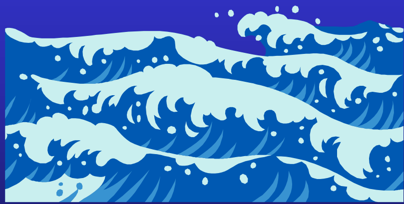
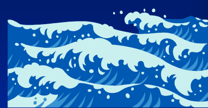
Lecture 1b

Energy equation and integrals of the motion, and two classical problems: one linear and one nonlinear



Energy equations



We start with our general governing equations:

$$\frac{\mathrm{D}\mathbf{u}'}{\mathrm{D}t'} = -\frac{1}{\rho'}\nabla' p' + \mathbf{F}'; \quad \nabla' \cdot \mathbf{u}' = 0$$

with $\rho' = \text{constant}$ **and a conservative body force:** $\mathbf{F}' = -\nabla'\Omega'$

(indeed, $\Omega' = g'z'$ **for our water waves).**

Using a vector identity leads to

$$\frac{\partial \mathbf{u}'}{\partial t'} + \nabla'\left(\frac{1}{2}\mathbf{u}' \cdot \mathbf{u}' + \frac{p'}{\rho'} + \Omega'\right) = \mathbf{u}' \wedge \boldsymbol{\omega}'.$$

We consider two cases, applicable to general flows:

Case 1

Steady (but rotational) flow, then

$$\frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + \Omega' = \text{constant},$$

being constant on lines parallel to $\mathbf{u}'$ (streamlines) and on lines parallel to $\boldsymbol{\omega}'$ (vortex lines); this is *Bernoulli's* equation (or theorem): conservation of energy.

Case 2

Irrotational flow (but unsteady), so $\mathbf{u}' = \nabla'\phi'$ and then

$$\frac{\partial\phi'}{\partial t'} + \frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + \Omega' = f(t'), \quad \textbf{everywhere;}$$

The *pressure* equation (or *unsteady Bernoulli* equation).

Integrated mass conservation (for water waves)

Start from $$\frac{\partial w}{\partial z} + \nabla_\perp\cdot\mathbf{u}_\perp = 0$$

and integrate from $z = b(\mathbf{x}_\perp)$ to $z = h(\mathbf{x}_\perp, t)$, using the kinematic boundary conditions to give

$$\frac{\partial d}{\partial t} + \nabla_\perp\cdot\bar{\mathbf{u}}_\perp = 0,$$

where $d = h - b$ is the local depth and $\bar{\mathbf{u}}_\perp = \int_b^h \mathbf{u}_\perp(\mathbf{x}_\perp, z, t)\,\mathrm{d}z.$

Informative special case is 1D motion: $\mathbf{u}_\perp = \big(u(x,z,t), 0\big)$, with $h = 1 + H(x,t)$ and undisturbed at infinity:

$$\bar{u} \to 0 \text{ and } H \to 0 \text{ as } |x| \to \infty,$$

then $\int_{-\infty}^{\infty} H(x,t)\,\mathrm{d}x = \text{constant}$ **: mass of fluid associated with the wave is conserved.**

Energy integral

From earlier, we start with

$$\frac{\partial \mathbf{u}'}{\partial t'} + \nabla'\left(\frac{1}{2}\mathbf{u}'\cdot\mathbf{u}' + \frac{p'}{\rho'} + \Omega'\right) = \mathbf{u}'\wedge\boldsymbol{\omega}'$$

and take the dot product with $\mathbf{u}'$; introducing ρ' and rewriting gives:

$$\frac{\partial}{\partial t'}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}' + \rho' g' z'\right) + \nabla'\cdot\left\{\mathbf{u}'\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}' + p' + \rho' g' z'\right)\right\} = 0.$$

Integration in z (from bottom to the surface) produces

$$\mathcal{E} + \nabla_{\perp}\cdot\mathcal{F} + \mathcal{P} = 0$$

where $\mathcal{E} = \int_{b'}^{h'}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}' + \rho' g' z\right)\mathrm{d}z$ is the energy/unit area,

$\mathcal{F} = \int_{b'}^{h'}\mathbf{u}'_{\perp}\left(\frac{1}{2}\rho'\mathbf{u}'\cdot\mathbf{u}' + p' + \rho' g' z\right)\mathrm{d}z$ the horizontal energy flux vector

and $\mathcal{P} = P_s'\,\partial h'/\partial t'$ is the nett energy input from the pressure forces doing work at the free surface, but can use $P_s' = \text{constant} = 0.$

Classical linear problem

We start from our general equations:

$$\frac{\mathrm{D}\mathbf{u}_{\perp}}{\mathrm{D}t} = -\nabla_{\perp}p;\ \ \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z};\ \ \nabla\cdot\mathbf{u} = 0$$

with

$$p = \eta\ \&\ w = \frac{\partial\eta}{\partial t} + \varepsilon(\mathbf{u}_{\perp}\cdot\nabla_{\perp})\eta \ \text{ on }\ z = 1 + \varepsilon\eta$$

and

$$w = (\mathbf{u}_{\perp}\cdot\nabla_{\perp})b \ \text{ on }\ z = b$$

(where $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon(\mathbf{u}\cdot\nabla)$.)

Consider arbitrary constant depth and arbitrary wave length, for small amplitude, i.e. $\varepsilon \to 0$, δ fixed, and for 1D waves propagating in the x-direction.

We measure the depth up from $z = 0$, so set $b = 0$; the (physical) depth is subsumed into the scale h_0' .

The classical linear 1D system is given by

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}; \quad \delta^2 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z}; \quad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

with $p = \eta - \delta^2 W_e \dfrac{\partial^2 \eta}{\partial x^2}$ and $w = \dfrac{\partial \eta}{\partial t}$ on $z = 1$

(surface tension included here)

and $w = 0$ on $z = 0.$

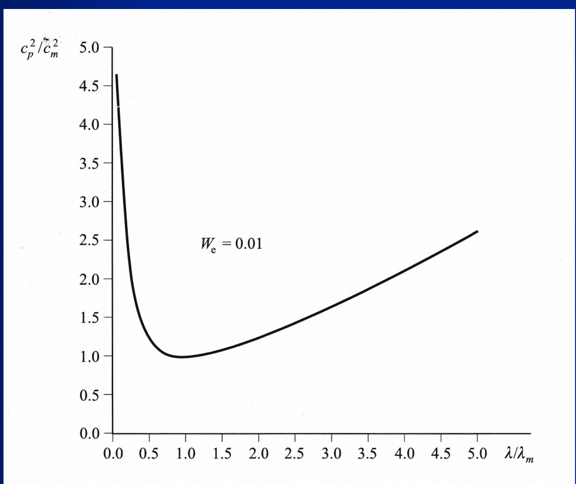
Seek harmonic-wave solution: $\eta = A\mathrm{e}^{\mathrm{i}(kx-\omega t)} (+c.c.)$

and write $(u, w, p) = \big(U(z), W(z), P(z)\big)\mathrm{e}^{\mathrm{i}(kx-\omega t)} (+c.c.)$

to give $\dfrac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \delta^2 k^2 W = 0$ **and so** $W = B\mathrm{e}^{\delta kz} + C\mathrm{e}^{-\delta kz}$ **;**

then $W = -i\omega A\left(\dfrac{\sinh(\delta kz)}{\sinh(\delta k)}\right)$ **and** $\left(\dfrac{\omega}{k}\right)^2 = c_p^2 = \left(1 + \delta^2 k^2 W_e\right)\dfrac{\tanh(\delta k)}{\delta k}.$

Dispersion relation (with surface tension):



where $\lambda = \delta k$ **and** ***m*** **denotes the value at the minimum point. The branch near the origin corresponds to** ***surface-tension*** **waves (*ripples*); the other branch describes** ***gravity*** **waves.**

For gravity waves (no surface tension): $c_p^2 = \dfrac{\tanh(\delta k)}{\delta k}.$

Short waves ($\delta k \to \infty$)**:** $c_p \sim \pm\dfrac{1}{\sqrt{\delta k}}$ $\quad (c_p' \sim \pm\sqrt{g'\lambda'}).$

Long waves ($\delta k \to 0$)**:** $c_p \sim \pm 1$ $\quad (c_p' \sim \pm\sqrt{g' h_0'}).$

Nonlinear, long waves

This problem is defined by $\delta \to 0$, for ε fixed;

this produces the (classical) long-wave problem ; we consider 1D propagation.

Thus we have the system:

$$u_t + \varepsilon(u u_x + w u_z) = -p_x; \quad p_z = 0; \quad u_x + w_z = 0,$$

with $\quad p = \eta \;\&\; w = \eta_t + \varepsilon u \eta_x \quad \text{on} \quad z = 1 + \varepsilon\eta$

and $\qquad w = 0 \quad \text{on} \quad z = 0.$

We have written $\mathbf{u}_\perp = (u, 0)$ with $\eta = \eta(x,t)$ and $b \equiv 0$.

Thus we have

$$p = \eta(x,t) \quad (\text{for } 0 \le z \le 1 + \varepsilon\eta)$$

and we seek the solution for which $u = u(x, t)$;

so the equation of mass conservation, with the boundary conditions included, requires

$$w = \left(\frac{\eta_t + \varepsilon u \eta_x}{1 + \varepsilon\eta} \right) z.$$

Write $h(x,t) = 1 + \varepsilon\eta(x,t)$ then we obtain the pair of coupled, nonlinear equations

$$u_t + u u_x + h_x = 0; \quad h_t + (u h)_x = 0.$$

These two equations can be solved by the *method of characteristics*; first we write them as

$$\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}(u+2c)=0;$$

$$\left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right\}(u-2c)=0 \qquad \left(c(x,t)=\sqrt{h}\right)$$

which show that

$$u+2c = \textbf{constant on lines } C^{+}: \frac{\mathrm{d}x}{\mathrm{d}t}=u+c;$$

$$u-2c = \textbf{constant on lines } C^{-}: \frac{\mathrm{d}x}{\mathrm{d}t}=u-c.$$

These constants are the *Riemann invariants*.

End of Lecture 1b