

Wave Turbulence: a theoretical physics perspective

Lecture 2: the wave kinetic equation

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Outline

- 1 Equations for correlation functions
- 2 Wave kinetic equation via multiple scale analysis
- 3 Stationary solutions of the wave kinetic equation
- 4 Numerical and experimental evidence

Statistical homogeneity

Statistical homogeneity means that moments of the wave field depend only on relative geometry:

$$\begin{aligned}\langle a(\mathbf{x}) a(\mathbf{x} + \mathbf{r}) \rangle &= M_2^{++}(\mathbf{r}) \\ \langle a(\mathbf{x}) a^*(\mathbf{x} + \mathbf{r}_1) a^*(\mathbf{x} + \mathbf{r}_2) \rangle &= M_3^{+--}(\mathbf{r}_1, \mathbf{r}_2),\end{aligned}$$

Writing the a 's via their Fourier transforms illustrates that Fourier space moments are proportional to delta functions:

$$\begin{aligned}\langle a(\mathbf{k}) a(\mathbf{k}') \rangle &= \hat{M}_2^{++}(\mathbf{k}, \mathbf{k}') \delta(\mathbf{k} + \mathbf{k}') \\ \langle a(\mathbf{k}) a^*(\mathbf{k}') a^*(\mathbf{k}'') \rangle &= \hat{M}_3^{+--}(\mathbf{k}, \mathbf{k}', \mathbf{k}'') \delta(\mathbf{k} - \mathbf{k}' - \mathbf{k}'') \quad \text{etc.}\end{aligned}$$

The wave spectrum is obtained from the second moment:

$$\langle a^*(\mathbf{k}) a(\mathbf{k}') \rangle = n(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}').$$

Moments and cumulants

Cumulants are an alternative to the moments and can be put in one-to-one correspondence with moments. For $\langle a_{\mathbf{k}} \rangle = 0$:

$$\langle a^{s_1}(\mathbf{k}_1) a^{s_2}(\mathbf{k}_2) \rangle = Q_2^{s_1 s_2}(\mathbf{k}_1) \delta(s_1 \mathbf{k}_1 + s_2 \mathbf{k}_2),$$

$$\langle a^{s_1}(\mathbf{k}_1) a^{s_2}(\mathbf{k}_2) a^{s_3}(\mathbf{k}_3) \rangle = Q_3^{s_1 s_2 s_3}(\mathbf{k}_1, \mathbf{k}_2) \delta(s_1 \mathbf{k}_1 + s_2 \mathbf{k}_2 + s_3 \mathbf{k}_3),$$

$$\begin{aligned} \langle a^{s_1}(\mathbf{k}_1) a^{s_2}(\mathbf{k}_2) a^{s_3}(\mathbf{k}_3) a^{s_4}(\mathbf{k}_4) \rangle &= Q_4^{s_1 s_2 s_3 s_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \delta(s_1 \mathbf{k}_1 + s_2 \mathbf{k}_2 + s_3 \mathbf{k}_3 + s_4 \mathbf{k}_4), \\ &+ Q_2^{s_1 s_2}(\mathbf{k}_1) Q_2^{s_3 s_4}(\mathbf{k}_3) \delta(s_1 \mathbf{k}_1 + s_2 \mathbf{k}_2) \delta(s_3 \mathbf{k}_3 + s_4 \mathbf{k}_4) \\ &+ Q_2^{s_1 s_3}(\mathbf{k}_1) Q_2^{s_2 s_4}(\mathbf{k}_2) \delta(s_1 \mathbf{k}_1 + s_3 \mathbf{k}_3) \delta(s_2 \mathbf{k}_2 + s_4 \mathbf{k}_4) \\ &+ Q_2^{s_1 s_4}(\mathbf{k}_1) Q_2^{s_2 s_3}(\mathbf{k}_2) \delta(s_1 \mathbf{k}_1 + s_4 \mathbf{k}_4) \delta(s_2 \mathbf{k}_2 + s_3 \mathbf{k}_3). \end{aligned}$$

They measure deviations from Gaussianity: for a Gaussian field, all cumulants of order 3 and above are zero.

Equations of motion (3-wave)

Consider a Hamiltonian $H = Q + U = \int \mathcal{H}_{\mathbf{k}} d\mathbf{k}$ where the Hamiltonian density, $\mathcal{H}_{\mathbf{k}} = \mathcal{Q}_{\mathbf{k}} + \mathcal{U}_{\mathbf{k}}$ is

$$\mathcal{H}_{\mathbf{k}} = \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* + \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} (a_{\mathbf{k}}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2} + a_{\mathbf{k}} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^*) \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}. \quad (1)$$

Here $\delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}$ means $\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)$. The equations of motion are

$$\dot{a}_{\mathbf{k}} = i \frac{\delta H}{\delta a_{\mathbf{k}}^*} = i \omega_{\mathbf{k}} a_{\mathbf{k}} + i \frac{\delta U}{\delta a_{\mathbf{k}}^*} \quad (2)$$

$$= i \omega_{\mathbf{k}} a_{\mathbf{k}} + i \int d\mathbf{k}_1 d\mathbf{k}_2 \left(W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} + 2 W_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} a_{\mathbf{k}_1} a_{\mathbf{k}_2}^* \delta_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} \right). \quad (3)$$

Closure problem

Generic issue with the statistical description of nonlinear equations:
the evolution of the second moment depends on the third:

$$\begin{aligned} \partial_t \langle a_{\mathbf{k}} a_{\mathbf{k}'}^* \rangle = & i \int d\mathbf{k}_1 d\mathbf{k}_2 \left(W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \langle a_{\mathbf{k}'}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2} \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} + 2 W_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} \langle a_{\mathbf{k}'}^* a_{\mathbf{k}_1} a_{\mathbf{k}_2}^* \rangle \delta_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} \right) \\ & - i \int d\mathbf{k}_1 d\mathbf{k}_2 \left(W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}'} \langle a_{\mathbf{k}} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}'} + 2 W_{\mathbf{k}' \mathbf{k}_2}^{\mathbf{k}_1} \langle a_{\mathbf{k}} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2} \rangle \delta_{\mathbf{k}' \mathbf{k}_2}^{\mathbf{k}_1} \right). \end{aligned}$$

The third depends on the fourth etc:

$$\partial_t \langle a_{\mathbf{k}} a_{\mathbf{k}_1}^* a_{\mathbf{k}_2}^* \rangle = \dots$$

Corresponding cumulant hierarchy suffers from the same issue.

Constant flux relation

Consider the Hamiltonian density:

$$\begin{aligned}\partial_t \langle \mathcal{H}_{\mathbf{k}} \rangle + \nabla_{\mathbf{k}} \cdot \langle J_{\mathbf{k}}^{(\mathcal{H})} \rangle &= \mathcal{F}_{\mathbf{k}} - \mathcal{D}_{\mathbf{k}} \\ &= 0 \quad (\text{in the inertial range})\end{aligned}$$

In the steady state we must have

$$\nabla_{\mathbf{k}} \cdot \langle J_{\mathbf{k}}^{(\mathcal{H})} \rangle = 0$$

in the inertial range. This is a constraint on any solution of the moment hierarchy.

Make this constraint explicit:

$$\begin{aligned}0 &= -\nabla_{\mathbf{k}} \cdot \langle J_{\mathbf{k}}^{(\mathcal{H})} \rangle = \langle \dot{\mathcal{H}}_{\mathbf{k}} \rangle \\ 0 &= \omega_{\mathbf{k}} \langle \dot{a}_{\mathbf{k}} a_{\mathbf{k}}^* \rangle + \omega_{\mathbf{k}} \langle a_{\mathbf{k}} \dot{a}_{\mathbf{k}}^* \rangle + \langle \dot{u}_{\mathbf{k}} \rangle \\ 0 &= \langle \dot{u}_{\mathbf{k}} - \dot{a}_{\mathbf{k}} \frac{\delta U}{\delta a_{\mathbf{k}}} - \dot{a}_{\mathbf{k}}^* \frac{\delta U}{\delta a_{\mathbf{k}}^*} \rangle.\end{aligned}$$

using eqns of motion, (2), to eliminate $\omega_{\mathbf{k}} a_{\mathbf{k}}$ terms.

Constant flux relation

Using Eqs. (1) for $u_{\mathbf{k}}$ and U some algebra gives

$$0 = 2 \int d\mathbf{k}_1 d\mathbf{k}_2 \left[W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \operatorname{Re} \langle a_{\mathbf{k}}^* \dot{a}_{\mathbf{k}_1} a_{\mathbf{k}_2} \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} - W_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} \operatorname{Re} \langle a_{\mathbf{k}_1}^* \dot{a}_{\mathbf{k}} a_{\mathbf{k}_2} \rangle \delta_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} \right].$$

Let us now assume isotropy and perform angle averaging:

$$C^{(\mathcal{H})}(k, k_1, k_2) = \int d\Omega_1 d\Omega_2 \operatorname{Re} \langle a_{\mathbf{k}}^* \dot{a}_{\mathbf{k}_1} a_{\mathbf{k}_2} \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}.$$

We now have

$$\int dk_1 dk_2 (k_1 k_2)^{d-1} \left[W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} C^{(\mathcal{H})}(k, k_1, k_2) - W_{\mathbf{k} \mathbf{k}_2}^{\mathbf{k}_1} C^{(\mathcal{H})}(k_1, k, k_2) \right] = 0.$$

Constant Flux relation

Now assume scale invariance:

$$W_{hk_1 hk_2}^{hk} = h^\gamma W_{k_1 k_2}^k$$

$$C^{(\mathcal{H})}(hk, hk_1, hk_2) = h^{-y} C^{(\mathcal{H})}(k, k_1, k_2).$$

Change of variables (Zakharov-Kraichnan) in second integral :

$$k_1 = \frac{k^2}{k_1'}, \quad k_2 = \frac{kk_2'}{k_1'}.$$

$$\int dk_1 dk_2 (k_1 k_2)^{d-1} W_{k_1 k_2}^k C^{(\mathcal{H})}(k, k_1, k_2) \left[1 - \frac{k}{k_1}^{\gamma+3d-y} \right] = 0.$$

We must therefore have $y = \gamma + 3d$.

Why is there no WT analogue of the $\frac{4}{5}$ -law?

If it is correct, the $\gamma + 3d$ scaling for the flux-carrying correlation function, $C^{(\mathcal{H})}(k, k_1, k_2)$, is exact:

- No weak nonlinearity or closure assumptions.
- Similar in spirit to Kolmogorov's $\frac{4}{5}$ -law.

Differences:

- No local \mathbf{x} -representation.
- $\mathcal{H}_{\mathbf{k}}$ is not quadratic.

Düring and Krstulovic 1-law:

For the Föppl-von Kármán eqn:

- H is quadratic
- Field equations are local in \mathbf{x} -space.

For this system, it can be shown (Düring and Krstulovic, 2018)

$$\langle \mathcal{J}[\delta\chi, \delta\zeta] \delta\dot{\zeta} \rangle \cdot \hat{\mathbf{r}} = -\epsilon r, \quad (4)$$

where $\delta\zeta = \zeta(\mathbf{x} + \mathbf{r}) - \zeta(\mathbf{x})$ etc.

Reminder: Föppl-von Kármán equation

Displacement, ζ , and stress, χ :

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{D}{h} \nabla^4 \zeta + \mathcal{L}[\chi, \zeta]$$

$$\nabla^4 \chi = -\frac{E}{2} \mathcal{L}[\zeta, \zeta],$$

Constants: ρ - density, h - thickness, D - bending stiffness, E - Young's modulus.

$$\mathcal{L}[f, g] = \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 g}{\partial y^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 g}{\partial x^2}.$$

Can show that

$$\mathcal{L}[f, g] = -\nabla \cdot \mathcal{J}[f, g]$$

with

$$\mathcal{J}[f, g] = \begin{pmatrix} f_y g_{yx} - f_x g_{yy} \\ f_x g_{xy} - f_y g_{xx} \end{pmatrix}.$$

The correlation function, $\langle \mathcal{J}[\delta\chi, \delta\zeta] \delta\zeta \rangle$, appearing in the 1-law is therefore quartic in amplitude variables. Consistent with leading interaction being 4-wave.

Weak wave turbulence and the wave kinetic equation

Wave spectrum, $n_{\mathbf{k}}$, relates to the second moment (cumulant):

$$\langle a^*(\mathbf{k}) a(\mathbf{k}') \rangle = n(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}').$$

Claim: when the nonlinearity is weak, the long time behaviour of $n_{\mathbf{k}}(t)$ is determined by the wave kinetic equation:

$$\begin{aligned} \frac{\partial n_{\mathbf{k}_1}}{\partial t} &= \pi \int \left| W_{\mathbf{k}_2 \mathbf{k}_3}^{\mathbf{k}_1} \right|^2 (n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}_1} n_{\mathbf{k}_3}) \\ &\quad \delta(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_2 d\mathbf{k}_3 \\ &+ \pi \int \left| W_{\mathbf{k}_1 \mathbf{k}_3}^{\mathbf{k}_2} \right|^2 (n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}_1} n_{\mathbf{k}_3}) \\ &\quad \delta(\omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1}) \delta(\mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_1) d\mathbf{k}_2 d\mathbf{k}_3 \\ &+ \pi \int \left| W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}_3} \right|^2 (n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}_1} n_{\mathbf{k}_2} + n_{\mathbf{k}_1} n_{\mathbf{k}_3}) \\ &\quad \delta(\omega_{\mathbf{k}_3} - \omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2}) \delta(\mathbf{k}_3 - \mathbf{k}_1 - \mathbf{k}_2) d\mathbf{k}_2 d\mathbf{k}_3 \end{aligned}$$

Wave kinetic equation derivation: interaction variables

We will sketch the derivation for the BPV equation since it is (slightly) less messy:

$$\frac{\partial a_{\mathbf{k}}}{\partial t} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}, \quad (5)$$

It is convenient to introduce "interaction" variables that incorporate the linear dynamics and formal small parameter ϵ :

$$\epsilon b_{\mathbf{k}} = a_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t}, \quad (6)$$

in which Eq. (5) takes the form

$$\frac{\partial b_{\mathbf{k}}}{\partial t} = \epsilon \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} b_{\mathbf{k}_1} b_{\mathbf{k}_2} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} e^{i\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} t}, \quad (7)$$

where $\Omega_{\mathbf{q}\mathbf{r}}^{\mathbf{p}}$ is shorthand notation for $\omega_{\mathbf{p}} - \omega_{\mathbf{q}} - \omega_{\mathbf{r}}$.

Wave kinetic equation derivation: naive perturbation theory

Exploit weak nonlinearity assumption to solve Eq. (7) perturbatively:

$$b_{\mathbf{k}}(t) = b_{\mathbf{k}}^{(0)}(t) + \epsilon b_{\mathbf{k}}^{(1)}(t) + \epsilon^2 b_{\mathbf{k}}^{(2)}(t) + \dots$$

Much of the algebraic complexity comes from the fact that it turns out to be necessary to go to second order in ϵ to obtain a non-trivial answer.

Wave kinetic equation derivation: naive perturbation theory

The first few terms in the expansion are

$$b_{\mathbf{k}}^{(0)}(t) = B_{\mathbf{k}} \quad (8)$$

$$b_{\mathbf{k}}^{(1)}(t) = \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} B_{\mathbf{k}_1} B_{\mathbf{k}_2} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \Delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}, t) \quad (9)$$

$$b_{\mathbf{k}}^{(2)}(t) = -2 \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} W_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{k}_1} B_{\mathbf{k}_2} B_{\mathbf{k}_3} B_{\mathbf{k}_4} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \delta_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{k}_1} E(\Omega_{\mathbf{k}}^{\mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}, \Omega_{\mathbf{k}}^{\mathbf{k}_1 \mathbf{k}_2}, t) \quad (10)$$

where the $B_{\mathbf{k}}$ are constants. All time-dependence is in the integrals

$$\Delta(x, t) = \int_0^t d\tau e^{i x \tau} = \frac{e^{i x t} - 1}{i x} \quad (11)$$

$$E(x, y, t) = \int_0^t d\tau \Delta(x - y, \tau) e^{i y \tau}. \quad (12)$$

Wave kinetic equation derivation: resonances

At this point we begin to see why *resonant* interactions play such a central role in weak wave turbulence: Riemann-Lebesgue lemma and Sokhotski–Plemelj theorem tell us that as t gets large

$$\Delta(x, t) \sim \pi\delta(x) + i\mathcal{P}\left(\frac{1}{x}\right)$$

Thus the terms in $b_{\mathbf{k}}^{(1)}(t)$ concentrate on the resonant sets:

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \quad \omega_{\mathbf{k}} = \omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}.$$

Likewise for $b_{\mathbf{k}}^{(2)}(t)$, one can show that

$$E(x, y, t) \sim \left(\pi\delta(x) + i\mathcal{P}\left(\frac{1}{x}\right) \right) \left(\pi\delta(y) + i\mathcal{P}\left(\frac{1}{y}\right) \right)$$

Wave kinetic equation derivation: perturbative $n_{\mathbf{k}}$

To order ϵ^2 , the wave spectrum is

$$\begin{aligned}
 n_{\mathbf{p}}(t) \delta_{\mathbf{p}'}^{\mathbf{p}} &= \langle B_{\mathbf{p}} B_{\mathbf{p}'}^* \rangle & (13) \\
 + \epsilon \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}'} \langle B_{\mathbf{p}} B_{\mathbf{k}_1}^* B_{\mathbf{k}_2}^* \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}'} \Delta(\Omega_{\mathbf{p}'}^{\mathbf{k}_1 \mathbf{k}_2}, t) \\
 + \epsilon \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \langle B_{\mathbf{p}'}^* B_{\mathbf{k}_1} B_{\mathbf{k}_2} \rangle \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \Delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}, t) \\
 - 4 \epsilon^2 \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{k}_1} \operatorname{Re} [\langle B_{\mathbf{p}'}^* B_{\mathbf{k}_2} B_{\mathbf{k}_3} B_{\mathbf{k}_4} \rangle] \\
 \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \delta_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{k}_1} E(\Omega_{\mathbf{p}}^{\mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4}, \Omega_{\mathbf{p}}^{\mathbf{k}_1 \mathbf{k}_2}, t) \\
 + \epsilon^2 \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{p}'} \langle B_{\mathbf{k}_1} B_{\mathbf{k}_2} B_{\mathbf{k}_3}^* B_{\mathbf{k}_4}^* \rangle \\
 \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \delta_{\mathbf{k}_3 \mathbf{k}_4}^{\mathbf{p}'} \Delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}, t) \Delta(\Omega_{\mathbf{p}'}^{\mathbf{k}_3 \mathbf{k}_4}, t).
 \end{aligned}$$

Wave kinetic equation derivation: averaging

Taking into account that $\langle B_{\mathbf{k}} \rangle = 0$ and using the fact that $b_{\mathbf{k}}^* = b_{-\mathbf{k}}$ and taking $s_i = \pm 1$, one can write a general third or fourth order correlation function as:

$$\begin{aligned} \langle B_{s_1 \mathbf{k}_1} B_{s_2 \mathbf{k}_2} B_{s_3 \mathbf{k}_3} \rangle &= Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3}^{(3)} \\ \langle B_{s_1 \mathbf{k}_1} B_{s_2 \mathbf{k}_2} B_{s_3 \mathbf{k}_3} B_{s_4 \mathbf{k}_4} \rangle &= n_{\mathbf{k}_1} n_{\mathbf{k}_2} \delta_{s_3 \mathbf{k}_3}^{s_1 \mathbf{k}_1} \delta_{s_4 \mathbf{k}_4}^{s_2 \mathbf{k}_2} + n_{\mathbf{k}_1} n_{\mathbf{k}_3} \delta_{s_2 \mathbf{k}_2}^{s_1 \mathbf{k}_1} \delta_{s_4 \mathbf{k}_4}^{s_3 \mathbf{k}_3} \\ &\quad + n_{\mathbf{k}_1} n_{\mathbf{k}_2} \delta_{s_4 \mathbf{k}_4}^{s_1 \mathbf{k}_1} \delta_{s_3 \mathbf{k}_3}^{s_2 \mathbf{k}_2} + Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3 s_4 \mathbf{k}_4}^{(4)} \end{aligned}$$

where $Q^{(3)}$ and $Q^{(4)}$ are the appropriate third and fourth order cumulants of the field $B_{\mathbf{k}}$.

Random phase "approximation" or Wick closure: neglect

$$Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3}^{(3)} \text{ and } Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3 s_4 \mathbf{k}_4}^{(4)}.$$

Wave kinetic equation derivation: averaging

Next steps:

- average Eq. (13) for $n_{\mathbf{p}}(t) \delta_{\mathbf{p}'}^{\mathbf{p}}$ and substitute moments.
- integrate out (the right) two delta functions from each term.
- use the symmetries of $W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}$ to group similar terms together.

$$\begin{aligned}
 n_{\mathbf{p}}(t) \delta_{\mathbf{p}'}^{\mathbf{p}} &= n_{\mathbf{p}}^{(0)} \delta_{\mathbf{p}'}^{\mathbf{p}} & (14) \\
 + 2 \epsilon^2 \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \Delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}, t) \Delta(\Omega_{\mathbf{p}}^{\mathbf{k}_1 \mathbf{k}_2}, t) \delta_{\mathbf{p}'}^{\mathbf{p}} \\
 - 8 \epsilon^2 \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{p} \mathbf{k}_2}^{\mathbf{k}_1} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} n_{\mathbf{p}} n_{\mathbf{k}_2} \operatorname{Re} \left[E(0, \Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}; t) \right] \delta_{\mathbf{p}'}^{\mathbf{p}}.
 \end{aligned}$$

Wave kinetic equation derivation: secular terms

Averaging introduced some singular integrals that diverge as $t \rightarrow \infty$. For large time, they behave as [Newell, 1968]:

$$\Delta(x; t) \Delta(-x; t) \sim 2\pi t \delta(x) + 2\mathcal{P}\left(\frac{1}{x}\right) \frac{\partial}{\partial x}$$

$$E(0, x; t) \sim \left[\pi \delta(x) + i\mathcal{P}\left(\frac{1}{x}\right)\right] t - i \left[\pi \delta(x) + i\mathcal{P}\left(\frac{1}{x}\right) \frac{\partial}{\partial x}\right],$$

We now see that our expansion breaks down at $t \sim \epsilon^{-2}$:

$$n_{\mathbf{p}}(t) = n_{\mathbf{p}}^{(0)} - (\epsilon^2 t) S [n_{\mathbf{k}}^0] + \epsilon^2 [\text{terms bounded in } t], \quad (15)$$

where the divergent (secular) part is

$$S [n_{\mathbf{k}}^0] = 4\pi \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}})$$

$$- 8\pi \int d\mathbf{k}_1 d\mathbf{k}_2 W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} W_{\mathbf{p} \mathbf{k}_2}^{\mathbf{k}_1} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}} n_{\mathbf{p}} n_{\mathbf{k}_2} \delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{p}}).$$

Note that the divergences concentrate on the resonant curves.

Wave kinetic equation derivation: method of multiple scales

Assume that $n_{\mathbf{p}}^{(0)}(t)$ varies slowly on the nonlinear timescale $T_2 = \epsilon^2 t$ and treat T_2 as an additional independent variable so that we have $n_{\mathbf{p}}^{(0)}(t, T_2)$. In our case, $n_{\mathbf{p}}^{(0)}$ is constant wrt t so we get:

$$\frac{dn_{\mathbf{p}}^{(0)}}{dt} = \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial t} + \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial T_2} \frac{dT_2}{dt} = \epsilon^2 \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial T_2}$$

Now differentiate Eq. (15):

$$\frac{dn_{\mathbf{p}}}{dt} = \epsilon^2 \frac{\partial n_{\mathbf{p}}^{(0)}}{\partial T_2} - \epsilon^2 S[n_{\mathbf{k}}^0] + \epsilon^2 \frac{d}{dt} [\text{terms bounded in } t],$$

Expansion is consistent to times $\sim \epsilon^4 t$ if

$$\frac{\partial n_{\mathbf{p}}^{(0)}}{\partial T_2} = S[n_{\mathbf{k}}^0] \quad (3\text{-wave kinetic equation}) \quad (16)$$

Wave kinetic equation derivation: remarks

- The kinetic equation is a consistency condition that must be satisfied by the second moment to account for the effect of resonant interactions on the timescale of $\epsilon^2 t$.
- Fundamentally non-perturbative: solution of the kinetic equation adds up an infinite number of terms in the original regular perturbation expansion. Which terms?
- In principle, the method of multiple scales can be extended to higher orders to describe behaviour on longer timescales, $\epsilon^4 t$ etc.
- It is not guaranteed that the *solution* of the kinetic equation is consistent with the assumption of weak nonlinearity / separation of timescales used in this derivation.

Nonlinear frequency correction

- We focused thus far on the the correlation function $\langle a^*(\mathbf{k}) a(\mathbf{k}') \rangle$. What about the expansion of $\langle a(\mathbf{k}) a(\mathbf{k}') \rangle$?
- The $i\omega_{\mathbf{k}} a_{\mathbf{k}}$ terms do not disappear at leading order for $\langle a(\mathbf{k}) a(\mathbf{k}') \rangle$ so a "fast" time dependence remains.
- We still find secular terms on the ϵt timescale in the expansion - the calculation is comparably messy as for $\langle a^*(\mathbf{k}) a(\mathbf{k}') \rangle$.
- However, the consistency condition for the removal of these secular terms is less complex and can be solved explicitly by correcting the frequency:

$$\omega_{\mathbf{k}} \rightarrow \omega_{\mathbf{k}} + \Omega[n_{\mathbf{k}}].$$

where

$$\Omega[n_{\mathbf{k}}] \sim \int \left| W_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \right|^2 n_{\mathbf{k}_1} \delta_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}} \delta(\Omega_{\mathbf{k}_1 \mathbf{k}_2}^{\mathbf{k}}) d\mathbf{k}_1 d\mathbf{k}_2.$$

The asymptotic closure argument of Newell

In deriving the kinetic equation I *assumed* the Wick closure. The asymptotic closure argument shows that the Wick closure arises naturally from the dynamics for long times.

Outline:

- Go back to Eq. (14) but **don't** neglect $Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3}^{(3)}$ and $Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3 s_4 \mathbf{k}_4}^{(4)}$.
- Write perturbative expansions for $Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3}^{(3)}(t)$ and $Q_{s_1 \mathbf{k}_1 s_2 \mathbf{k}_2 s_3 \mathbf{k}_3 s_4 \mathbf{k}_4}^{(4)}(t)$ and identify any additional secular terms that should be included in the consistency condition, Eq. (16).
- Surprise: no **new** secular terms appear that are not already accounted for in the consistency conditions for $\langle a^*(\mathbf{k}) a(\mathbf{k}') \rangle$ (kinetic equation) and $\langle a(\mathbf{k}) a(\mathbf{k}') \rangle$ (nonlinear frequency correction).

The asymptotic closure argument of Newell

Claim: this finding is general: the dynamics generates correlations in such a way that the secular (divergent) terms appearing in the higher order cumulants are functions of lower order cumulants only.

- Doesn't say that there is no closure problem. Rather, the asymptotic consistency conditions for the removal of divergences from perturbation theory (for **all** cumulants) are closed.
- May provide useful insights for mathematical treatments of wave kinetics. [Deng & Hani (2021), Staffilani & Tran (2021)]
- Accounting for the nonlinear frequency correction is essential to asymptotic closure (although I didn't detail the calculation here).

Symmetric form of general 3-wave kinetic equation

$$\partial_t n_{\mathbf{k}_1} = S[n_{\mathbf{k}}] = \int_{\mathbb{R}^{2d}} (R_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} - R_{\mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_1} - R_{\mathbf{k}_3 \mathbf{k}_1 \mathbf{k}_2}) d\mathbf{k}_2 d\mathbf{k}_3 \quad (17)$$

where

$$R_{\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3} = 4\pi \left| W_{\mathbf{k}_2 \mathbf{k}_3}^{\mathbf{k}_1} \right|^2 (n_{\mathbf{k}_2} n_{\mathbf{k}_3} - n_{\mathbf{k}_1} n_{\mathbf{k}_3} - n_{\mathbf{k}_1} n_{\mathbf{k}_2}) \delta(\omega_{\mathbf{k}_1} - \omega_{\mathbf{k}_2} - \omega_{\mathbf{k}_3}) \delta_{\mathbf{k}_2 \mathbf{k}_3}^{\mathbf{k}_1}$$

The (quadratic) energy and wave action are,

$$E = \int \omega_{\mathbf{k}} n_{\mathbf{k}} d\mathbf{k} \quad \text{and} \quad N = \int n_{\mathbf{k}} d\mathbf{k}.$$

E is conserved by Eq. (17) but not necessarily by the original dynamical equation.

Frequency space representation for isotropic wave turbulence

For isotropic systems, $\omega_{\mathbf{k}} = ck^\alpha$, the angle-averaged frequency spectrum, N_ω , is more convenient than the \mathbf{k} -space spectrum, $n_{\mathbf{k}}$:

$$\begin{aligned} \int_{\mathbb{R}^d} n_{\mathbf{k}} d\mathbf{k} &= \Omega^{(d)} \int_0^\infty n_{\mathbf{k}} k^{d-1} \frac{dk}{d\omega} d\omega = \frac{\Omega^{(d)}}{\alpha} c^{-\frac{\alpha}{d}} \int_0^\infty n_\omega \omega^{\frac{d-\alpha}{\alpha}} d\omega \\ &= \int_0^\infty N_\omega d\omega. \end{aligned}$$

The (quadratic) energy and wave action are

$$\begin{aligned} E &= \int_0^\infty \omega N_\omega d\omega \\ N &= \int_0^\infty N_\omega d\omega \end{aligned}$$

In terms of N_ω , the integrals in the kinetic equation become one-dimensional integrals of ω 's rather than d -dimensional integrals over \mathbf{k} 's.

Kinetic equation in frequency space

$$\frac{\partial N_{\omega_1}}{\partial t} = S_1[N_{\omega}] + S_2[N_{\omega}] + S_3[N_{\omega}] \quad (18)$$

where

$$\begin{aligned} S_1[N_{\omega}] = & \int K_1(\omega_2, \omega_3) N_{\omega_2} N_{\omega_3} \delta(\omega_1 - \omega_2 - \omega_3) d\omega_{23} \\ & - \int K_1(\omega_3, \omega_1) N_{\omega_1} N_{\omega_3} \delta(\omega_2 - \omega_3 - \omega_1) d\omega_{23} \\ & - \int K_1(\omega_1, \omega_2) N_{\omega_1} N_{\omega_2} \delta(\omega_3 - \omega_1 - \omega_2) d\omega_{23}, \end{aligned} \quad (19)$$

and $K_1(\omega_1, \omega_2)$ is a homogeneous function of degree

$$\lambda = \frac{2\gamma - \alpha}{\alpha}.$$

Kinetic equation in frequency space

$$\begin{aligned} S_2[N_\omega] = & - \int K_2(\omega_2, \omega_3) N_{\omega_1} N_{\omega_2} \delta(\omega_1 - \omega_2 - \omega_3) d\omega_{23} \\ & + \int K_2(\omega_3, \omega_1) N_{\omega_2} N_{\omega_3} \delta(\omega_2 - \omega_3 - \omega_1) d\omega_{23} \quad (20) \\ & + \int K_2(\omega_1, \omega_2) N_{\omega_1} N_{\omega_3} \delta(\omega_3 - \omega_1 - \omega_2) d\omega_{23} \end{aligned}$$

with

$$K_2(\omega_1, \omega_2) = K_1(\omega_1, \omega_2) \left(\frac{\omega_1 + \omega_2}{\omega_2} \right)^{\frac{\alpha-d}{\alpha}}$$

Kinetic equation in frequency space

and

$$\begin{aligned} S_3[N_\omega] = & - \int K_3(\omega_2, \omega_3) N_{\omega_1} N_{\omega_3} \delta(\omega_1 - \omega_2 - \omega_3) d\omega_{23} \\ & + \int K_3(\omega_3, \omega_1) N_{\omega_1} N_{\omega_2} \delta(\omega_2 - \omega_3 - \omega_1) d\omega_{23} \quad (21) \\ & + \int K_3(\omega_1, \omega_2) N_{\omega_2} N_{\omega_3} \delta(\omega_3 - \omega_1 - \omega_2) d\omega_{23}. \end{aligned}$$

with

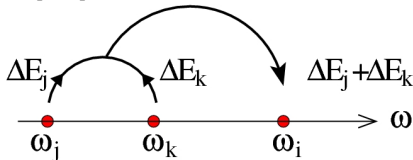
$$K_3(\omega_1, \omega_2) = K_1(\omega_1, \omega_2) \left(\frac{\omega_1 + \omega_2}{\omega_1} \right)^{\frac{\alpha-d}{\alpha}}.$$

Kinetic equation in frequency space

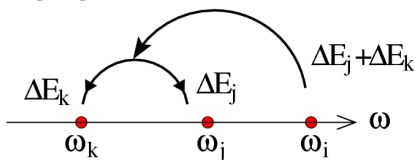
Advantages of this long-form representation of the collision integral:

- Only a single scaling parameter, λ , instead of 3 (γ , α and d).
- S_1 , S_2 and S_3 have natural physical interpretations:

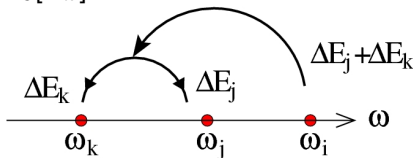
$S_1[N_\omega]$: Forward transfer



$S_2[N_\omega]$: Backscatter



$S_3[N_\omega]$: Backscatter



“Cheat-sheet”: 3-wave turbulence on one slide

Spectra:

- Kolmogorov–Zakharov: $N_\omega = c_{KZ} \sqrt{J} \omega^{-\frac{\lambda+3}{2}}$.
- Generalised Phillips (critical balance): $N_\omega = c_P \omega^{-\lambda}$.
- Thermodynamic: $N_\omega \sim \omega^{-2+\frac{d}{\alpha}}$.

Phase transitions:

- Infinite capacity: $\lambda < 1$.
Finite capacity: $\lambda > 1$.
- Breakdown at small scales: $\lambda > 3$.
Breakdown at large scales: $\lambda < 3$.

Locality (later): if $K_1(\omega_i, \omega_j)$ has asymptotics $K_1(\omega_i, \omega_j) \sim \omega_i^\mu \omega_j^\nu$ with $\mu + \nu = \lambda$ for $\omega_1 \gg \omega_2$, the KZ spectrum is local if

- $\mu < \nu + 3$ and $x_{KZ} > x_T$.

Zakharov-Kraichnan transformations

First consider the $S_1[N_\omega]$ term only. Seek a stationary solution

$N_\omega = c_{KZ} \omega^{-x}$ such that $S_1[N_\omega] = 0$:

$$\begin{aligned}
 0 = & c_{KZ}^2 \int K_1(\omega_2, \omega_3) (\omega_2 \omega_3)^{-x} \delta(\omega_1 - \omega_2 - \omega_3) d\omega_{23} \\
 & - c_{KZ}^2 \int K_1(\omega_3, \omega_1) (\omega_1 \omega_3)^{-x} \delta(\omega_2 - \omega_3 - \omega_1) d\omega_{23} \quad (22) \\
 & - c_{KZ}^2 \int K_1(\omega_1, \omega_2) (\omega_1 \omega_2)^{-x} \delta(\omega_3 - \omega_1 - \omega_2) d\omega_{23}.
 \end{aligned}$$

Apply the changes of variables to the second and third integrals:

$$(\omega_2, \omega_3) \rightarrow \left(\frac{\omega_1^2}{\omega_2'}, \frac{\omega_1 \omega_3'}{\omega_2'} \right) \quad (23)$$

and

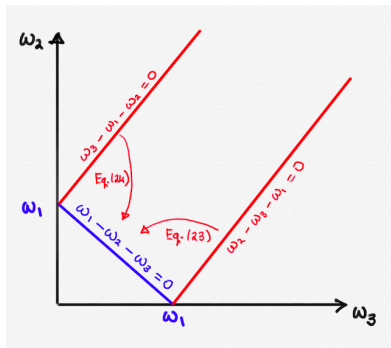
$$(\omega_2, \omega_3) \rightarrow \left(\frac{\omega_1 \omega_2}{\omega_3}, \frac{\omega_1^2}{\omega_3} \right) \quad (24)$$

Zakharov-Kraichnan transformations

Jacobians are $\left(\frac{\omega_1}{\omega_2}\right)^3$ and $\left(\frac{\omega_1}{\omega_3}\right)^3$ respectively. The trick:

$$\begin{aligned} K(\omega_3, \omega_1) &= K\left(\frac{\omega_1 \omega'_3}{\omega'_2}, \omega_1\right) \\ &= \left(\frac{\omega_1}{\omega'_2}\right)^\lambda K(\omega'_1, \omega'_2). \end{aligned}$$

Support ends up concentrated on $\delta(\omega_1 - \omega_2 - \omega_3)$.



Graphical representation of Zakharov-Kraichnan transformations

Zakharov-Kraichnan transformations

The result is a single integral,

$$0 = c_{KZ}^2 \int K_1(\omega_2, \omega_3) (\omega_2 \omega_3)^{-x} \delta(\omega_1 - \omega_2 - \omega_3) \quad (25)$$

$$\omega_1^{\lambda+2-2x} \left[\omega_1^{2x-\lambda-2} - \omega_2^{2x-\lambda-2} - \omega_3^{2x-\lambda-2} \right] d\omega_{23}.$$

It is now easy to see that the right hand side vanishes when $2x - \lambda - 2 = 1$. This yields the KZ exponent:

$$x = \frac{\lambda + 3}{2}$$

Identical analysis applies to S_2 and S_3 . The equilibrium exponent $x = \frac{\alpha-d}{\alpha} + 1$ appears only in the sum $S_1 + S_2 + S_3$ (detailed balance).

Calculation of the Zakharov constant

Conservation of energy in the inertial range:

$$\partial_t (\omega N_\omega) = -\partial_\omega J_\omega = \omega S[N_\omega] \quad (26)$$

where J_ω is the energy flux at frequency ω . From previous analysis, on power law spectrum $N_\omega = c\omega^{-x}$, we have

$$\partial_\omega J_\omega = -c^2 \omega^{\lambda-2x+2} I(x), \quad (27)$$

where

$$I(x) = \int_0^1 \left[K_1(u, 1-u) (u(1-u))^{-x} \right. \\ \left. - K_2(u, 1-u) u^{-x} - K_3(u, 1-u) (1-u)^{-x} \right] \\ \left[1 - (1-u)^{2x-\lambda-2} - u^{2x-\lambda-2} \right] du. \quad (28)$$

Calculation of the Zakharov constant

Integrating once gives

$$J_\omega = -\omega^{\lambda-2x+3} \frac{c^2 I(x)}{\lambda - 2x + 3}.$$

We know that $J_\omega \rightarrow J$ (constant) as $x \rightarrow x_{KZ} = \frac{\lambda+3}{2}$ so

$$J = \lim_{x \rightarrow \frac{\lambda+3}{2}} -\omega^{\lambda-2x+3} \frac{c_{KZ}^2 I(x)}{\lambda - 2x + 3} = \frac{1}{2} c_{KZ}^2 \left. \frac{dI}{dx} \right|_{x=x_{KZ}}.$$

The K-Z constant is therefore

$$c_{KZ} = \sqrt{2J \left. \frac{dI}{dx} \right|_{x=x_{KZ}}^{-1}} \quad (29)$$

The (spectral) locality criterion

Previous analysis assumes that the collision integral is *convergent* on the KZ spectrum. This needs to be checked a-posteriori.

Need to know the (asymptotic) form of $K(\omega_1, \omega_2)$. Consider the models:

$$K(\omega_1, \omega_2) = \frac{1}{2} [\omega_1^\mu \omega_2^\nu + \omega_1^\nu \omega_2^\mu]$$

$$K(\omega_1, \omega_2) = \min(\omega_1, \omega_2)^\mu \max(\omega_1, \omega_2)^\nu$$

Must have $\mu + \nu = \lambda$.

Strategy:

- Use δ -function to integrate out ω_3 .
- Determine integrability of integrand as $\omega_2 \rightarrow 0$ and $\omega_2 \rightarrow \infty$.

Conclusion: see [Connaughton (2009)]: Collision integral is convergent provided:

- $x > 1 + \frac{\alpha-d}{\alpha}$
- $\mu < x < \nu + 3$

Universality of the KZ spectrum

The spectral locality criteria have a very important physical meaning: they tell us when the KZ spectrum is *universal* (independent of forcing and dissipation in the turbulence limit.)

Two requirements:

- χ_{KZ} must be steeper than the equilibrium spectrum.
- an interval of locality, $[\mu, \nu + 3]$, must exist - ie $\mu < \nu + 3$.

Comments:

- When an interval of locality exists, the KZ exponent is at the midpoint.
- (Spectral) non-locality does **not** present a problem for the applicability of the kinetic equation. However forcing and/or dissipation must be included explicitly.

Should we expect to see the KZ spectrum "in the wild"?

Although WT theory has been proposed to apply to many physical systems, these applications are fraught with danger. Some potential problem scenarios:

- The asymptotic limit in which the KE arises is not mathematically consistent.
- KE is consistent but system of interest is not in the relevant regime (weak nonlinearity, long time) for it to apply.
- KE is consistent and applicable but KZ spectrum is not local.
- KE is consistent and applicable and KZ spectrum is local but exponent $\frac{\lambda+3}{2}$ is inconsistent with the assumption of weak nonlinearity as $\omega \rightarrow \infty$.

Experimental observations of weak capillary wave turbulence

Capillary wave turbulence is one of the best experimental candidates but effects of gravity waves limit the scaling range available under normal conditions. So ...

Experimental observations of weak capillary wave turbulence

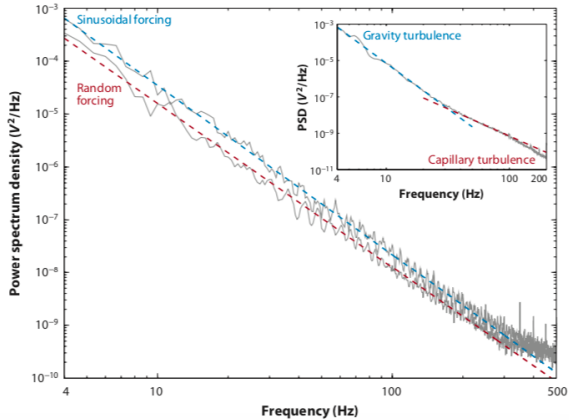
Capillary wave turbulence is one of the best experimental candidates but effects of gravity waves limit the scaling range available under normal conditions. So ...



Claudio Falcón, Eric Falcon & Stéphan Fauve during a parabolic flight on an A300 Airbus.

... why not do the experiments in space? Or at least in microgravity...

Experimental observations of weak capillary wave turbulence



Numerical observations of weak wave turbulence

It has proven difficult to realise "pure" weak wave turbulence in numerical simulations too.

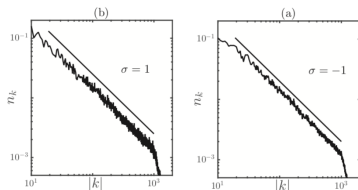
MMT model:

$$i\partial_t\psi = \mathcal{L}^{(1/2)}\psi + \sigma\psi|\psi|^2 + \mathcal{FD}$$

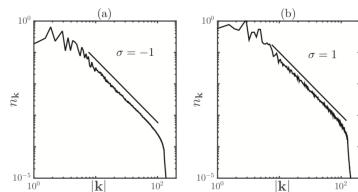
where $\mathcal{L}^{(1/2)}\psi \sim \sqrt{k}\hat{\psi}_{\mathbf{k}}$ and $\sigma = \pm 1$.

KZ spectrum is:

$$n_{\mathbf{k}} = c_{KZ} J^{\frac{1}{3}} k^{-d}.$$



Simulations in 1D from Sheffield & Rumpf (2017)



Simulations in 2D from Sheffield & Rumpf (2017)

Numerical validation of WT closure mechanism

In MMT the second moment depends on the 4th-order correlation function:

$$\partial_t n_{\mathbf{k}} = 2\sigma \int \text{Im} J_{123\mathbf{k}} \delta_{\mathbf{k}_3\mathbf{k}}^{\mathbf{k}_1\mathbf{k}_2} d\mathbf{k}_{123}.$$

where

$$J_{123\mathbf{k}} \delta_{\mathbf{k}_3\mathbf{k}}^{\mathbf{k}_1\mathbf{k}_2} = \langle \hat{\psi}_{\mathbf{k}_1} \hat{\psi}_{\mathbf{k}_2} \hat{\psi}_{\mathbf{k}_3}^* \hat{\psi}_{\mathbf{k}}^* \rangle.$$

If the WT closure is correct:

$$\text{Im} J_{123\mathbf{k}} \sim 2\pi\sigma F_{123\mathbf{k}} \delta(\Omega_{\mathbf{k}_3\mathbf{k}}^{\mathbf{k}_1\mathbf{k}_2})$$

where

$$F_{123\mathbf{k}} = n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}_3} + n_{\mathbf{k}_1} n_{\mathbf{k}_2} n_{\mathbf{k}} - n_{\mathbf{k}_1} n_{\mathbf{k}_3} n_{\mathbf{k}} - n_{\mathbf{k}_2} n_{\mathbf{k}_3} n_{\mathbf{k}}.$$

Testable predictions:

- $\text{Im} J_{123\mathbf{k}}$ sharply peaked at resonances.
- $\text{sgn} \text{Im} J_{123\mathbf{k}} = \text{sgn} \sigma$.
- $\text{Im} J_{123\mathbf{k}} \sim n^3$.
- Time evolution of $\text{Im} J_{123\mathbf{k}}$ from uncorrelated initial state.

Numerical validation of WT closure mechanism

Example: $\text{Im}J_{123k}$ with $k_1 = 49$, $k_2 = -4$ and $k_3 = 9$ (satisfies resonance conditions for $k = 36$.)

