Wave Turbulence: a theoretical physics perspective

Lecture 4: Dynamical phenomena and toy models

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Outline

The 3-wave kinetic equation

Evolution of wave spectrum, $n_{\mathbf{k}}$, given by:

$$\begin{aligned}
\frac{\partial n_{\mathbf{k}_1}}{\partial t} &= \pi \int V^2_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}(n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}_1}n_{\mathbf{k}_3}) \\
&\qquad \delta(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3})\,\delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)\,d\mathbf{k}_2 d\mathbf{k}_3 \\
&+\pi \int V^2_{\mathbf{k}_2\mathbf{k}_1\mathbf{k}_3}(n_{\mathbf{k}_2}n_{\mathbf{k}_3} + n_{\mathbf{k}_1}n_{\mathbf{k}_2} - n_{\mathbf{k}_1}n_{\mathbf{k}_3}) \\
&\qquad \delta(\omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1})\,\delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_1)\,d\mathbf{k}_2 d\mathbf{k}_3 \\
&+\pi \int V^2_{\mathbf{k}_3\mathbf{k}_1\mathbf{k}_2}(n_{\mathbf{k}_2}n_{\mathbf{k}_3} - n_{\mathbf{k}_1}n_{\mathbf{k}_2} + n_{\mathbf{k}_1}n_{\mathbf{k}_3}) \\
&\qquad \delta(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2})\,\delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2)\,d\mathbf{k}_2 d\mathbf{k}_3
\end{aligned}$$

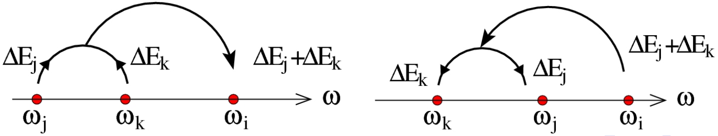
Scaling parameters : $(d,\, \alpha,\, \gamma)$	Dimension, d:	$\mathbf{k} \in \mathbf{R}^d$
	Dispersion, α:	$\omega_{\mathbf{k}} \sim k^{\alpha}$
	Nonlinearity, γ:	$V_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \sim k^{\gamma}$

Isotropic case: frequency space representation

We consider only isotropic systems: all functions of $\mathbf{k}$ depend only on $k = |\mathbf{k}|$.
Dispersion relation, $\omega_{\mathbf{k}} = \omega(k)$, can be used to pass to a frequency-space description where the kinetic equation is:

$$\frac{\partial N_\omega}{\partial t} = S_1[N_\omega] + S_2[N_\omega] + S_3[N_\omega].$$

- $N_\omega = \frac{\Omega_d}{\alpha}\omega^{\frac{d-\alpha}{\alpha}} n_\omega$ is the frequency spectrum.
- The RHS has been split into forward-transfer terms ($S_1[N_\omega]$) and backscatter terms ($S_2[N_\omega]$ and $S_3[N_\omega]$).



Isotropic case: frequency space representation

The forward term, $S_1[N_\omega]$ ($S_2[N_\omega]$ and $S_3[N_\omega]$ look similar) is:

$$
\begin{aligned}
S_1[N_{\omega_1}] = & \int L_1(\omega_2,\omega_3)\, N_{\omega_2}\, N_{\omega_3}\, \delta(\omega_1-\omega_2-\omega_3)\, d\omega_2 d\omega_3 \\
& - \int L_1(\omega_3,\omega_1)\, N_{\omega_3}\, N_{\omega_1}\, \delta(\omega_2-\omega_3-\omega_1)\, d\omega_2 d\omega_3 \\
& - \int L_1(\omega_1,\omega_2)\, N_{\omega_1}\, N_{\omega_2}\, \delta(\omega_3-\omega_1-\omega_2)\, d\omega_2 d\omega_3,
\end{aligned}
$$

Advantage of this notation: only 1 scaling parameter, λ:

$$
L_1(\omega_1,\omega_2) \sim \omega^\lambda, \qquad \lambda = \frac{2\gamma-\alpha}{\alpha}
$$

Disadvantage: $S_1[N_\omega]$, $S_2[N_\omega]$ and $S_3[N_\omega]$ have slightly different $L(\omega_1,\omega_2)$'s.

"Cheat-sheet": 3-wave turbulence on one slide

Spectra:

- Kolmogorov–Zakharov: $N_\omega = c_{\mathrm{KZ}}\sqrt{J}\,\omega^{-\frac{\lambda+3}{2}}$.
- Generalised Phillips (critical balance): $N_\omega = c_P\omega^{-\lambda}$.
- Thermodynamic: $N_\omega \sim \omega^{-2+\frac{d}{\alpha}}$.

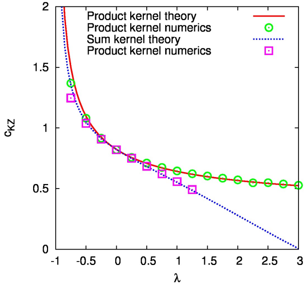
Phase transitions:

- Infinite capacity: $\lambda < 1$.
  Finite capacity: $\lambda > 1$.
- Breakdown at small scales: $\lambda > 3$.
  Breakdown at large scales: $\lambda < 3$.

Locality: if $L_1(\omega_i,\omega_j)$ has asymptotics $K_1(\omega_i,\omega_j) \sim \omega_i^\mu \omega_j^\nu$ with $\mu+\nu=\lambda$ for $\omega_1 \gg \omega_2$, the KZ spectrum is local if

- $\mu < \nu + 3$ and $x_{KZ} > x_T$.

Some model problems ($d = \alpha$):



- Sometimes c_{KZ} can be calculated exactly.
- Product kernel:
$$L(\omega_1, \omega_2) = (\omega_1\, \omega_2)^{\frac{\lambda}{2}}.$$
- Sum kernel:
$$L(\omega_1, \omega_2) = \frac{1}{2}\left(\omega_1^{\lambda} + \omega_2^{\lambda}\right).$$

$$c_{\mathrm{KZ}} = \sqrt{2}\left(\left.\frac{dI}{dx}\right|_{x=\frac{\lambda+3}{2}}\right)^{-1/2}$$

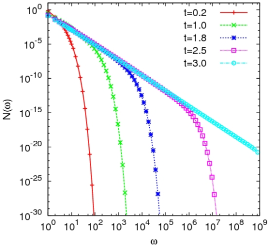
where

$$I(x) = \frac{1}{2}\int_0^1 L(y, 1-y)\,(y(1-y))^{-x}\,(1 - y^x - (1-y)^x)$$
$$(1 - y^{2x-\lambda-2} - (1-y)^{2x-\lambda-2})\,dy.$$

Cascades without backscatter (cluster aggregation)

3WKE → Smoluchowski eqn:

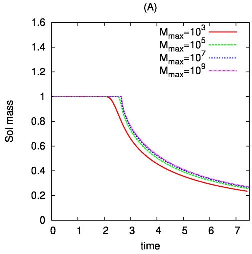
$$
\begin{aligned}
\partial_t N_m &= \int_0^m dm_1 K(m_1, m - m_1) N_{m_1} N_{m-m_1} \\
&- 2N_m \int_0^\infty dm_1 K(m, m_1) N_{m_1} + J\,\delta(m-1)
\end{aligned}
$$



- $N_m(t)$: cluster mass distribution. $K(m_1, m_2)$: kernel.
- $K(am_1, am_2) = a^{\mu+\nu} K(m_1, m_2)$
  $K(m_1, m_2) \sim m_1^{\mu} m_2^{\nu} \quad m_1 \ll m_2$
- Typical size, $s(t)$:

$$
\begin{aligned}
N(m,t) &\sim s(t)^a F(m/s(t)) \\
F(x) &\sim x^{-y} \quad x \ll 1
\end{aligned}
$$

Gelation Transition (Lushnikov [1977], Ziff [1980])



$M(t)$ for $K(m_1, m_2) = (m_1 m_2)^{3/4}$.

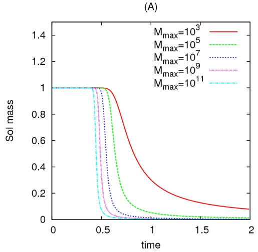
- $M(t) = \int_0^\infty m\, N(m,t)\, dm$ is formally conserved. However for $\mu + \nu > 1$:
$$M(t) < \int_0^\infty m\, N(m,0)\, dm \quad t > t^*$$
- Best studied by introducing cut-off, $M_{\max}$, and studying limit $M_{\max} \to \infty$. (Laurencot [2004])

"Instantaneous" Gelation

- If $\nu > 1$ then $t^* = 0$. (Van Dongen & Ernst [1987])
- May be *complete*: $M(t) = 0$ for $t > 0$. Example : $K(m_1, m_2) = m_1^{1+\epsilon} + m_2^{1+\epsilon}$ for $\epsilon > 0$.
- Mathematically pathological?

Instantaneous Gelation in Regularised System

Yet there *are* applications with $\nu > 1$: differential sedimentation.

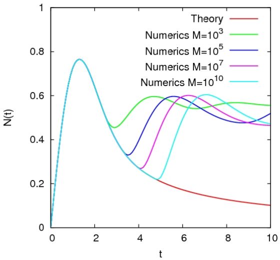


$M(t)$ for $K(m_1, m_2) = (m_1 + m_2)^{4/3}$.

- With cut-off, $M_{\max}$, regularized gelation time, $t^*_{M_{\max}}$, is clearly identifiable.
- $t^*_{M_{\max}}$ *decreases* as $M_{\max}$ increases.
- Van Dongen & Ernst recovered in limit $M_{\max} \to \infty$.

- Decrease of $t^*_{M_{\max}}$ as $M_{\max}$ is very slow (numerics suggest logarithmic decrease). This suggests such models are physically reasonable.
- Consistent with related results of Krapivsky and Ben-Naim and Krapivsky [2003] on exchange-driven growth.

Nonlocal Interactions Drive Instantaneous Gelation



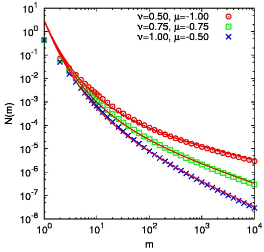
Total density, $\int_0^\infty N(m,t)\,dm$ for $K(m_1, m_2) = m_1 + m_2$ and source.

- Instantaneous gelation is driven by the runaway absorbtion of small clusters by large ones.
- This is clear from the analytically tractable (but non-gelling) marginal kernel, $m_1 + m_2$, with source of monomers.
- $M(t) \sim t$ (due to source) but $N(t) \sim 1/t$.

If the exponent $\nu > 1$ then big clusters are so “hungry" that they eat all smaller clusters at a rate which diverges with the cut-off, $M_{\rm max}$. (cf “addition model" (Brilliantov & Krapivsky [1992], Laurencot [1999])).

Asymptotic solution of the nonlocal case

With cut-off, a stationary state may be reached if a source of monomers is present (Horvai et al [2007]) even though no such state exists in the unregularized system.



Stationary state (theory vs numerics).

- Stationary state has the asymptotic form for $M \gg 1$:

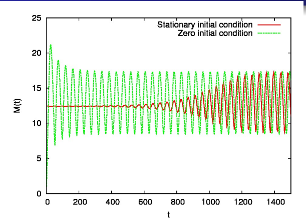
$$N_m = \frac{\sqrt{2J\gamma\,\log\, M}}{M}\, M^{m^{-\gamma}}\, m^{-\nu}.$$

$\gamma = \nu - \mu - 1.$

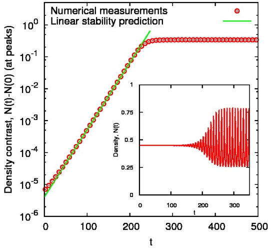
- Stretched exponential for small m, power law for large m.
- Agrees well with numerics without any adjustable parameters.

Note: the stationary state **vanishes** as $M \to \infty$. What happens to the mass flux?

Hopf bifurcation of stationary state as M increased



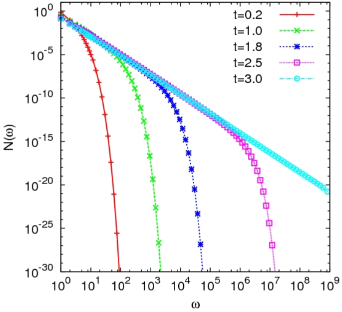
Total density, $N(t)$, vs time for $\nu = \frac{3}{2}$, $\mu = -\frac{3}{2}$.



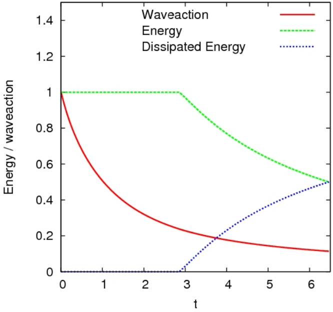
- Semi-analytic linear stability analysis of the exact stationary state shows that the nonlocal stationary state is linearly unstable for large enough M.
- Constant mass flux is replaced by time-periodic pulses.
- Oscillatory behaviour due to an attracting limit cycle embedded in this very high-dimensional dynamical system.
- Recent mathematical work by Pego, Valazquez, Niethammer...

Developing wave turbulence

- Developing wave turbulence refers to the evolution of the spectrum *before* the onset of dissipation.
- We shall focus on the unforced case.



$K(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\frac{3}{2}}$



Energy balance.

Distinguish Finite and Infinite Capacity Cases

Stationary KZ spectrum:

$$N_\omega = c_{\mathrm{KZ}} \sqrt{J}\, \omega^{-\frac{\lambda+3}{2}}.$$

Total energy contained in the spectrum:

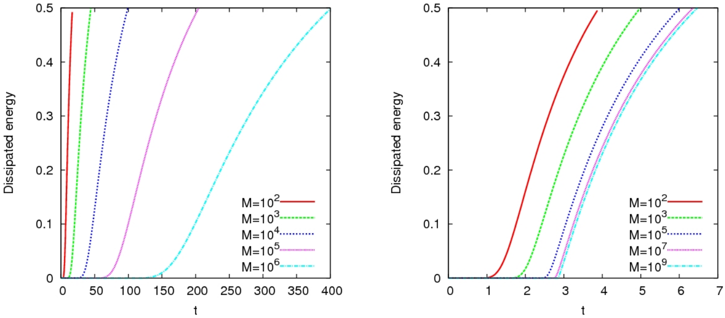
$$E = c_{\mathrm{KZ}} \sqrt{J} \int_1^\Omega d\omega\, \omega^{-\frac{\lambda+1}{2}}.$$

- E diverges as $\Omega \to \infty$ if $\lambda \leq 1$: *Infinite Capacity* .
- E finite as $\Omega \to \infty$ if $\lambda > 1$: *Finite Capacity* .

Transition occurs at $\lambda = 1$.

Dissipative Anomaly

Finite capacity systems exhibit a dissipative anomaly as the dissipation scale $\to \infty$, infinite capacity systems do not:



$\lambda = 3/4$ $\qquad\qquad$ $\lambda = 3/2$

See recent paper by Soffer & Tran (2020).

Transient solutions (Falkovich and Shafarenko, 1991)

Scaling hypothesis: there exists a typical scale, $s(t)$, such that $N_\omega(t)$ is asymptotically of the form

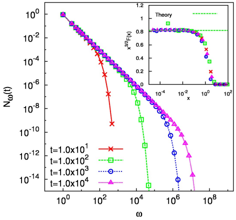
$$N_\omega(t) \sim s(t)^{-a}\, F(\xi) \qquad \xi = \frac{\omega}{s(t)}.$$

Typical frequency, $s(t)$, and the scaling function, $F(\xi)$, satisfy:

$$\begin{aligned} \frac{ds}{dt} &= s^{\lambda - a + 2} \\ -a\,F - \xi \frac{dF}{d\xi} &= S_1[F(\xi)] + S_2[F(\xi)] + S_3[F(\xi)]. \end{aligned}$$

Transient scaling exponent given by the small ξ divergence of the scaling function, $F(\xi) \sim A\,\xi^{-x}$.

Self-similarity: forced case



Self-similar evolution for the forced case with $L(\omega_1, \omega_2) = 1$. Inset shows scaling function $F(\xi)$ compensated by $x^{3/2}$ and predicted amplitude.

Forced case: energy grows linearly:

$$J\,t = \int_0^\infty \omega\, N_\omega\, d\omega = s(t)^{a+2} \int_0^\infty \xi\, F(\xi)\, d\xi$$

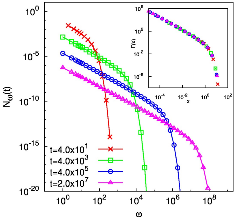
Assuming $F(\xi) \sim A\,\xi^{-x}$, and balancing leading order terms in scaling equation leads to:

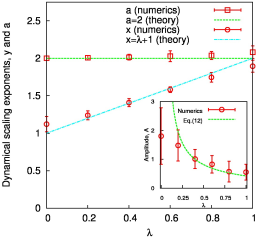
$$x = \frac{\lambda+3}{2}$$

$$A = \sqrt{2}\left(\left.\frac{dI}{dx}\right|_{x=\frac{\lambda+3}{2}}\right)^{-1/2} = c_{KZ}$$

Convergence problem at $\lambda = 1$.

Self-similarity: unforced case





Unforced case: energy conserved:

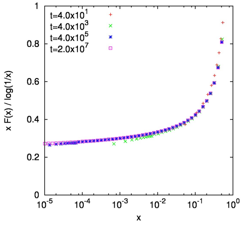
$$1 = \int_0^\infty \omega \, N_\omega \, d\omega = s(t)^{a+2} \int_0^\infty \xi \, F(\xi) \, d\xi$$

Assuming $F(\xi) \sim A\,\xi^{-x}$, and balancing leading order terms in scaling equation leads to:

$$\begin{aligned} x &= \lambda + 1 \\ A &= \frac{\lambda - 1}{I(\lambda+1)}. \end{aligned}$$

Convergence problem at $\lambda = 1$.

Logarithmic corrections to scaling at $\lambda = 0$



Scaling function $F(\xi)$ compensated by $x \ln(1/\xi)$ for the case $L(\omega_1, \omega_2) = 1$.

- For $\lambda = 0$ the *transient* exponent, $x = \lambda + 1$, coincides with the equipartition exponent $x = 1$ and A diverges.
- Correct leading order balance is:

$$F(\xi) \sim \xi^{-1} \ln(1/\xi) \quad \text{as } \xi \to 0,$$

  c.f. 2D optical turbulence (Dyachenko et al, 1992)
- Anomalously slow development of the turbulence. For example, the total wave action is:

$$N(t) \sim \frac{3}{\pi^2} \frac{(\ln t)^2}{t}.$$

Finite capacity case

- Previous argument fails at $\lambda = 1$ for both forced and unforced cases (boundary of finite capacity regime).
- $\int_0^\infty \xi \, F(\xi) \, d\xi$ diverges at 0 so that conservation of energy does not determine dynamical exponent, a.
- For finite capacity systems, $s(t) \to \infty$ as $t \to t^*$. If $F(\xi) \sim \xi^{-x}$ as $\xi \to 0$ then scaling implies that

$$N(\omega, t) \sim s(t)^a \left(\frac{\omega}{s(t)} \right)^{-x} = s(t)^{a+x} \, \omega^{-x}$$

  as $t \to t^*$. Hence the transient exponent, x must be taken equal to $-a$ if the spectrum is to remain finite as $t \to t^*$.
- This is true independent of whether we force or not.

The finite capacity anomaly

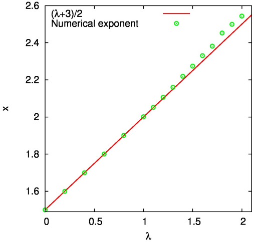
Dynamical exponent, a, must be determined by self consistently solving the scaling equation:

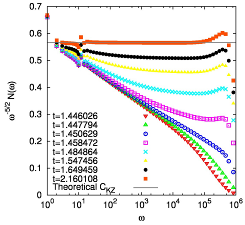
$$-a\,F - \xi \frac{dF}{d\xi} = S_1[F(\xi)] + S_2[F(\xi)] + S_3[F(\xi)].$$

Dimensionally, one might guess $a = -\frac{\lambda+3}{2}$, the KZ value. Overwhelming numerical evidence suggests that this is not so:

- Cluster-cluster aggregation (Lee, 2000).
- MHD turbulence (Galtier et al., 2000).
- Non-equilibrium BEC (Lacaze et al, 2001)
- Leith model (CC & Nazarenko, 2004)
- Generic 3-wave turbulence (CC & Newell, 2010)

Finite capacity anomaly in 3WKE





- For 3WKE with $K(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$, the transient spectrum is consistently (slightly) steeper than the KZ value.
- KZ spectrum is set up from right to left.
- Finite capacity in itself is not sufficient for the anomaly (eg Smoluchowski eqn with product kernel).
- Anisotropy is not necessary but it might help! (c.f. Galtier et al. 2005).

Spectral truncation of the 3WKE

- Introduce a maximum frequency $\omega = \Omega$: modes having $\omega > \Omega$ have $N_\omega = 0$.
- In sum over triads we only include $\omega_j \leq \omega_i < \omega_k \leq \Omega$.
- However we must *choose* what to do with triads having $\omega_j \leq \omega_i < \Omega < \omega_k$ (only relevant for $S_1[N_\omega]$).
    - Include them: open truncation (dissipative)
    - Remove them: closed truncation (conservative)
    - Damp them by a factor $0 < \nu < 1$: partially open truncation (dissipative)
- "Boundary conditions" on the energy flux are not local for integral collision operators.

In case someone asks for formulae...

$$\frac{dN_i}{dt} = \quad S_i^{(1)}[\mathbf{N},\Omega] + S_i^{(2)}[\mathbf{N},\Omega] + S_i^{(3)}[\mathbf{N},\Omega] - \gamma\, T_i[\mathbf{N},\Omega]$$

$$i = 1, 2, \dots \Omega$$

with the truncated collision integrals given by

$$S_i^{(1)}[\mathbf{N},\Omega] = \sum_{j=1}^{i-1} K_1(j, i-j)\, N_j N_{i-j} - \sum_{j=i+1}^{\Omega} K_1(j-i, i)\, N_i N_{j-i} - \sum_{j=1}^{\Omega-i} K_1(i,j)\, N_i N_j,$$

In case someone asks for formulae...

$$S_i^{(2)}[\mathbf{N}, \Omega] = -\sum_{j=1}^{i-1} K_2(j, i-j)\, N_i N_j$$
$$+ \sum_{j=i+1}^{\Omega} K_2(j-i, i)\, N_j N_{j-i}$$
$$+ \sum_{j=1}^{\Omega-i} K_2(i, j)\, N_i N_{i+j}$$

In case someone asks for formulae...

and

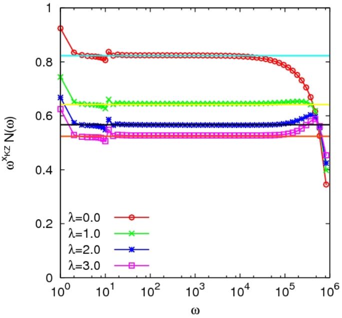
$$
\begin{aligned}
S_i^{(3)}[\mathbf{N}, \Omega] = & -\sum_{j=1}^{i-1} K_3(j, i-j)\, N_j N_{i-j} \\
& + \sum_{j=i+1}^{\Omega} K_3(j-i, i)\, N_j N_i \\
& + \sum_{j=1}^{\Omega-i} K_3(i, j)\, N_j N_{i+j}
\end{aligned}
$$

Dissipative "boundary condition"

The modified collision integrals $S_i^{(1)}$, $S_i^{(2)}$ and $S_i^{(3)}$ conserve energy. The dissipative "boundary" terms that transfer energy across the cutoff are

$$T_i[\mathbf{N}, \Omega] = \gamma \left(\sum_{j=\Omega+1}^{\Omega+i} K_1(j-i, i)\, N_i N_{j-i} + \sum_{j=\Omega-i+1}^{\Omega} K_1(i, j)\, N_i N_j \right),$$

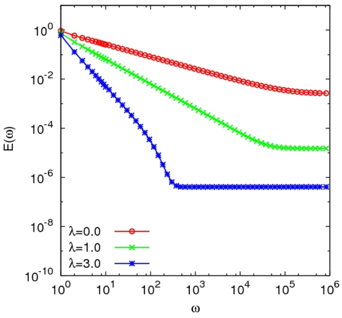
Open Truncation : Bottleneck Phenomenon



Compensated stationary spectra with open truncation.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Open truncation can produce a bottleneck as the solution approaches the dissipative cut-off (Falkovich 1994).
- Bottleneck does not occur for all $L(\omega_1, \omega_2)$. Why?
- Energy flux at Ω is 1.

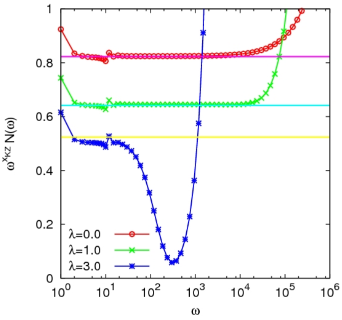
Closed Truncation: Thermalisation



Quasi-stationary spectra with closed truncation.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).
- Thermalisation occurs for all $L(\omega_1, \omega_2)$.
- Energy flux at Ω is 0.

Closed Truncation: Thermalisation



Quasi-stationary spectra with closed truncation compensated by KZ scaling.

- Product kernel: $L(\omega_1, \omega_2) = (\omega_1 \omega_2)^{\lambda/2}$.
- Closed truncation produces thermalisation near the cut-off (CC and Nazarenko (2004), Cichowlas et al (2005)).
- Thermalisation occurs for all $L(\omega_1, \omega_2)$.
- Energy flux at Ω is 0.