \eta}{\partial \tau} + \varepsilon\eta\frac{\partial \eta}{\partial \xi} + \cdots \end{aligned}\right\} \quad \text{on} \quad z = 1$$

and

$$w = 0 \quad \text{on} \quad z = 0.$$

The asymptotic structure follows the earlier pattern, but now written in (ξ, τ)-variables; we obtain

$$p_0 = \eta_0,\ u_0 = \eta_0,\ w_0 = -z\frac{\partial u_0}{\partial \xi} = -z\frac{\partial \eta_0}{\partial \xi} \quad \text{each for } 0 \le z \le 1,$$

and for arbitrary $\eta_0(\xi,\tau)$ at this order; here, we have selected that solution for which all perturbations vanish if there is no wave present.

At the next order we find that $p_1 = \eta_1 + \frac{1}{2}\left(1 - z^2\right)\frac{\partial^2 u_0}{\partial \xi^2}$;

$$\frac{\partial u_1}{\partial \xi} = \frac{\partial u_0}{\partial \tau} + u_0\frac{\partial \eta_0}{\partial \xi} + \frac{\partial \eta_1}{\partial \xi} + + \frac{1}{2}\left(1 - z^2\right)\frac{\partial^3 u_0}{\partial \xi^3},$$

and

$$w_1 = -\left(\frac{\partial \eta_0}{\partial \tau} + \eta_0\frac{\partial \eta_0}{\partial \xi} + \frac{\partial \eta_1}{\partial \xi}\right)z + + \frac{1}{2}\left(1 - \frac{1}{3}z^2\right)z\frac{\partial^3 \eta_0}{\partial \xi^3}$$

which satisfies the bottom boundary condition.

In summary we have, for example,

$$p \sim \eta_0 + \varepsilon\left\{\eta_1 + \tfrac{1}{2}(1 - z^2)\eta_{0\xi\xi}\right\}$$

and

$$w \sim -z\eta_{0\xi} + \varepsilon\left\{-\left(\eta_{1\xi} + \eta_{0\tau} + \eta_0\eta_{0\xi} + \tfrac{1}{2}\eta_{0\xi\xi\xi}\right)z + \tfrac{1}{6}z^3\eta_{0\xi\xi\xi}\right\},$$

both for $0 \le z \le 1$.

Finally, the kinematic condition at the surface gives the equation for $\eta_0(\xi,\tau)$, with $\eta_1(\xi,\tau)$ arbitrary at this order; we have

$$2\eta_{0\tau} + 3\eta_0\eta_{0\xi} + \tfrac{1}{3}\eta_{0\xi\xi\xi} = 0,$$

the <u>Korteweg-de Vries equation</u>.

N.B. All higher-order equations describing the surface profile are linear.

The asymptotic expansion is uniformly valid for $\tau \to \infty$, for bounded and suitably smooth initial data.

Indeed, rigorous developments based on this approach have demonstrated that the KdV equation constitutes a proper approximation to the problem, for $0 \le \varepsilon < \varepsilon_0$, and for times $t = \varepsilon^{-1}\tau_0$, for some fixed τ_0, independent of ε.

The KdV equation is the archetypal equation for the development of 'soliton'/inverse scattering theory.

Soliton theory for the KdV equation: a very brief overview

Treat the unknown function – a solution of the KdV equation – as the time-dependent potential of a one-dimensional, linear scattering problem. The associated inverse scattering problem, with time evolution consistent with the KdV equation, can be solved to recover a solution of this equation.

Consider

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

and write

$$u(x,t) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x;t)$$

where

$$K(x,z;t)+F(x,z,t)+\int_{x}^{\infty} K(x,y;t)F(y,z,t)\,\mathrm{d}y=0$$

with $F(x, z, t)$ satisfying

$$\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial z^2}=0 \quad \text{and} \quad \frac{\partial F}{\partial t}+4\left(\frac{\partial^3 F}{\partial x^3}+\frac{\partial^2 F}{\partial z^2}\right)=0;$$

the relevant choice is $F=F(x+z,t)$.

The solitary-wave solution is given by writing

$$F=\mathrm{e}^{-k(x+z)+\omega t+\alpha}, \quad \omega=8k^3;$$

the two-soliton solution is obtained from

$$F=\mathrm{e}^{\theta_1}+\mathrm{e}^{\theta_2},\ \theta_i=-k_i(x+z)+8k_i^3 t+\alpha_i\ (k_1\neq k_2).$$

Other soliton equations arising from the classical water-wave problem

Making suitable choices of coordinates and scalings, we can obtain:

2D KdV/KP:

$$\frac{\partial}{\partial \xi}\left(2\frac{\partial \eta}{\partial \tau}+3\eta\frac{\partial \eta}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \eta}{\partial \xi^3}\right)+\frac{\partial^2 \eta}{\partial Y^2}=0$$

cylindrical/ concentric KdV:

$$2\frac{\partial \eta}{\partial r}+\frac{1}{r}\eta+3\eta\frac{\partial \eta}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \eta}{\partial \xi^3}=0$$

Boussinesq:

$$\frac{\partial^2 H}{\partial t^2}-\frac{\partial^2 H}{\partial x^2}+3\frac{\partial^2}{\partial x^2}\left(H^2\right)-\frac{\partial^4 H}{\partial x^4}=0$$

$$(H=\eta-\varepsilon\eta^2)$$

and there are equations that are close to being soliton equations:

nearly-concentric KdV

$$\frac{\partial}{\partial \xi}\left(2\frac{\partial \eta}{\partial r}+\frac{1}{r}\eta+3\eta\frac{\partial \eta}{\partial \xi}+\frac{1}{3}\frac{\partial^3 \eta}{\partial \xi^3}\right)+\frac{1}{r^2}\frac{\partial^2 \eta}{\partial \theta^2}=0$$

which is integrable for initial data which decays rapidly at infinity

2D Boussinesq

$$\frac{\partial^2 H}{\partial t^2}-\frac{\partial^2 H}{\partial x^2}+3\frac{\partial^2}{\partial x^2}\left(H^2\right)-\frac{\partial^4 H}{\partial x^4}-\frac{\partial^2 H}{\partial Y^2}=0$$

which possesses a general 2-soliton solution, but only special N-soliton solutions.

Furthermore, there is the Nonlinear Schrödinger (NLS) family of equations, starting with

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+u\left|u\right|^2=0$$

and the Camassa-Holm (CH) family, starting with

$$\frac{\partial u}{\partial t}+2\omega\frac{\partial u}{\partial x}-\frac{\partial^3 u}{\partial x^2 \partial t}+3u\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}.$$

An important and extensive description of water waves can be developed following the 'soliton' route – but we now take a different direction in these lectures.

End of Lecture 3a