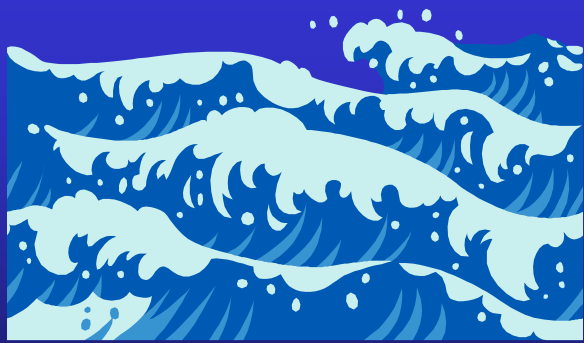
Lecture 2a

The solitary wave and Gerstner's exact solution



The solitary wave

First observed by J. Scott Russell in 1834: a wave which retains its shape over large distances/times.

Consider irrotational, 1D travelling waves, moving at constant speed with fixed shape, in water of finite depth and with rest-conditions at infinity.

Introduce the velocity potential and use the pressure equation to describe the surface dynamic (pressure) boundary condition; this gives (nondimensional)

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\varepsilon\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial \phi}{\partial z}\right)^2\right] + \eta = 0 \quad on \quad z = 1 + \varepsilon\eta.$$

The nondimensional problem is then

$$\frac{\partial^2 \phi}{\partial z^2} + \delta^2 \frac{\partial^2 \phi}{\partial x^2} = 0$$

with

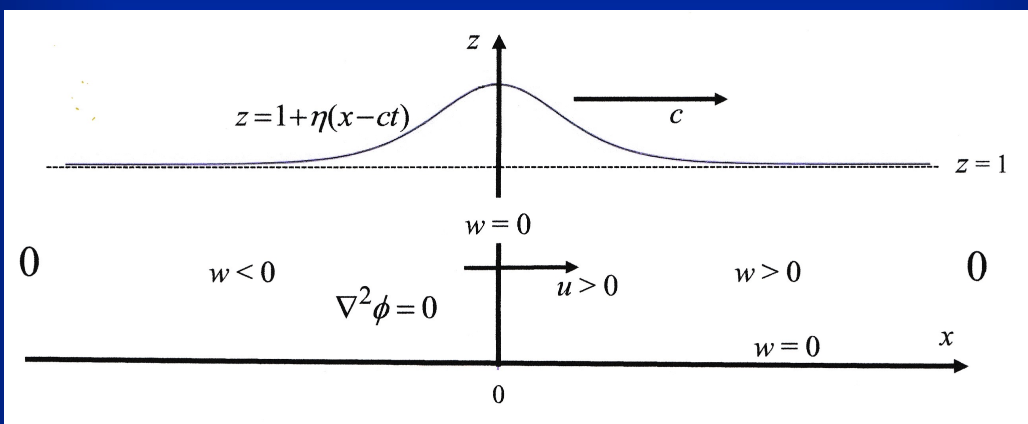
$$\left.\begin{aligned} \frac{\partial \phi}{\partial z} &= \delta^2 \left(\frac{\partial \eta}{\partial t} + \varepsilon \frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x} \right) \\ \frac{\partial \phi}{\partial t} + \frac{1}{2}\varepsilon &\left[\left(\frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{\delta^2} \left(\frac{\partial \phi}{\partial z} \right)^2 \right] + \eta = 0 \end{aligned} \right\} \quad \text{on} \quad z = 1 + \varepsilon \eta$$

and

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0.$$

Because here we consider arbitrary amplitudes, it is convenient to set $\varepsilon = 1$.

The resulting problem, and some properties, are summarised by:



For irrotational flow, the existence of a steady solitary wave (moving into stationary conditions) has been proved, although no closed-form solution can be written down.

There are some intriguing and useful integral relations for the solitary wave. We introduce:

$$M=\int_{-\infty}^{\infty}\eta(\xi)\,\mathrm{d}\xi$$ **total mass associated with the wave** $(\xi=x-ct)$

$$I=\int_{-\infty}^{\infty}\int_{0}^{1+\eta(\xi)}\frac{\partial\phi}{\partial\xi}(z,\xi)\,\mathrm{d}z\,\mathrm{d}\xi$$ **total momentum (or impulse) of the motion**

$$T=\frac{1}{2}\int_{-\infty}^{\infty}\int_{0}^{1+\eta(\xi)}\left[\left(\frac{\partial\phi}{\partial\xi}\right)^2+\frac{1}{\delta^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right](z,\xi)\,\mathrm{d}z\,\mathrm{d}\xi$$ **total kinetic energy of the motion**

$$V=\frac{1}{2}\int_{-\infty}^{\infty}\eta^2(\xi)\,\mathrm{d}\xi$$ **potential energy of the motion**

$$C=\int_{-\infty}^{\infty}\mathbf{u}\cdot\mathrm{d}\mathbf{s}=[\phi]_{-\infty}^{\infty}$$ **circulation, the integral being taken along any streamline**

Then, for example, starting with the equation of mass conservation:

$$\frac{\partial}{\partial\xi}(u-c)+\frac{\partial w}{\partial z}=0$$ **(since we are in the moving frame)**

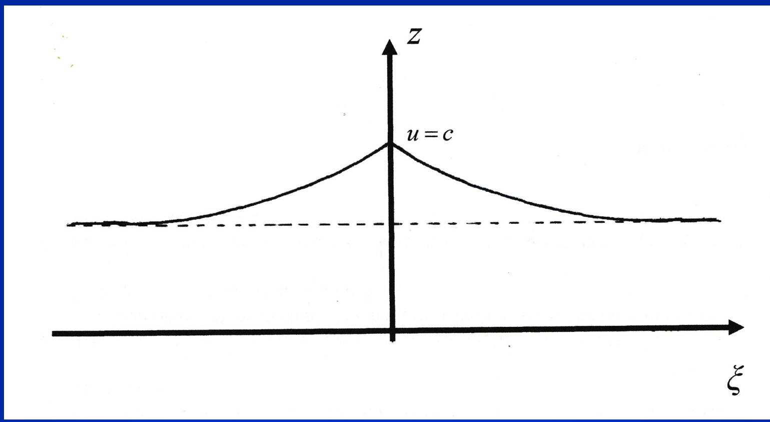
we obtain $$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\int_{0}^{1+\eta}(u-c)\,\mathrm{d}z\right)=0,$$

which gives $$\int_{0}^{1+\eta}(u-c)\,\mathrm{d}z=\text{constant}=\int_{0}^{1}(-c)\,\mathrm{d}z=-c.$$

Then $$\int_{0}^{1+\eta}u\,\mathrm{d}z\left(=\int_{0}^{1+\eta}\frac{\partial\phi}{\partial\xi}\mathrm{d}z\right)=c\eta:\int_{-\infty}^{\infty}\left(\int_{0}^{1+\eta}\frac{\partial\phi}{\partial\xi}\mathrm{d}z\right)\mathrm{d}\xi=c\int_{-\infty}^{\infty}\eta\,\mathrm{d}\xi$$

and so $$I=cM$$ **(Starr, 1947) (and see notes).**

There is a wave of greatest height, which corresponds to $u = c$ at the peak:



This has an included angle of $2\pi/3$ at the peak; away from the peak, $u < c$; $\eta_{\max} \approx 0\cdot 83$.

Solution cannot be written in closed form, but many asymptotic properties can be proved; the detailed shape is obtained by numerical integration.

The small-amplitude, long-wave (necessarily, of course) approximation can be written in closed form:

$$\varepsilon\eta = 2\varepsilon a \operatorname{sech}^2\left[\frac{1}{\delta}\sqrt{\frac{3\varepsilon a}{2}}\left(x - t - \varepsilon a t\right)\right]$$

This is the famous solitary-wave profile that is important in 'soliton' theory; we will outline its derivation and mention its relevance in Lecture 3a.

We see that the wave speed is $1 + \varepsilon a$: larger waves travel faster; also they are narrower (by virtue of the dependence on $\sqrt{\varepsilon a}$).

Gerstner's exact solution (1802)

This is the only known, non-trivial, exact solution for gravity waves.

We consider a 2D, steadily propagating wave in an infinite depth of water; it will turn out that the flow has non-zero vorticity.

The approach uses the Lagrangian description, specified by two parameters: $x' = X'(t'; a, b), \quad z' = Z'(t'; a, b)$

which gives

$$\frac{\partial}{\partial x'} \equiv \frac{1}{J}\left(\frac{\partial Z'}{\partial b}\frac{\partial}{\partial a} - \frac{\partial Z'}{\partial a}\frac{\partial}{\partial b}\right), \frac{\partial}{\partial z'} \equiv \frac{1}{J}\left(\frac{\partial X'}{\partial a}\frac{\partial}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial}{\partial a}\right)$$

where *J* is the Jacobian of the transformation:

$$J = \frac{\partial X'}{\partial a}\frac{\partial Z'}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial Z'}{\partial a} \ (\neq 0).$$

The velocity components are

$$u' = \frac{\mathrm{d}X'}{\mathrm{d}t'}, \quad w' = \frac{\mathrm{d}Z'}{\mathrm{d}t'}$$

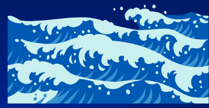
and then mass conservation $\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0$ **becomes**

$$\frac{1}{J}\left(\frac{\partial Z'}{\partial b}\frac{\partial u'}{\partial a} - \frac{\partial Z'}{\partial a}\frac{\partial u'}{\partial b} + \frac{\partial X'}{\partial a}\frac{\partial w'}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial w'}{\partial a}\right)$$

$$= \frac{1}{J}\left(\frac{\partial Z'}{\partial b}\frac{\partial^2 X'}{\partial t'\partial a} - \frac{\partial Z'}{\partial a}\frac{\partial^2 X'}{\partial t'\partial b} + \frac{\partial X'}{\partial a}\frac{\partial^2 Z'}{\partial t'\partial b} - \frac{\partial X'}{\partial b}\frac{\partial^2 Z'}{\partial t'\partial a}\right) = \frac{1}{J}\frac{\partial J}{\partial t'} = 0$$

which gives

$$\frac{\partial}{\partial t'}\left(\frac{\partial X'}{\partial a}\frac{\partial Z'}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial Z'}{\partial a}\right) = 0.$$



The two components of Euler's equation, (x', z'), written in Lagrangian form, are

$$\frac{\mathrm{d}^2 X'}{\mathrm{d}t'^2} = -\frac{1}{\rho' J}\left(\frac{\partial Z'}{\partial b}\frac{\partial p'}{\partial a} - \frac{\partial Z'}{\partial a}\frac{\partial p'}{\partial b}\right); \quad \frac{\mathrm{d}^2 Z'}{\mathrm{d}t'^2} = -\frac{1}{\rho' J}\left(\frac{\partial X'}{\partial a}\frac{\partial p'}{\partial b} - \frac{\partial X'}{\partial b}\frac{\partial p'}{\partial a}\right) - g',$$

which give

$$\frac{\partial p'}{\partial a} = -\rho'\left[\frac{\mathrm{d}^2 X'}{\mathrm{d}t'^2}\frac{\partial X'}{\partial a} + \left(\frac{\mathrm{d}^2 Z'}{\mathrm{d}t'^2} + g'\right)\frac{\partial Z'}{\partial a}\right];$$

$$\frac{\partial p'}{\partial b} = -\rho'\left[\frac{\mathrm{d}^2 X'}{\mathrm{d}t'^2}\frac{\partial X'}{\partial b} + \left(\frac{\mathrm{d}^2 Z'}{\mathrm{d}t'^2} + g'\right)\frac{\partial Z'}{\partial b}\right],$$

and then $\dfrac{\partial^2 p'}{\partial a \partial b} = \dfrac{\partial^2 p'}{\partial b \partial a}$ **requires**

$$\frac{\mathrm{d}^3 X'}{\mathrm{d}t'^2 \partial b}\frac{\partial X'}{\partial a} + \frac{\mathrm{d}^3 Z'}{\mathrm{d}t'^2 \partial b}\frac{\partial Z'}{\partial a} = \frac{\mathrm{d}^3 X'}{\mathrm{d}t'^2 \partial a}\frac{\partial X'}{\partial b} + \frac{\mathrm{d}^3 Z'}{\mathrm{d}t'^2 \partial a}\frac{\partial Z'}{\partial b}.$$

So far, we have a general formulation using the Lagrangian approach; the Gerstner wave is an exact solution of this system. So

the position of every particle, at time *t*, is described by a circular path (with labels *a*, *b*):

$$X'(t'; a, b) = a - \frac{1}{k}\mathrm{e}^{kb}\sin\left(ka - t'\sqrt{g'k}\right);$$

$$Z'(t'; a, b) = b + \frac{1}{k}\mathrm{e}^{kb}\cos\left(ka - t'\sqrt{g'k}\right),$$

where $k > 0$ is a constant wave number, and

$$-\infty < a < \infty, \quad -\infty < b \le b_0 \quad (b_0 \text{ constant}).$$

This maps to a wave in $X' \in (-\infty, \infty)$, $Z' \in (-\infty, h']$, where $Z' = h'$ is the free surface.

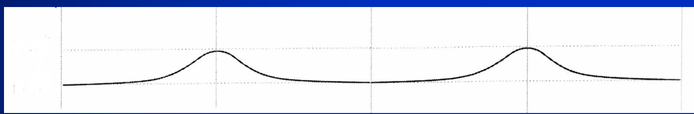
The free surface is given parametrically (parameter *a*) by

$$(X', Z') = \left(a - \frac{1}{k} e^{kb_0} \sin\left(ka - t'\sqrt{g'k}\right), b_0 + \frac{1}{k} e^{kb_0} \cos\left(ka - t'\sqrt{g'k}\right) \right);$$

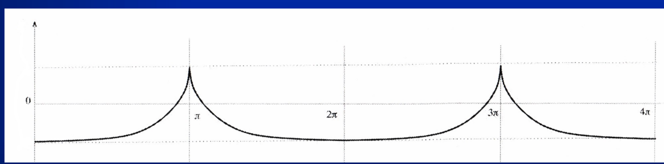
surface kinematic condition is directly satisfied, and then satisfying the surface pressure condition gives

$$p'(t'; a, b) = p'_a - \rho' g'(b - b_0) + \frac{\rho' g'}{2k}\left(e^{2kb} - e^{2kb_0}\right).$$

Typical surface profiles:



the general profile: $b_0 < 0$



the limiting profile: $b_0 = 0$

(a cusped wave)

In summary:

- **Lagrangian formulation**
- **exact solution (but written parametrically)**
- **particle paths are circles**
- **rotational – non-zero vorticity (decaying with depth)**
- **there is a special cusped solution**

End of Lecture 2a