Introduction to the mathematical description of water waves

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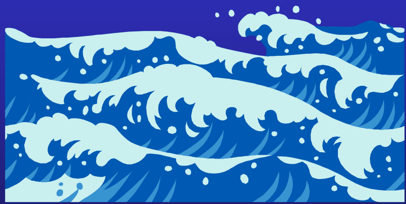


Outline of lectures

1a governing equations, boundary conditions, etc.

1b integrals of the motion, simple wave problems

2a solitary wave and Gerstner's exact solution

2b introduction to asymptotic (parameter) methods

3a Korteweg-de Vries (KdV) eqn, 'soliton' theory

3b KdV eqn for variable depth and with background vorticity

4a asymptotic theory for 'cat's eyes' and critical levels

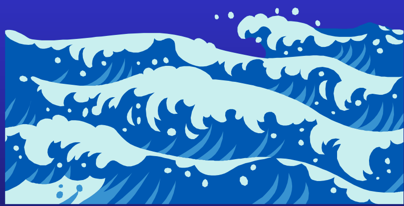
4b edge waves over variable depth

5a periodic waves with vorticity: example of breakdown

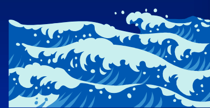
5b oceanic example: the Equatorial Under Current

Lecture 1a

Governing equations, boundary conditions, non-dimensionalisation and scaling



Modelling the fluid



Treat the water as an incompressible, inviscid fluid with zero surface tension (but will include a brief comment on this later).

We invoke the continuum hypothesis.

The water moves over an impermeable, stationary bed, under the action of constant acceleration of gravity (g') and with constant pressure (atmospheric pressure) at the surface.

The fundamental governing equations are therefore Euler's equation and the appropriate equation of mass conservation:

$$\frac{\mathrm{D}\mathbf{u}'}{\mathrm{D}t'} = -\frac{1}{\rho'}\nabla' p' + \mathbf{F}'; \quad \nabla' \cdot \mathbf{u}' = 0$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t'} \equiv \frac{\partial}{\partial t'} + \mathbf{u}' \cdot \nabla' \quad \text{and} \quad \mathbf{F}' = (0, 0, -g') .$$

We write $\mathbf{x}' = (\mathbf{x}'_{\perp}, z')$ and $\mathbf{u}' = (\mathbf{u}'_{\perp}, w')$, with p', ρ' the pressure and density, respectively. ($\perp$ is perp. to z)

The primes here denote dimensional (physical) variables; these will be removed shortly.

By virtue of the continuum hypothesis, we take all functions to be continuous, and suitably differentiable.

Before we proceed with the general formulation, we introduce two important descriptors of flows:

A streamline is an imaginary line in the fluid which everywhere has the velocity vector as its tangent, at any instant in time; let this line be $\mathbf{x}' = \mathbf{X}'(s, t')$

then $\dfrac{\mathrm{d}\mathbf{X}'}{\mathrm{d}s} = \mathbf{u}'(\mathbf{X}', t')$ with $\mathbf{X}' \in C^1$ at fixed t'.

A particle path is the path, $\mathbf{x}' = \mathbf{X}'(t')$, followed by a point (particle) as it moves in the fluid according to the given velocity vector, so $\dfrac{\mathrm{d}\mathbf{X}'}{\mathrm{d}t'} = \mathbf{u}'$

providing a determination of $\mathbf{X}'(t')$, given $\mathbf{u}'(\mathbf{X}', t')$.

Comment: for steady flow, the particle paths and the streamlines coincide.

An important additional observation: The description of a fluid as outlined so far, defining the properties of the fluid at any point, at any time – the most common one in use – is called the *Eulerian* description.
The alternative is to follow a particular point (particle) as it moves in the fluid, and then determine how the properties change on this particle; this is the *Lagrangian* description.

For most of our work, we use the Eulerian description, but there will be one important example that requires the Lagrangian approach.

We now construct the boundary conditions.

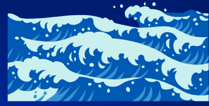
<u>Kinematic condition</u>: Points in a bounding surface of the fluid remain in the surface (in the absence of mixing); let such a surface be $S(\mathbf{x}', t') = 0$ **then, as the fluid moves, particles remain in the surface if** $\dfrac{\mathrm{D}S}{\mathrm{D}t'} = 0$.

So at the free surface, $z' = h'(\mathbf{x}'_\perp, t')$, **we obtain** $w' = \dfrac{\mathrm{D}h'}{\mathrm{D}t'}$.
On the bottom, $z' = b'(\mathbf{x}'_\perp)$, **we have** $w' = (\mathbf{u}'_\perp \cdot \nabla_\perp) b'$.

<u>Dynamic condition</u>: This prescribes the stress at the free surface; so $p' = p'_a = \text{constant}$ **on** $z' = h'(x'_\perp, t')$.

In addition, we require initial data; however, for much that we do here, we assume that suitable data exist for the solutions that we develop.

Summary of the classical water-wave problem:



$$\frac{\mathrm{D}\mathbf{u}'}{\mathrm{D}t'} = -\frac{1}{\rho'}\nabla' p' + \mathbf{F}'; \quad \nabla' \cdot \mathbf{u}' = 0 \quad \textbf{where } \mathbf{F}' = (0, 0, -g')$$

with $w' = \dfrac{\mathrm{D}h'}{\mathrm{D}t'},\ p' = p'_a = \text{constant}$ **on** $z' = h'(\mathbf{x}'_\perp, t')$

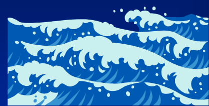
and $w' = (\mathbf{u}'_\perp \cdot \nabla'_\perp) b'$ **on** $z' = b'(\mathbf{x}'_\perp)$. **(+ initial data?)**

Seek a solution in $z' \in [b'(\mathbf{x}'_\perp), h'(\mathbf{x}'_\perp, t')],\ \mathbf{x}'_\perp \in D, t' > 0.$

We proceed on the assumption that, in a suitable domain and with given initial data, we have a well-posed problem.

Observe that a function of particular interest – the surface wave – appears in a boundary condition.

One final property of the flow: Vorticity



This is defined as $\boldsymbol{\omega}' = \nabla' \wedge \mathbf{u}'$ **and measures the local spin of the fluid.**

It is observed that many flows have almost zero vorticity almost everywhere (but there are important exceptions, and water waves propagating in the presence of a background vorticity is obviously one of them).

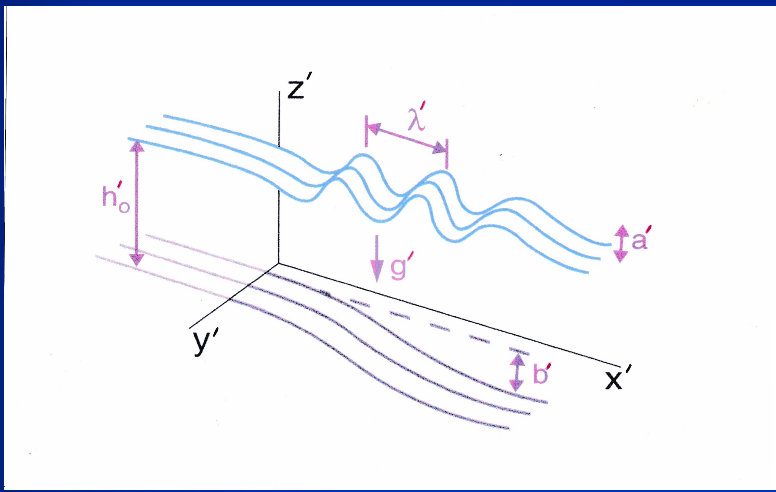
Nevertheless, a good starting point is to assume zero vorticity, then $\boldsymbol{\omega}' \equiv 0$ **and so** $\mathbf{u}' = \nabla' \phi'$**, where** $\phi(\mathbf{x}', t')$ **is the velocity potential; for incompressible flow (as here)**

$$\nabla'^2 \phi' = 0 \text{ – } \textbf{Laplace's equation.}$$

For 2D flow, we also have a stream function, e.g. from

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 \quad \textbf{then } u' = \frac{\partial \psi'}{\partial y'},\ v' = -\frac{\partial \psi'}{\partial x'}.$$

Non-dimensionalisation



Sketch showing the typical scales

Introduce

$$h' = h_0' + a'\eta \quad \text{and} \quad p' = p_a' + \rho' g'(h_0' - z') + \rho' g' h_0' p$$

and the scales

$$\lambda' \text{ (length)}, h_0' \text{ (length)}, \sqrt{g'h_0'} \text{ (speed)}, \lambda'/\sqrt{g'h_0'} \text{ (time)}.$$

Define new (nondimensional) variables:

$$\mathbf{x}_\perp' = \lambda' \mathbf{x}_\perp ; \; z' = h_0' z ; \; t' = \left(\lambda'/\sqrt{g'h_0'}\right) t ;$$

$$\mathbf{u}_\perp' = \sqrt{g'h_0'}\, \mathbf{u}_\perp ; \; w' = \left(h_0' \sqrt{g'h_0'}/\lambda'\right) w ; \; b' = h_0' b,$$

(where the definition of *w* ensures that mass conservation is satisfied, consistent with the existence of a stream function in 2D).

So we get $\dfrac{\mathrm{D}\mathbf{u}_\perp}{\mathrm{D}t} = -\nabla_\perp p ; \quad \delta^2 \dfrac{\mathrm{D}w}{\mathrm{D}t} = -\dfrac{\partial p}{\partial z} ; \quad \nabla \cdot \mathbf{u} = 0,$

with $p = \varepsilon\eta \;\; \& \;\; w = \varepsilon \left\{ \dfrac{\partial \eta}{\partial t} + (\mathbf{u}_\perp \cdot \nabla_\perp)\eta \right\} \quad \text{on} \quad z = 1 + \varepsilon\eta(\mathbf{x}_\perp, t)$

and $w = (\mathbf{u}_\perp \cdot \nabla_\perp) b \quad \text{on} \quad z = b.$

We have introduced the two fundamental parameters used in classical water-wave theory:

$\varepsilon = a'/h_0'$ **: amplitude parameter;**

$\delta = h_0'/\lambda'$ **: long wavelength (or shallowness) parameter.**

Important observation:

The case $\varepsilon = 0$ corresponds to no waves; indeed, disturbances vanish as $\varepsilon \to 0$. Our equations must be consistent with this choice.

Thus, we redefine ('scale') according to

$$(\mathbf{u}_\perp, w, p) \to \varepsilon(\mathbf{u}_\perp, w, p)$$

(where the underlying flow is assumed to be stationary – we will allow an existing background flow later, requiring an adjustment to this scaling property).

Thus we work, initially, with this form of the governing equations:

$$\frac{\mathrm{D}\mathbf{u}_\perp}{\mathrm{D}t} = -\nabla_\perp p; \quad \delta^2 \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z}; \quad \nabla \cdot \mathbf{u} = 0$$

with

$$p = \eta \;\&\; w = \frac{\partial \eta}{\partial t} + \varepsilon(\mathbf{u}_\perp \cdot \nabla_\perp)\eta \quad \text{on} \quad z = 1 + \varepsilon\eta$$

and $$w = (\mathbf{u}_\perp \cdot \nabla_\perp) b \quad \text{on} \quad z = b$$

where $$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + \varepsilon(\mathbf{u} \cdot \nabla).$$

Simplified system: irrotational flow

In this case we introduce the velocity potential, so

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \mathbf{0} \ \text{ gives } \ \mathbf{u} = \nabla \phi$$

and so

$$\phi_{zz} + \delta^2 \nabla_\perp^2 \phi = 0,$$

with

$$\phi_z = \delta^2 \left\{ \eta_t + \varepsilon \left(\mathbf{u}_\perp \cdot \nabla_\perp \right) \eta \right\} \ \&$$

$$\phi_t + \eta + \frac{1}{2} \varepsilon \left\{ (\nabla_\perp \phi)^2 + \frac{1}{\delta^2} \phi_z^2 \right\} = 0 \ \text{ on } \ z = 1 + \varepsilon \eta$$

and

$$\phi_z = \delta^2 (\mathbf{u}_\perp \cdot \nabla_\perp b) \ \text{ on } \ z = b.$$

N.B. We have used subscripts to denote partial derivatives.

Final comment on the governing equations:

Some problems can be reformulated so that they depend on only one parameter, taken to be *ε* for arbitrary *δ*. These are problems where only one length scale is relevant.

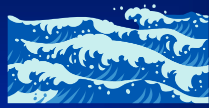
This is possible by carrying out one further transformation:

$$\mathbf{x}_\perp = \frac{\delta}{\sqrt{\varepsilon}} \mathbf{X}_\perp, \ t = \frac{\delta}{\sqrt{\varepsilon}} T, \ w = \frac{\sqrt{\varepsilon}}{\delta} W$$

which produces our earlier non-dim., scaled equations with δ^2 replaced by *ε*, for arbitrary *δ*.

N.B. Then $\delta \to 0$ at fixed *ε* is not accessible.

There are two problems of classical interest, based on two limits:



$\varepsilon \to 0$ (δ fixed): the linearised problem;

$\delta \to 0$ (ε fixed): the long-wave or shallow-water problem.

The first case recovers the most general ***linear*** problem;

the second is ***fully nonlinear***, but the pressure correction (due to the passage of the wave) is missing: there is no dispersion in this case.

We will examine these two classical problems in the second half of the next lecture.

End of Lecture 1a