Boussinesq, non-hydrostatic, incompressible

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\nabla)\underline{u} + \underline{f} \times \underline{u} = -\frac{\nabla p'}{\rho_0} - \frac{g\rho'}{\rho_0}\underline{e}_z$$

$$\nabla . \underline{u} = 0$$

$$\frac{\partial \rho'}{\partial t} + (\underline{u}.\nabla)\rho' = 0$$

ρ' is density anomaly w.r.t. constant value ρ_0

Background state: $\underline{u} = \underline{0}$, $\rho' = \rho_s(z)$

p' s.t. $-\frac{\partial p}{\partial z} - g\rho_s(z) = 0$

Now consider small disturbances about this background state; apply linearization

$$\frac{\partial \underline{\tilde{u}}}{\partial t} + \underline{f} \times \underline{\tilde{u}} = -\frac{\nabla \tilde{p}}{\rho_0} - \frac{g\tilde{\rho}}{\rho_0}\underline{e}_z \quad (1)$$

$$\nabla . \underline{\tilde{u}} = 0 \quad (2)$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{w}\frac{d\rho_s}{dz} = 0 \quad (3)$$

NHSIGW2

Assume plane waves;

all fields $\propto \exp(i(\underline{k}\cdot\underline{x} - \omega t))$ $\underline{k} = (k, l, m)$

For economy of notation replace $\underset{\sim}{\tilde{u}}$ by $\hat{\tilde{u}}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

etc.

Write equations out in components:

$$-i\omega\tilde{u} - f\tilde{v} = -\frac{ik\tilde{p}}{\bar{\rho}_0} \qquad A$$

$$-i\omega\tilde{v} + f\tilde{u} = -\frac{il\tilde{p}}{\bar{\rho}_0} \qquad B$$

$$-i\omega\tilde{w} = -\frac{im\tilde{p}}{\bar{\rho}_0} \cancel{- \frac{\tilde{\rho}}{\rho_0} g} + \tilde{\sigma} \qquad C$$

Convenient to replace $\tilde{\rho}$ by

$$-i\omega\tilde{\rho} + \tilde{w}\frac{d\rho_s}{dz} = 0$$

$$\tilde{\sigma} = -\frac{\tilde{\rho} g}{\rho_0}$$

$$\downarrow$$

$$-i\omega\tilde{\sigma} + \tilde{w}N^2 = 0 \qquad D$$

$$k\tilde{u} + l\tilde{v} + m\tilde{w} = 0 \qquad E$$

Now start the algebra!

$$i\omega \times A - f B : \quad (\omega^2 - f^2)\tilde{u} = (k\omega + ilf)\frac{\tilde{p}}{\rho_0}$$

$$i\omega \times B + f A \quad (\omega^2 - f^2)\tilde{v} = (l\omega - ikf)\frac{\tilde{p}}{\rho_0}$$

Now use E

$$\tilde{w} = -\frac{k}{m}\tilde{u} - \frac{l}{m}\tilde{v} = \left\{-\frac{k}{m}(k\omega + ilf) - \frac{l}{m}(l\omega - ikf)\right\} \times \frac{\tilde{p}}{\rho_0(\omega^2 - f^2)}$$

$$= -\frac{(k^2+l^2)\tilde{p}\omega}{\rho_0(\omega^2 - f^2)m}$$

Now substitute for $\tilde{p}$ in C

$$-i\omega\tilde{w} = \frac{im^2\tilde{w}(\omega^2 - f^2)}{(k^2+l^2)\omega} + \tilde{\sigma}$$

Hence

$$\tilde{w} = \frac{\omega(k^2+l^2)\tilde{\sigma}}{-i\{\omega^2(k^2+l^2+m^2) - f^2m^2\}}$$

From D we have

$$\tilde{w} = \frac{i\omega}{N^2}\tilde{\sigma}$$

For consistency:

NHSIGW4

$$\frac{(k^2+l^2)}{\{\omega^2(k^2+l^2+m^2) - f^2m^2]} = \frac{1}{N^2}$$

Hence: $$\omega^2 = \frac{N^2(k^2+l^2) + f^2m^2}{k^2+l^2+m^2}$$

[In dispersion relation I wrote in lecture I suppressed the y-component of $\underline{k}$, i.e. I took $l=0$]