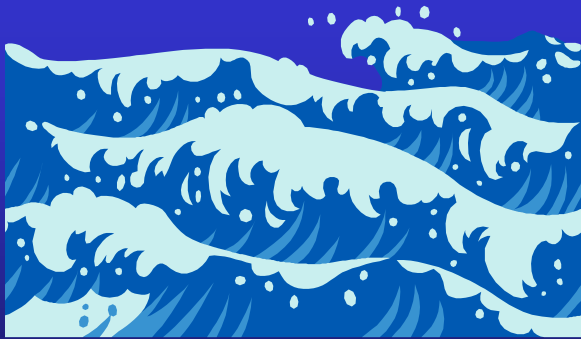
Lecture 2b

Introduction to parameter asymptotics: ideas and methods



Asymptotic expansions with a parameter

Consider (for simplicity) a real-valued, scalar function of one real scalar variable and one real parameter: *f* (*x*;*ε*).

Construct an expansion of *f*, in *ε* at fixed *x*; this based on a suitable asymptotic sequence: $\{\phi_n(\varepsilon)\},\ n = 0, 1, 2, \ldots ,$

defined such that

$$\lim_{\varepsilon \to 0} \left[\phi_{n+1}(\varepsilon) / \phi_n(\varepsilon) \right] = 0 \ \text{ for every } \ n = 0, 1, 2, \ldots$$

This is usually expressed as

$$\phi_{n+1}(\varepsilon) = \mathrm{o}\left[\phi_n(\varepsilon) \right] \ \text{ as } \ \varepsilon \to 0.$$

For an appropriate asymptotic sequence, we construct

$$f(x;\varepsilon) = \sum_{n=0}^{N} a_n(x)\phi_n(\varepsilon) + \mathrm{O}\left[\phi_{N+1}(\varepsilon)\right]$$

which is often written as

$$f(x;\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\phi_n(\varepsilon) \ \text{ as } \ \varepsilon \to 0.$$

Note: the asymptotic sequence $\{\varepsilon^n\}$ is appropriate for e.g. $\sqrt{1+x+\varepsilon}$ but not for $\exp(-x/\varepsilon)$.

Also, the asymptotic sequence is not unique; e.g. could use $\{\sin^n\varepsilon\}$ or $\{\tan^n\varepsilon\}$ or $\{\ln(1+\varepsilon^n)\}$ instead of $\{\varepsilon^n\}$.

Suppose that $f(x;\varepsilon)$ is defined for $x \in D$;

we construct the asymptotic expansion of *f* with respect to the asymptotic sequence $\{\phi_n(\varepsilon)\}$:

$$f(x;\varepsilon) \sim \sum_{n=0}^{N} a_n(x)\phi_n(\varepsilon) \ \text{ at fixed } x, \ \text{ for every } N \geq 0.$$

Is this asymptotic representation, for $\forall N$, also valid for $\forall x \in D$?

The asymptotic expansion is *uniformly valid* if

$$a_{n+1}(x)\phi_{n+1}(\varepsilon) = \mathrm{o}\left[a_n(x)\phi_n(\varepsilon)\right]$$

for every n = 0, 1, 2, … and $\forall x \in D$, as $\varepsilon \to 0$.

Example 1: $f(x;\varepsilon) = \sqrt{1 + x + \varepsilon}, \quad x \ge 0$

has an asymptotic expansion that begins

$$f(x;\varepsilon) \sim \sqrt{1+x} + \frac{\varepsilon}{2\sqrt{1+x}}$$

which is uniformly valid on the given domain.

Example 2: $f(x;\varepsilon) = \sqrt{x^2 + x + \varepsilon}, \quad x \ge 0$

then $$f(x;\varepsilon) \sim \sqrt{x + x^2} + \frac{\varepsilon}{2\sqrt{x+x^2}}$$

which is *not* uniformly valid for $x \to 0^+$.

An asymptotic expansion 'breaks down' for any *x*s (in the given domain) for which

$$a_{n+1}(x)\phi_{n+1}(\varepsilon) = O\left[a_n(x)\phi_n(\varepsilon)\right]$$

for any *n* as $\varepsilon \to 0$.

In the previous example (2), this occurs for $x = O(\varepsilon)$;

so we introduce a new 'scaled' *x*: $x = \varepsilon X$ and then

$$f(x;\varepsilon) = f(\varepsilon X;\varepsilon) \equiv F(X;\varepsilon) = \sqrt{\varepsilon(1+X) + \varepsilon^2 X^2}$$

which gives $$F(X;\varepsilon) \sim \sqrt{\varepsilon}\left\{\sqrt{1+X} + \frac{\varepsilon X^2}{2\sqrt{1+X}}\right\}.$$

This new asymptotic expansion of the same function is uniformly valid as $X \to 0^+$

but not as $X \to \infty$ (which produces a breakdown that takes us back to the variable *x*).

The two expansions that we have now generated are said to 'match': they satisfy the 'matching principle':

The expansion of the first (in *x*), written in terms of the second variable (*X*), and expanded, agrees precisely with the corresponding expansion from the second (in *X*) written in terms of *x* (and expanded).

Let us check:

So we consider the expansion of *f*(*x*;*ε*):

$$\sqrt{x+x^2}+\frac{\varepsilon}{2\sqrt{x+x^2}} = \sqrt{\varepsilon X+\varepsilon^2 X^2}+\frac{\varepsilon}{2\sqrt{\varepsilon X+\varepsilon^2 X^2}}$$

$$\sim \sqrt{\varepsilon}\left\{\sqrt{X}+\frac{1}{2\sqrt{X}}+\varepsilon\left(\frac{1}{2}X^{3/2}-\frac{1}{4}\sqrt{X}\right)\right\}$$

and then the expansion of *F*(*X*;*ε*):

$$\sqrt{\varepsilon}\left\{\sqrt{1+X}+\frac{\varepsilon X^2}{2\sqrt{1+X}}\right\} = \sqrt{\varepsilon}\left\{\sqrt{1+x/\varepsilon}+\frac{1}{2}\frac{x^2/\varepsilon}{\sqrt{1+x/\varepsilon}}\right\}$$

$$\sim \sqrt{x}\left(1+\frac{1}{2}\frac{\varepsilon}{x}\right)+\frac{1}{2}x^{3/2}\left(1-\frac{1}{2}\frac{\varepsilon}{x}\right).$$

The final expressions are *identical* (when expressed in the same *x*).

Some comments about asymptotic expansions

Our expansions are generated as representations of solutions to differential equations – not of given functions!

Uniqueness of a solution to the underlying problem is assumed, but existence of an asymptotic solution is less clear:

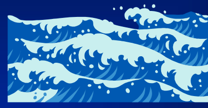
- **Our approach is purely formal; existence within a class of functions allowed within the asymptotic expansion constitutes existence (with the proviso of possible breakdowns).**
- **Existence in the sense of convergence for some $\varepsilon < \varepsilon_0$ is rarely the case; viewed conventionally, they are usually divergent.**

For convergent expansions, the error can be made vanishingly small by increasing the number of terms indefinitely.

For a divergent expansion, for a given *x* and *ε*, there is a choice of the number of terms that will minimise the error (which often gives useful results).

The asymptotic methods typically used in wave-propagation problems give the general structure of the solution and regions of validity, and are usually valid on a finite-time domain (measured by the relevant scaled time), for suitable initial data; see Lecture 3a.

Here is an example for you to investigate, following the procedure just outlined:



$$f(x;\varepsilon) = \frac{1}{1+\varepsilon x + e^{-x/\varepsilon}}$$

for $x \geq 0$ and $\varepsilon > 0$.

Some details are presented in the additional notes.

End of Lecture 2b