Wave Turbulence: a theoretical physics perspective

Lecture 3: Case study - the Charney-Hasegawa-Mima model

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Outline

Systems with 2 conserved quantities

Example: 2D NS. Let stream function be $\psi(x, y, t)$:

$$\frac{\partial \nabla^2 \psi}{\partial t} + (\mathbf{v} \cdot \nabla)\nabla^2 \psi = \nu \nabla^4 \psi,$$

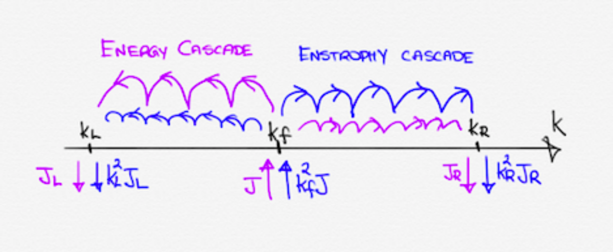
where $v_x = \partial_y \psi$ and $v_y = -\partial_x \psi$.
There are 2 formally conserved quantities if $\nu \to 0$:

- Energy: $E = \int d\mathbf{x} \, |\mathbf{v}|^2 = \int E(k) dk$.
- Enstrophy: $\Omega = \int d\mathbf{x} \, |\nabla \times \mathbf{v}|^2 = \int k^2 E(k) dk$.

Other examples:

- Nonlinear Schrodinger equation : energy and mass.
- Potential vorticity equation: energy and potential enstrophy.
- Surface water waves: energy and wave action.

The dual cascade: direct and inverse cascades



Use $\Omega(k) = k^2 E(k)$ to write global balance conditions for fluxes:

Energy: $J = J_R + J_L$ Enstrophy: $k_f^2 J = k_R^2 J_R + k_L^2 J_L$.

Solve for energy fluxes:

$$J_L = \frac{k_R^2 - k_f^2}{k_R^2 - k_L^2} J, \qquad J_R = \frac{k_f^2 - k_L^2}{k_R^2 - k_L^2} J.$$

The dual cascade: direct and inverse cascades

Energy fluxes:

$$J_L = \frac{k_R^2 - k_f^2}{k_R^2 - k_L^2} J, \qquad J_R = \frac{k_f^2 - k_L^2}{k_R^2 - k_L^2} J.$$

Enstrophy fluxes:

$$J_L^\Omega = k_L^2 \frac{k_R^2 - k_f^2}{k_R^2 - k_L^2} J, \qquad J_R^\Omega = k_R^2 \frac{k_f^2 - k_L^2}{k_R^2 - k_L^2} J.$$

Note:

- $J_L \to J$ and $J_R \to 0$ as $k_R \to \infty$.
- $J_L^\Omega \to 0$ and $J_R^\Omega \to k_f^2 J$ as $k_L \to 0$.

These scalings make the "pure" dual cascade very difficult to realise.

The dual cascade: Fjortoft argument

Characteristic scales (centroids) of the spectral energy and enstrophy densities:

$$k_E = \frac{\int_0^\infty kE(k)dk}{E}, \qquad k_\Omega = \frac{\int_0^\infty k^3 E(k)dk}{\Omega},$$

where E and Ω are the total energy and enstrophy.

$$\begin{aligned}\left(\int_0^\infty kE(k)dk\right)^2 &= \left(\int_0^\infty (k\sqrt{E(k)}dk)(\sqrt{E(k)}dk)\right)^2 \\ &\leq \left(\int_0^\infty k^2 E(k)dk\right)\left(\int_0^\infty E(k)dk\right) \\ &= \Omega E.\end{aligned}$$

$$\Rightarrow k_E \leq \sqrt{\frac{\Omega}{E}}. \qquad (1)$$

The dual cascade: Fjortoft argument

Similarly

$$\begin{aligned}\Omega^2 = \left(\int_0^\infty k^2 E(k)dk\right)^2 &= \left(\int_0^\infty (k^{\frac{3}{2}}\sqrt{E(k)}dk)(\sqrt{kE(k)}dk)\right)^2 \\ &\leq \left(\int_0^\infty k^3 E(k)dk\right)\left(\int_0^\infty kE(k)dk\right) \\ &= (\Omega k_\Omega)(E k_E).\end{aligned}$$

$$\Rightarrow k_\Omega k_E \geq \frac{\Omega}{E}. \tag{2}$$

Eq. (1) & Eq.(2):

$$k_\Omega \geq \sqrt{\frac{\Omega}{E}}. \tag{3}$$

The dual cascade: Fjortoft argument

Summary:

$$k_E \leq \sqrt{\frac{\Omega}{E}}$$

$$k_\Omega \geq \sqrt{\frac{\Omega}{E}}$$

$$k_\Omega k_E \geq \frac{\Omega}{E}.$$

From these we conclude that

- $k_E \leq k_\Omega$.
- if k_E decreases then k_Ω must increase.

Converse can also be shown - see Nazarenko's book (2011).

Reminder: Barotropic Potential Vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi - F\psi) + \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0\,.$$

Velocity ("geostrophic wind"):

$$\mathbf{v} = \nabla\times\psi(x,y,t)\,\hat{\mathbf{z}}.$$

Two conservation laws:

$$E = \frac{1}{2}\int d\mathbf{x}\left[|\nabla\psi|^2 + F\psi^2\right],$$
$$Q = \frac{1}{2}\int d\mathbf{x}\left[\nabla^2\psi - F\psi\right]^2.$$

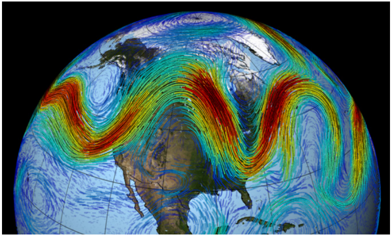


Image: NASA/GSFC.

Dual cascade: inverse cascade of energy, direct cascade of enstrophy.

Weak and strong turbulence regimes

Nonlinearity measured by dimensionless amplitude:

$$M = \frac{\Psi_0 k^3}{\beta}$$

where k typical scale, Ψ_0 is typical amplitude. Two limits:

- $M \gg 1$: "Euler" limit.
- $M \ll 1$: Wave turbulence limit.

Wave turbulence limit should be analytically tractable:

- Evolution of turbulence spectrum described by kinetic equation.
- Kinetic equation has stationary solutions describing cascades.

Structure of resonant curves

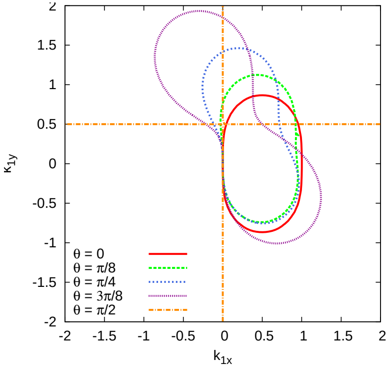
Rossby wave dispersion relation:

$$\omega_{\mathbf{k}} = -\frac{\beta k_x}{k^2 + F}$$

Resonance conditions:

$$\omega_{\mathbf{k}} - \omega_{\mathbf{k_1}} - \omega_{\mathbf{k}-\mathbf{k_1}} = 0.$$

If we fix $\mathbf{k} = (\cos\theta, \sin\theta)$ have a curve in (k_{1x}, k_{1y}) plane:



Shape of resonant curves.

$$\frac{k_{1x}}{k_{1x}^2 + k_{1y}^2 + F} + \frac{k_x - k_{1x}}{(k_x - k_{1x})^2 + (k_y - k_{1y})^2 + F} - \frac{k_x}{k_x^2 + k_y^2 + F} = 0.$$

Structure of quasiresonant sets

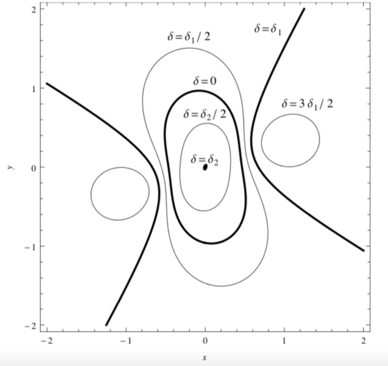
Quasiresonant set is the set of $\mathbf{k}_1$ that satisfy the resonance condition approximately:

$$|\omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k} - \mathbf{k}_1)| \leq \delta.$$

Detuning parameter, δ, due to noise or nonlinear resonance broadening.

Boundaries of quasiresonant set:

$$\begin{aligned} \omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k} - \mathbf{k}_1) &= \delta\,, \\ \omega(\mathbf{k}) - \omega(\mathbf{k}_1) - \omega(\mathbf{k} - \mathbf{k}_1) &= -\delta. \end{aligned}$$



$\mathbf{k} = (\cos\theta, \sin\theta)$ with $\theta = \pi/6$, $\beta = 1$ and $F = 1/5$.

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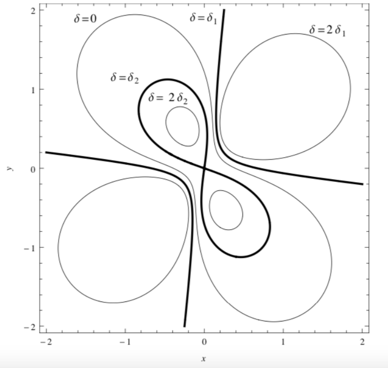
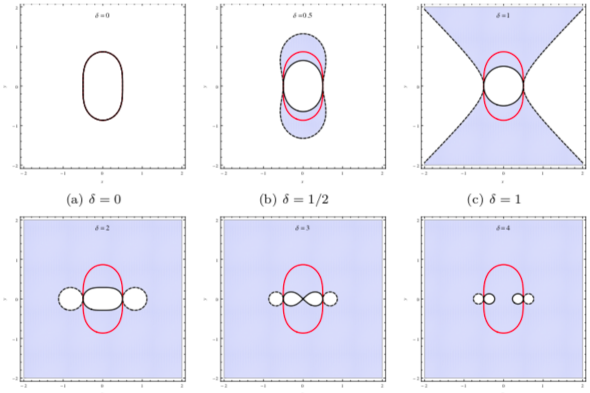
Self-intersection is possible.



$\mathbf{k} = (\cos\theta, \sin\theta)$ with $\theta = 3\pi/7$, $\beta = 1$ and $F = 1/5$.

Quasi resonant set for a meridional wavevector: $\mathbf{k} = (1, 0)$

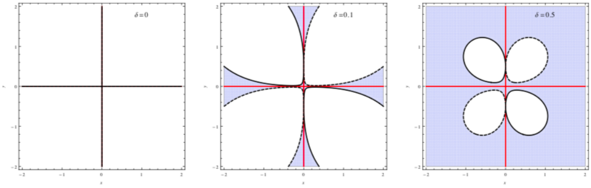


(a) $\delta = 0$ (b) $\delta = 1/2$ (c) $\delta = 1$

(d) $\delta = 2$ (e) $\delta = 3$ (f) $\delta = 4$

$\mathbf{k} = (1, 0)$ with $\beta = 1$ and $F = 0$.

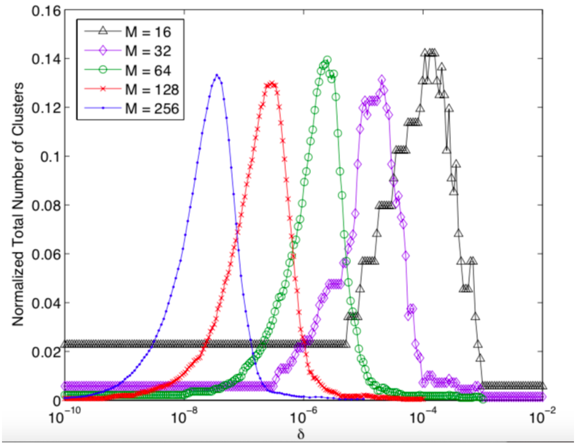
Quasi resonant set for a zonal wavevector: $\mathbf{k} = (0, 1)$



(a) $\delta = 0$ (b) $\delta = 1/10$ (c) $\delta = 1/2$

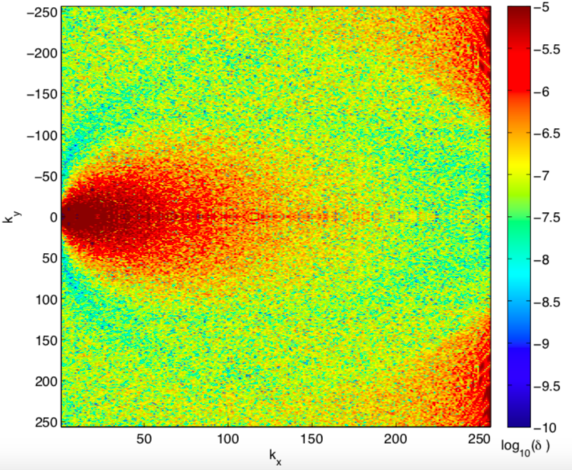
$\mathbf{k} = (0, 1)$ with $\beta = 1$ and $F = 0$.

Clusters of quasiresonant triads



Clusters of (quasi) resonances facilitate long range spectral transport.

Anisotropy of quasiresonant clustering



Minimum detuning required for modes to join a cluster.

Rossby wave kinetic equation

Another form of the Rossby wave kinetic equation:

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 4\pi \int \left| V^{\mathbf{k}}_{\mathbf{q}\,\mathbf{r}} \right|^2 \delta(\mathbf{k}-\mathbf{q}-\mathbf{r})\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{q}}-\omega_{\mathbf{r}})\times$$
$$[n_{\mathbf{q}} n_{\mathbf{r}} - n_{\mathbf{k}} n_{\mathbf{q}} \,\mathrm{sgn}(\omega_{\mathbf{k}}\omega_{\mathbf{r}}) - n_{\mathbf{k}} n_{\mathbf{r}} \,\mathrm{sgn}(\omega_{\mathbf{k}}\omega_{\mathbf{q}})]\, d\mathbf{q} d\mathbf{r} + \gamma_{\mathbf{k}}\, n_{\mathbf{k}}.$$

where

$$\omega_{\mathbf{k}} = -\frac{\beta\, k_x}{k^2+F}$$

$$V^{\mathbf{k}}_{\mathbf{q}\,\mathbf{r}} = \frac{i}{2}\sqrt{\beta\,|k_x q_x r_x|}\left(\frac{q_y}{q^2+F}+\frac{r_y}{r^2+F}-\frac{k_y}{k^2+F}\right).$$

Invariants of the kinetic equation

The kinetic equation has two conserved quantities inherited from the BPV equation:

$$E = \int_{k_x>0} |\omega_{\mathbf{k}}| n_{\mathbf{k}} \, d\mathbf{k},$$
$$Q = \int k_x \, n_{\mathbf{k}} \, d\mathbf{k}\,.$$

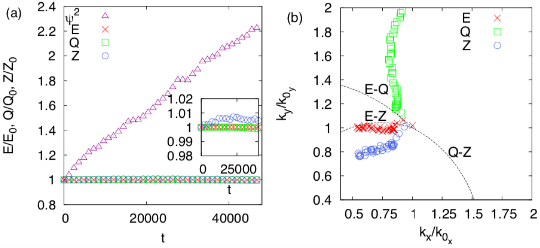
It has is an additional invariant ("zonostrophy") which is not conserved by BPV:

$$Z = \int_{k_x>0} \eta_{\mathbf{k}} n_{\mathbf{k}} \, d\mathbf{k}\,,$$

where

$$\eta_{\mathbf{k}} = \arctan \frac{k_y + k_x\sqrt{3}}{\rho k^2} - \arctan \frac{k_y - k_x\sqrt{3}}{\rho k^2}\,. \qquad (4)$$

Triple cascade phenomenon



Fjortoft argument suggests direction of flow of centroids of E, Q and Z [Nazarenko & Quinn (2009)]

Anisotropic scale invariance

We can generalise notion of scale invariance to include anisotropic systems by assuming bi-homogeneity:

$$\omega_{(h_x k_x, h_y k_y)} = h_x^a\, h_y^b\, \omega_{(k_x, k_y)}$$

$$V^{(h_x p_x, h_y p_y)}_{(h_x q_x, h_y q_y)\,(h_x r_x, h_y r_y)} = h_x^u\, h_y^v\, V^{(p_x, p_y)}_{(q_x, q_y)\,(r_x, r_y)}.$$

The Rossby wave frequency and interaction coefficient do not have this property in general but there are two limiting cases:

- Long wave limit: $F \to \infty$.
- Short wave limit: $F \to 0$.

Long wave and short wave limits

In the long wave limit, $F \gg k$:

$$\omega_{\mathbf{k}} = -\frac{k_x}{F}\left[1 - \frac{k_x^2 + k_y^2}{F}\right] \approx -\frac{1}{F}\left(k_x - \frac{1}{F}k_x k_y^2\right)$$

$$V_{\mathbf{qr}}^{\mathbf{p}} \sim F^{-2}\sqrt{|p_x q_x r_x|}\,\left(q_y^3 + r_y^3 - p_y^3\right).$$

Neglect leading k_x in $\omega_{\mathbf{k}}$ on resonant manifold.
Bihomogeneity exponents:

$$(a, b) = (1, 2) \qquad (u, v) = (\frac{3}{2}, 3).$$

Long wave and short wave limits

In the short wave limit, $F \ll k$ and we also assume that $k_x \ll k_y$ (zonal):

$$\omega_{\mathbf{k}} \sim -k_x\, k_y^{-2}$$
$$V_{\mathbf{qr}}^{\mathbf{p}} \sim \sqrt{|p_x q_x r_x|}\, \left(q_y^{-1} + r_y^{-1} - p_y^{-1}\right).$$

Bihomogeneity exponents:

$$(a, b) = (1, -2) \qquad (u, v) = (\frac{3}{2}, -1).$$

Anisotropic KZ spectra

The machinery developed for homogeneous functions works identically for bihomogeneous functions by assuming

$$n_{\mathbf{k}} = c\, k_x^x \, |k_y|^y$$

Results:

$$n_{\mathbf{k}} = c\, k_x^{-\frac{5}{2}} \, |k_y|^0 \quad \text{– Energy cascade (short wave limit),}$$

$$n_{\mathbf{k}} = c\, k_x^{-\frac{5}{2}} \, |k_y|^{-1} \quad \text{– Enstrophy cascade (short wave limit).}$$

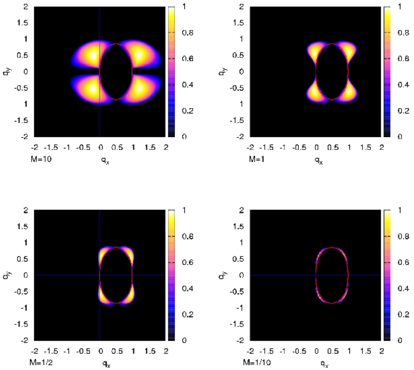
$$n_{\mathbf{k}} = c\, k_x^{-\frac{5}{2}} \, |k_y|^{-4} \quad \text{– Energy cascade (long wave limit),}$$

$$n_{\mathbf{k}} = c\, k_x^{-\frac{5}{2}} \, |k_y|^{-3} \quad \text{– Enstrophy cascade (long wave limit).}$$

However: all spectra turn out to be nonlocal.

Modulational instability of Rossby waves

Small scale Rossby waves are unstable to large scale modulations (Lorenz 1972, Gill 1973)



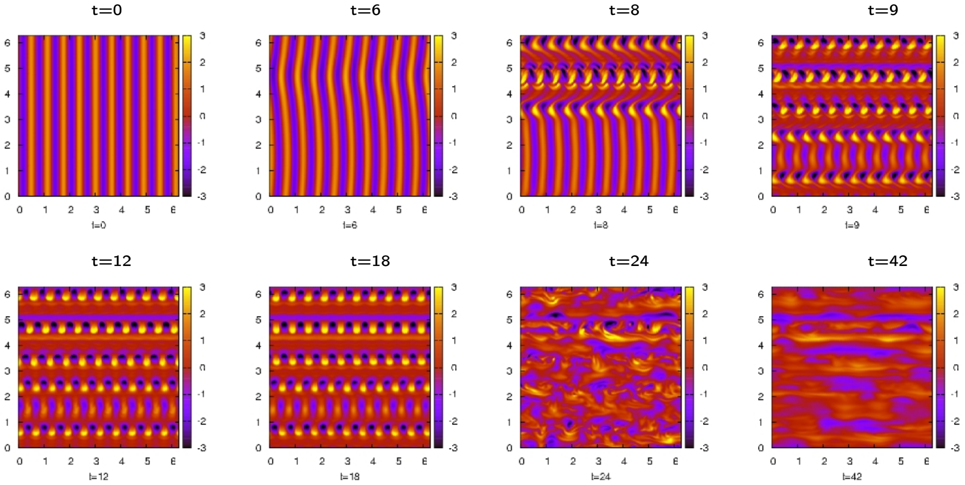
$\mathbf{p} = (1, 0)$.

- In wave turbulence limit, $M \ll 1$, unstable perturbations concentrate on resonant manifold:

$$\begin{aligned} \mathbf{p} &= \mathbf{q} + \mathbf{r} \\ \omega(\mathbf{p}) &= \omega(\mathbf{q}) + \omega(\mathbf{p}_{-}). \end{aligned}$$

- Perturbations with fastest growth rate become close to zonal for $M \ll 1$.

Generation of jets by modulational instability



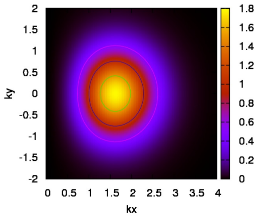
Charney-Hasegawa-Mima equation driven by small scale "instability"

$$(\partial_t - \mathcal{L})\,(\Delta\psi - F\psi) + \beta\,\partial_x\psi + J[\psi, \Delta\psi] = 0.$$

Operator $\mathcal{L}$ has Fourier-space representation

$$\mathcal{L}_{\mathbf{k}} = \gamma_{\mathbf{k}} - \nu k^{2m}$$

which mimics a small scale instability (and hyperviscosity).



- $\gamma_{\mathbf{k}}$ peaked around $(k_f, 0)$.
- Choose $k_f^2 \sim F$.
- Forcing excites meridionally propagating Rossby waves with wavelength comparable to deformation radius.

Large scale – small scale feedback loop

Initialise vorticity field with white noise of very low amplitude. What are the dynamics? The following scenario was proposed by Zakharov and coworkers (1990's):

- Waves having $\gamma_{\mathbf{k}} > 0$ initially grow exponentially.
- As amplitudes grow, nonlinearity initiates cascades.
- Inverse cascade transfers energy to large scales leading to zonal jets.
- Jets shear small scale waves generating negative feedback. "Switches off" the forcing.

Is this what really happens?

Distortion of small scales by large scales in weakly nonlinear regime

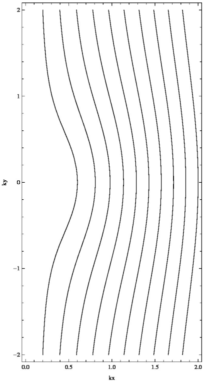
Modulational instability generates large scales directly from small scale waves (non-locality).
Subsequent evolution of the small scales can be described by a nonlocal turbulence theory:

- Assume major main contribution to collision integral for small scales comes from modes **q** having $q \ll k$ (scale separation).
- Taylor expand in **q**. Leading order equation for small scales is an anisotropic diffusion equation in **k**-space:

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\partial}{\partial k_i} S_{ij}(k_1, k_2) \frac{\partial n_{\mathbf{k}}}{\partial k_j}.$$

- Diffusion tensor, S_{ij}, depends on structure of the large scales: more intense zonal flows give faster diffusion.

More on the spectral diffusion approximation

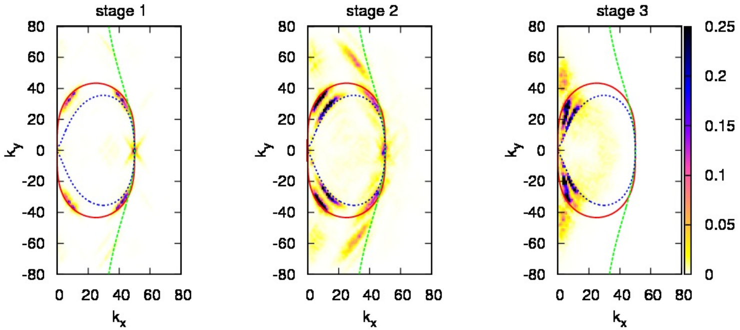


- Diffusion tensor, S_{ij} turns out to be *degenerate*
- A change of variables shows that diffusion is along 1-d curves:

$$\Omega = \frac{\beta\, k^2\, k_x}{k^2 + F} = const$$

- *Wave-action*, $n_{\mathbf{k}}$, of a small scale wavepackets is conserved by this motion.
- Energy, $\omega_{\mathbf{k}}\, n_{\mathbf{k}}$, of small scale wavepackets is not conserved.

Numerical view of spectral transport



Wave-action diffuses along the Ω curves from the forcing region until it is dissipated at large wave-numbers.

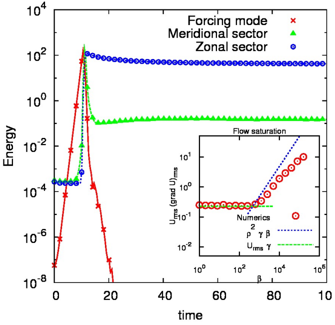
Mechanism for saturation of the large scales

- Energy lost by the small scales is transferred to the large scale zonal flows which grow more intense.
- More intense large scales results in a larger diffusion coefficient.
- Increases the rate of dissipation of the small scale wave-action $\rightarrow$ negative feedback.

Prediction:

Growth of the large scales should suppress the small scale instability which turns off the energy source for the large scales leading to a saturation of the large scales *without large scale dissipation*.

Numerical observations of feedback loop



- Feedback loop is very evident in weakly nonlinear regime.
- Scaling arguments allow one to predict the saturation level of the large scales in terms of the small scale grow rate:

$$[U\,(\nabla U)]_{\mathrm{LS}} \sim \gamma\,\beta \qquad M \ll 1$$

$$[U\,(\nabla U)]_{\mathrm{LS}} \sim \gamma\,U \qquad M \gg 1$$

These estimates are reasonably supported by numerics - especially the cross-over from the weak to strong turbulence regimes (see inset).