Wave Turbulence: a theoretical physics perspective

Lecture 1: Phenomenology of turbulence and cascade phenomena

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Outline

The Navier-Stokes equation

$\mathbf{v}(\mathbf{x}, t)$, incompressible fluid velocity in $\mathbb{R}^3$, satisfies

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} &= -\nabla p + \nu \nabla^2 \mathbf{v} + f(\mathbf{x}, t) \\ \nabla \cdot \mathbf{v} &= 0, \end{aligned}$$

where

- $p(\mathbf{x}, t)$ is the pressure,
- $f(\mathbf{x}, t)$ is external forcing,
- ν is the kinematic viscosity.

Kinetic energy:

$$E = \int d\mathbf{x}\, \frac{\rho}{2}\, \mathbf{v}(\mathbf{x}, t)^2.$$

E is conserved if $\nu = f = 0$. We take density, $\rho = 1$.

The Navier-Stokes equation: Reynolds number

Non-dimensionalise:

- $\mathbf{v} \to V\,\mathbf{v}$ (V: characteristic velocity scale)
- $\mathbf{x} \to L\,\mathbf{x}$ (L: characteristic length scale)
- $t \to \frac{L}{V}\,t$ (also $p \to V^2 p$ and $f \to \frac{L}{V^2} f$)

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} &= -\nabla p + \frac{1}{R}\,\nabla^2\mathbf{v} + f(\mathbf{x},t) \\ \nabla\cdot\mathbf{v} &= 0, \end{aligned}$$

where Reynolds number is

$$R = \frac{L\,V}{\nu}.$$

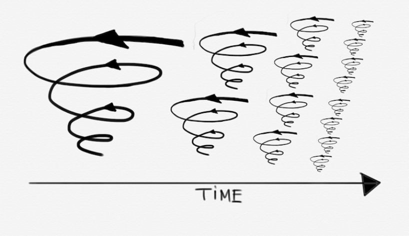
Turbulence: spatio-temporally disordered flow emerging as $R \to \infty$.

Visualisation of fluid turbulence in 3D

https://gfycat.com/@BlackByte

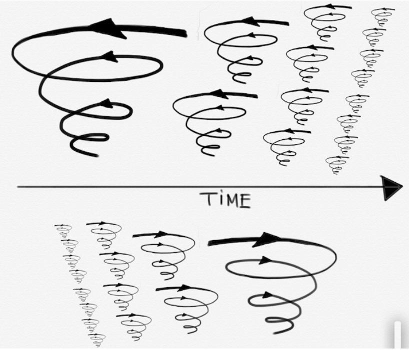
- Energy is supplied to the flow at large scales. However, when $R \gg 1$, viscosity can only dissipate energy at small scales.
- **Key idea**: separation of forcing and dissipation requires transfer of energy across scales via the inertial term.

The Richardson cascade: simple picture



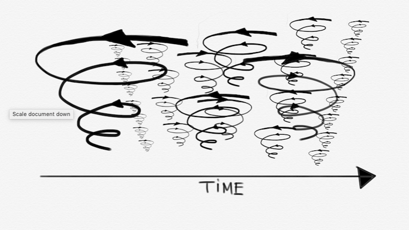
Cartoon conceptualisation of the mechanism of transfer of energy from large scales to small in turbulence: hierarchical vortex instabilities.

The Richardson cascade: backscatter



In practice, the inverse process (“backscatter") is almost as prevalent.

The Richardson cascade: less simple picture



The Richardson cascade should really be thought of more like this. Energy is transferred from large scales to small *on average* - but the fluctuations are strong.

Rationale for a statistical theory of turbulence

Turbulent flows are chaotic: $\mathbf{v}(\mathbf{x}, t)$ is not predictable.
However, the *statistical properties* of $\mathbf{v}(\mathbf{x}, t)$ is are predictable - for example *moments* of $\mathbf{v}(\mathbf{x}, t)$.
Some important moments:

$$
\begin{aligned}
E &= \langle \frac{1}{2} \left|\mathbf{v}\right|^2 \rangle \qquad \text{(mean energy)} \\
\Omega &= \langle \frac{1}{2} \left|\nabla \times \mathbf{v}\right|^2 \rangle \qquad \text{(mean enstrophy)}
\end{aligned}
$$

- Mathematically, $\langle . \rangle$ usually denotes an ensemble average wrt realisations of the initial condition.
- Experimentally, $\langle . \rangle$ usually denotes a spatial or time average.

Stationary homogeneous isotropic turbulence

For turbulence in a bi-periodic box (no boundaries), we assume that a statistically steady state is eventually reached in which forcing and dissipation balance on average.

It is believed that at small scales, symmetries of the NS equation are restored in a statistical sense [Frisch, 1995]. Moments of $\mathbf{v}$ then have the properties:

- **Stationarity:** $\langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t')\rangle = f_{ij}(\mathbf{x}, \mathbf{x}', t - t')$.
- **Homogeneity:** $\langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t')\rangle = f_{ij}(\mathbf{x} - \mathbf{x}', t, t')$.
- **Isotropy:** $\langle v_i(\mathbf{x}, t)\, v_j(\mathbf{x}', t')\rangle = f(\left|\mathbf{x} - \mathbf{x}'\right|, t, t')$.

Global energy balance and dissipative anomaly

Average kinetic energy satisfies the global energy budget relation

$$\frac{1}{2}\partial_t \langle v^2 \rangle = \langle \mathbf{f} \cdot \mathbf{v} \rangle + \nu \langle \mathbf{v} \cdot \nabla^2 \mathbf{v} \rangle. \tag{1}$$

No contribution from the nonlinear terms.
In absence of forcing, mean rate of energy dissipation is:

$$\epsilon(\nu) = -\frac{1}{2}\, \partial_t \langle v^2 \rangle = -\nu \langle \mathbf{v} \cdot \nabla^2 \mathbf{v} \rangle. \tag{2}$$

Dissipative anomaly (conjecture)

The mean rate of energy dissipation remains finite (and positive) in the limit $\nu \to 0$.

$$\lim_{\nu \to 0} \epsilon(\nu) = \epsilon > 0.$$

Fourier space representation of the Navier-Stokes equations

To progress beyond global energy budget we need to quantify Richardson's notion of motion at particular scales. The Fourier transform is used to (imperfectly) do this:

$$\mathbf{v}(\mathbf{x},t) = \int d\mathbf{k}\; \hat{\mathbf{v}}(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{x}} \qquad \hat{\mathbf{v}}(\mathbf{k},t) = \frac{1}{(2\pi)^d}\int d\mathbf{x}\; \mathbf{v}(\mathbf{x},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}.$$

After some work, the i^{th} component of $\hat{\mathbf{v}}(\mathbf{k},t)$ satisfies

$$(\partial_t + \nu k^2)\,\hat{\mathbf{v}}_i(\mathbf{k}) = \int d\mathbf{p}\, d\mathbf{q}\;\; T_{ijm}(\mathbf{k},\mathbf{p},\mathbf{q})\,\hat{\mathbf{v}}_j(\mathbf{p})\,\hat{\mathbf{v}}_m(\mathbf{q}).$$

Here the summation convention is implied and

$$T_{ijm}(\mathbf{k},\mathbf{p},\mathbf{q}) \;= -ik_j\left(\delta_{im} - \frac{k_i k_m}{|\mathbf{k}|^2}\right)\delta\left(\mathbf{k}-\mathbf{p}-\mathbf{q}\right).$$

Spectral energy density

$$
\begin{aligned}
E &= \int d\mathbf{x} \sum_{i=1}^{3} \frac{\rho}{2}\, v_i(\mathbf{x})^2 \\
&= \frac{\rho}{2} \int d\mathbf{x} \sum_{i=1}^{3} \left(\int d\mathbf{k}\, \hat{v}_i(\mathbf{k})\, e^{2\pi i \mathbf{k}\cdot\mathbf{x}} \right)^2 \\
&= \frac{\rho}{2} \int d\mathbf{k}\, d\mathbf{k}' \sum_{i=1}^{3} \hat{v}_i(\mathbf{k})\ \hat{v}_i(\mathbf{k}')\, \delta(\mathbf{k}+\mathbf{k}') \\
&= \frac{\rho}{2} \int d\mathbf{k}\ |\hat{\mathbf{v}}(\mathbf{k})|^2 \\
&\equiv \int E(k)\, dk \qquad \text{(assuming isotropy)}
\end{aligned}
$$

$E(k)$ is the spectral energy density.
Describes how energy is distributed across scales of motion.

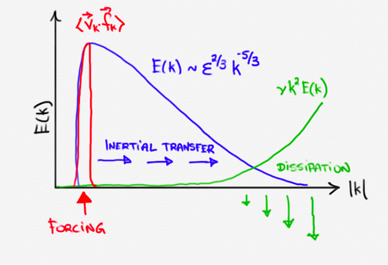
Kolmogorov 1941 phenomenology

If forcing injects energy at mean rate $\langle \mathbf{f} \cdot \mathbf{v} \rangle = \epsilon > 0$, global energy balance implies the mean rate of energy dissipation in the steady state is also ϵ.

Kolmogorov 1941

When $\nu \to 0$, in the inertial range, $E(k)$ depends only on ϵ and k.

Dimensional analysis: $[k] = L^{-1}$, $[E(k)] = L^3 T^{-2}$, $[\epsilon] = L^2 T^{-3}$.



This suggests

$$E(k) = C\, \epsilon^{\frac{2}{3}}\, k^{-\frac{5}{3}}.$$

Filtering

A spectral band-pass filter can split **v** into large and small scale contributions at a selected length scale, $l = K^{-1}$:

$$\mathbf{v}(\mathbf{x},t) = \mathbf{U}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t),$$

where

$$\mathbf{U}(\mathbf{x},t) = \int\limits_{|\mathbf{k}|<K} d\mathbf{k}\;\hat{\mathbf{v}}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}} \quad \mathbf{u}(\mathbf{x},t) = \int\limits_{|\mathbf{k}|>K} d\mathbf{k}\;\hat{\mathbf{v}}(\mathbf{k},t)\,e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Cumulative mean energy between wavenumbers 0 and K:

$$E^{(K)} = \frac{1}{2}\langle |U|^2 \rangle = \int\limits_{k<K} E(k)\,dk. \tag{3}$$

Scale-resolved energy balance - cumulative version

Cumulative energy satisfies the scale-resolved energy budget relation:

$$\partial_t E^{(K)} + \Pi^{(K)} = -2\nu\Omega^{(K)} + F^{(K)} \tag{4}$$

where

$$\Omega^{(K)} = \frac{1}{2}\langle|\nabla \times U|^2\rangle = \int\limits_{k<K} k^2\, E(k)\, dk \qquad \text{(cumulative enstrophy)}$$

$$F^{(K)} = \langle \mathbf{f} \cdot U\rangle = \int\limits_{|\mathbf{k}|<K} \hat{\mathbf{f}}(\mathbf{k}) \cdot \hat{\mathbf{v}}(\mathbf{k})\, d\mathbf{k} \qquad \text{(cumulative energy input)}$$

$$\Pi^{(K)} = \langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla\mathbf{U}) + \mathbf{u} \cdot (\mathbf{U} \cdot \nabla\mathbf{U})\rangle \qquad \text{(energy transfer)}$$

see Frisch, Eq.(2.52)

Scale-resolved energy balance - local version

Differentiating with respect to K gives a local equation for the spectral energy density, $E(k)$:

$$\frac{\partial E(k)}{\partial t} + \frac{\partial \Pi^{(k)}}{\partial k} = -2k^2 E(k) + \mathcal{F}(k). \tag{5}$$

where

$$\mathcal{F}(k) = \frac{\partial F^{(k)}}{\partial k}.$$

The scale resolved energy budget equations express the physically reasonable fact that in the inertial range, where forcing and dissipation are negligible, the rate of change of energy at a given scale is equal to the flux of energy through that scale due to the energy-conserving nonlinear interactions.

Constant flux condition

Assume that:

- We reach the stationary state: $\partial_t E^{(K)} = 0$.
- K is outside the forcing range so the cumulative energy input becomes the total energy input, ϵ.

Scale resolved energy budget relation becomes

$$\Pi^{(K)} = -2\nu\Omega^{(K)} + \epsilon,$$

Now take the limit $\nu \to 0$. The energy transfer is now constant (as a function of K) and equal to ϵ:

$$\Pi^{(K)} = \epsilon. \tag{6}$$

This relationship is exact for stationary, homogeneous, isotropic turbulence.

Energy transfer and velocity increments

Consider velocity increments in physical space:

$$\delta\mathbf{v}(\mathbf{r},t) = \mathbf{v}(\mathbf{x}+\mathbf{r},t) - \mathbf{v}(\mathbf{x},t)$$
$$\delta\mathbf{v}_{\|}(\mathbf{r},t) = [\mathbf{v}(\mathbf{x}+\mathbf{r},t) - \mathbf{v}(\mathbf{x},t)]\cdot\frac{\mathbf{r}}{|\mathbf{r}|}.$$

It is possible (not obvious - see Frisch Eqs. (6.17) &(6.24)) to show that $\Pi^{(K)}$ can be expressed as a Fourier sine transform of a particular 3rd order correlation function of velocity increments:

$$\begin{aligned}\Pi^{(K)} &= -\frac{1}{8\pi^2}\int d\mathbf{r}\frac{\sin(Kr)}{r}\,\nabla_{\mathbf{r}}\cdot\left\langle|\delta\mathbf{v}(\mathbf{r})|^2\,\delta\mathbf{v}(\mathbf{r})\right\rangle \\ &= -\frac{1}{6\pi}\int_0^\infty dr\frac{\sin(Kr)}{r}\,(1+r\partial_r)(3+r\partial_r)(5+r\partial_r)\frac{\langle\delta\mathbf{v}_{\|}(\mathbf{r})^3\rangle}{r}\end{aligned}$$

Kolmogorov's $\frac{4}{5}$-law

Letting $x = Kr$, the relation $\Pi^{(K)} = \epsilon$ becomes

$$\frac{1}{6\pi}\int_0^\infty dx \frac{\sin x}{x} F\left(\frac{x}{K}\right) = -\epsilon$$

where

$$F(r) = (1 + r\partial_r)(3 + r\partial_r)(5 + r\partial_r)\frac{S_3(r)}{r}.$$

$F(r)$ must become constant, c, as $K \to \infty$. Noting that

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

we need $c = -12\epsilon$.

Thus in the inertial range limit $K \to \infty$, $S_r(r)$ satisfies an ODE:

$$(1 + r\partial_r)(3 + r\partial_r)(5 + r\partial_r)\frac{S_3(r)}{r} = -12\epsilon$$

The general solution is

$$S_3(r) = c_1 r^{-4} + c_2 r^{-2} + c_3 - \frac{4}{5}\epsilon r.$$

Solution vanishing as $r \to 0$ is

$$S_3(r) = -\frac{4}{5}\epsilon r. \qquad (7)$$

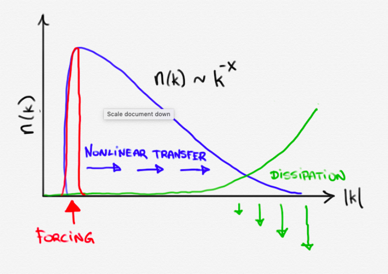
This is Kolmogorov's $\frac{4}{5}$-law.

Nonlinear wave equations and wave turbulence

Besides N-S, many nonlinear PDEs have conserved quantities. Expect similar cascade phenomena when subjected to forcing and dissipation that are separated in scale.
Wave turbulence is the theory of cascade phenomena in nonlinear dispersive wave equations:

$$\frac{\partial\psi}{\partial t} = \mathcal{L}(\psi) + \mathcal{N}(\psi) + \Gamma,$$

where Γ represents forcing and dissipation.



Wave spectrum

$$n_{\mathbf{k}} = \langle \hat{\psi}_{\mathbf{k}} \hat{\psi}^{*}_{\mathbf{k}} \rangle,$$

analogous to $E(k)$ in N-S.

Example 1: Nonlinear Schrodinger equation

NLS is a Hamiltonian equation for complex field, $\psi(\mathbf{x}, t)$:

$$i\frac{\partial\psi}{\partial t} = -\nabla^2\psi + V(\mathbf{x})\psi + g\left|\psi\right|^2\psi$$
$$= \frac{\delta\mathcal{H}}{\delta\psi^*},$$

where (we take $V(\mathbf{x}) = 0$)

$$\mathcal{H} = \int d\mathbf{x}\left[\left|\nabla\psi\right|^2 + \frac{g}{2}\left|\psi\right|^4\right].$$

Two conservation laws:

- $\mathcal{H}$.
- $\mathcal{N} = \int d\mathbf{x}\left|\psi\right|^2$.

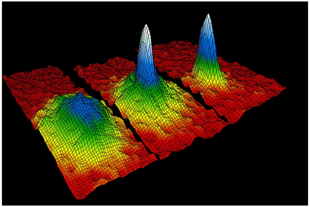


Image: NIST/JILA/CU-Boulder.

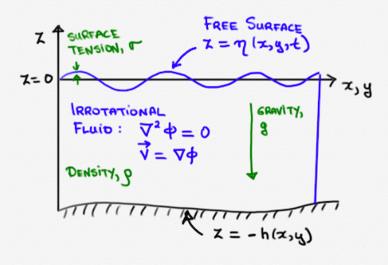
Applications:

- Bose-Einstein condensates.
- Nonlinear optics.
- Modulation of monochromatic dispersive wave trains.

Example 2: Water wave equations

Motion of free fluid surface subject to gravity and surface tension:

Mathematical model: incompressible, irrotational fluid with free surface $z = \eta(x, y, t)$:



Two limiting regimes: capillary waves (short wavelengths) and gravity waves (long wavelengths).

Example 2: Water wave equations

$$\begin{aligned}
\nabla^2\phi &= 0 && \text{on } -h(x,y) < z < \eta(x,y,t)\\
\frac{\partial\phi}{\partial z} + \nabla\phi\cdot\nabla h &= 0 && \text{on } z = h(x,y)\\
\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta &= \frac{\partial\eta}{\partial z} && \text{on } z = \eta(x,y,t)\\
\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + g\eta &= \frac{\sigma}{\rho}\nabla\cdot\left(\tfrac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right) && \text{on } z = \eta(x,y,t)
\end{aligned}$$

Conservation law: $H = \int \mathcal{H}\, dxdy$

$$\mathcal{H} = \int_{-h(x,y)}^{\eta(x,y,t)} \frac{1}{2}\left|\nabla\phi\right|^2\, dz + \frac{1}{2}\eta^2 + \frac{\sigma}{\rho}\left(\sqrt{1+|\nabla\eta|^2}-1\right)$$

2D Hamiltonian structure is non-obvious (Zakharov, 1968)

Example 3: Barotropic Potential Vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2\psi - F\psi) + \beta\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\nabla^2\psi}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\nabla^2\psi}{\partial x} = 0\,.$$

Velocity ("geostrophic wind"):

$$\mathbf{v} = \nabla \times \psi(x,y,t)\,\hat{\mathbf{z}}.$$

Two conservation laws:

$$E = \frac{1}{2}\int d\mathbf{x}\left[|\nabla\psi|^2 + F\psi^2\right],$$
$$Q = \frac{1}{2}\int d\mathbf{x}\left[\nabla^2\psi - F\psi\right]^2.$$

E is the energy. Q is the "potential enstrophy".

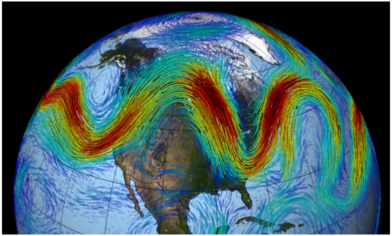


Image: NASA/GSFC.

BPV equation models the large scale dynamics of Rossby waves.

Example 4: Föppl-von Kármán equation

Displacement, ζ, and 2D stress, χ, in a thin elastic plate satisfy:

$$\rho \frac{\partial^2 \zeta}{\partial t^2} = -\frac{D}{h}\nabla^4 \zeta + \mathcal{N}\left[\chi, \zeta\right]$$
$$\nabla^4 \chi = -\frac{E}{2}\,\mathcal{L}\left[\zeta, \zeta\right],$$

Constants: ρ - density, h - thickness, D - bending stiffness, E - Young's modulus.

$$\mathcal{N}\left[f, g\right] = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 g}{\partial y^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 g}{\partial x^2}.$$

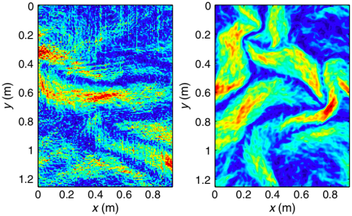


Image: Miquel et al., PRL (2013).

Fourier space representation of equations of motion

With appropriate choice of (complex) variables, $a_{\mathbf{k}}$, the equations for each of these examples can be written (roughly) as:

$$\partial_t a_{\mathbf{k}} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2\, T^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}\, a_{\mathbf{k}_1} a_{\mathbf{k}_2}\, \delta^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2}$$

or

$$\partial_t a_{\mathbf{k}} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\, T^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}\, a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3}\, \delta^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}$$

Here we use compactified notation for the Dirac delta function:

$$\begin{aligned} \delta^{\mathbf{k}}_{\mathbf{k}_1\mathbf{k}_2} &= \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \\ \delta^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3} &= \delta(\mathbf{k} + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3). \end{aligned}$$

Example: the Nonlinear Schrodinger Equation

NLS is equivalent to

$$\partial_t a_{\mathbf{k}} + i\omega_{\mathbf{k}} a_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \, T^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3} \, a^*_{\mathbf{k}_1} a_{\mathbf{k}_2} a_{\mathbf{k}_3} \, \delta^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3}$$

where

$$\begin{aligned} a_{\mathbf{k}} &= \hat{\psi}_{\mathbf{k}} \\ \omega_{\mathbf{k}} &= k^2 \\ T^{\mathbf{k}\mathbf{k}_1}_{\mathbf{k}_2\mathbf{k}_3} &= -i\, g \end{aligned}$$

The cubic nonlinearity means that waves interact in quartets. The NLS is an example of a 4-wave interaction.

Example: the Barotropic Potential Vorticity Equation

The BPV equation is equivalent to

$$\frac{\partial a_{\mathbf{k}}}{\partial t} + i\,\omega_{\mathbf{k}}\,a_{\mathbf{k}} = \int d\mathbf{k}_1 d\mathbf{k}_2\, W^{\mathbf{k}}_{\mathbf{k}_1\,\mathbf{k}_2}\, a_{\mathbf{k}_1}\, a_{\mathbf{k}_2}\, \delta^{\mathbf{k}}_{\mathbf{k}_1\,\mathbf{k}_2},$$

where

$$a_{\mathbf{k}} = \frac{k^2+F}{\sqrt{|k_x|}}\,\hat{\psi}_{\mathbf{k}}$$

$$\omega_{\mathbf{k}} = -\frac{k_x}{k^2+F}$$

$$W^{\mathbf{p}}_{\mathbf{q}\,\mathbf{r}} = -\frac{1}{2}\sqrt{\frac{|q_x|\,|r_x|}{|p_x|}}\;\frac{(\mathbf{q}\times\mathbf{r})_z\,(q^2-r^2)}{(q^2+F)\,(r^2+F)}.$$

The quadratic nonlinearity means that waves interact in triads. The BPV is an example of a 3-wave interaction.

Wave action variables

The variables $a_{\mathbf{k}}$ are chosen so

1. the (quadratic part of the) spectral energy density takes diagonal form:
$$E_{\mathbf{k}} = |\omega_{\mathbf{k}}|\; n_{\mathbf{k}}.$$
2. lowest order “non-resonant" terms are removed from the equations of motion.

Point 1 can be messy but is usually straightforward. Point 2, although not always needed, can be very non-trivial.

The $a_{\mathbf{k}}$ for water waves are non-trivial in this respect:

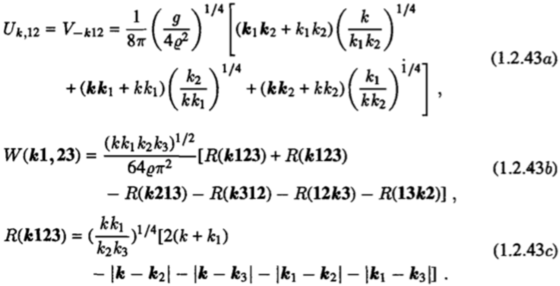
$$a_{\mathbf{k}} = \frac{1}{\sqrt{2\lambda_{\mathbf{k}}}}\eta_{\mathbf{k}} + i\sqrt{\frac{\lambda_{\mathbf{k}}}{2}}\varphi_{\mathbf{k}}$$

where $\eta_{\mathbf{k}}$ and $\varphi_{\mathbf{k}}$ are the (2D) Fourier transforms of $\eta(x, y, t)$ and $\phi(x, y, z = \eta(x, y, t), t)$ respectively and

$$\lambda_{\mathbf{k}} = \frac{\omega_{\mathbf{k}}}{g + \sigma k^2/\rho}$$
$$\omega_{\mathbf{k}} = \sqrt{k(g + \sigma k^2/\rho)\tanh(kh)}.$$

Interaction coefficients in wave action variables

For the water wave example, the transition to wave action variables results in some very complicated expressions for the wave interaction coefficients (from ZLF book):



$$U_{\boldsymbol{k},12} = V_{-\boldsymbol{k}12} = \frac{1}{8\pi}\left(\frac{g}{4\varrho^2}\right)^{1/4}\left[(\boldsymbol{k}_1\boldsymbol{k}_2 + k_1k_2)\left(\frac{k}{k_1k_2}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_1 + kk_1)\left(\frac{k_2}{kk_1}\right)^{1/4} + (\boldsymbol{k}\boldsymbol{k}_2 + kk_2)\left(\frac{k_1}{kk_2}\right)^{1/4}\right], \qquad (1.2.43a)$$

$$W(\boldsymbol{k}\mathbf{1},\mathbf{23}) = \frac{(kk_1k_2k_3)^{1/2}}{64\varrho\pi^2}[R(\boldsymbol{k}\mathbf{123}) + R(\boldsymbol{k}\mathbf{123}) - R(\boldsymbol{k}\mathbf{213}) - R(\boldsymbol{k}\mathbf{312}) - R(\mathbf{12}\boldsymbol{k}\mathbf{3}) - R(\mathbf{13}\boldsymbol{k}\mathbf{2})], \qquad (1.2.43b)$$

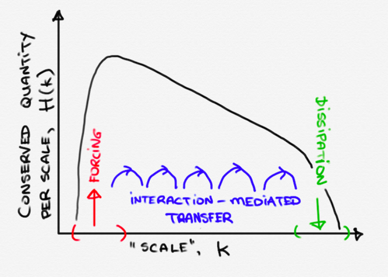
$$R(\boldsymbol{k}\mathbf{123}) = \left(\frac{kk_1}{k_2k_3}\right)^{1/4}[2(k + k_1) - |\boldsymbol{k} - \boldsymbol{k}_2| - |\boldsymbol{k} - \boldsymbol{k}_3| - |\boldsymbol{k}_1 - \boldsymbol{k}_2| - |\boldsymbol{k}_1 - \boldsymbol{k}_3|]. \qquad (1.2.43c)$$

The Föppl-von Kármán example is also quite complicated. Going forward we will take the equation for $a_{\mathbf{k}}$ as given.

Cascade phenomena in general

Hydrodynamics is one a of a broad class of driven-dissipative systems that exhibit cascade phenomena. Examples:

- Wave turbulence.
- Condensed matter: BECs
- Soft matter: Aggregation-fragmentation
- Self-organised criticality

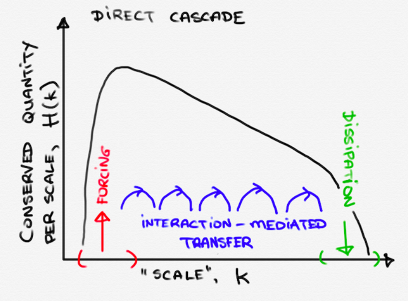


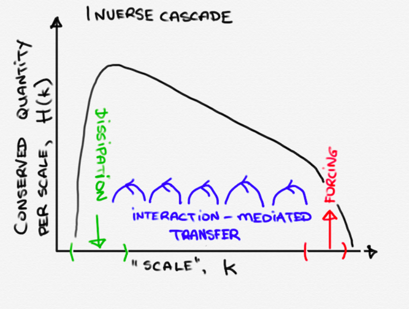
Common features:

- a conserved quantity, H, preserved by interactions,
- sources and sinks of H that are *strongly separated*.

Direct and inverse cascades

Distinguish between direct and inverse cascades depending of the "direction" of transfer:





Schematic structure of WT problems

Schematically, the equation of motion in Fourier space is:

$$\frac{\partial a_{\mathbf{k}}}{\partial t} \sim -i\,\omega_{\mathbf{k}}\,a_{\mathbf{k}} + \int (d\mathbf{k})^{n-1}\, W^{(n)}_{\mathbf{k}} (a_{\mathbf{k}})^{n-1}\,\delta(\mathbf{k}). \qquad (8)$$

Many systems of interest are scale invariant. We can write

$$\omega_{\mathbf{k}} = c\,k^{\alpha}$$

$$W^{(n)}_{\mathbf{k}\,\mathbf{k}_1\ldots\mathbf{k}_{n-1}} = g\,k^{\gamma_n}\,f_{\mathbf{k}\,\mathbf{k}_1\ldots\mathbf{k}_{n-1}}.$$

Here α and γ_n are scaling exponents and c and g are dimensional constants.

Spectral energy density in isotropic case:

$$\begin{aligned} E &= \int |\omega_{\mathbf{k}}|\; n_{\mathbf{k}}\, d\mathbf{k} \\ &= \int E_{\mathbf{k}}\, d\mathbf{k} = \int_0^\infty E_k\, dk. \end{aligned}$$

Scale-by-scale energy budget:

$$\frac{\partial E_k}{\partial t} = -\frac{\partial J_k}{\partial k}$$

Physical dimensions of relevant quantities

Dimensional analysis allows us to orient ourselves before doing any detailed calculations.

$$[k] = L^{-1} \quad [d\mathbf{k}] = L^{-d} \quad [\delta(\mathbf{k})] = L^{d} \quad [\omega_{\mathbf{k}}] = T^{-1}.$$

Since $E = \int \omega_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}} d\mathbf{k}$:

$$[a_{\mathbf{k}}] = E^{\frac{1}{2}}\, T^{\frac{1}{2}}\, L^{\frac{d}{2}}.$$

From the dynamical equation we can then infer that

$$[c] = L^{\alpha}\, T^{-1}$$

$$[g] = E^{\frac{2-n}{2}}\, T^{-\frac{n}{2}}\, L^{\left(\frac{n-2}{2}\right)d+\gamma_n}.$$

From the definition of the spectrum $\langle a_{\mathbf{k}} a^*_{\mathbf{k}'} \rangle = n(\mathbf{k})\, \delta(\mathbf{k}-\mathbf{k}')$ we find

$$[n_{\mathbf{k}}] = E T.$$

Final ingredient is the energy dissipation per unit volume, J:

$$[J] = E T^{-1} L^{-d}.$$

Look for universal steady states

Assume that in the inertial range, $n_{\mathbf{k}}$ depends on c, g, J and k only:

$$n_{\mathbf{k}} = c^u \, g^v \, J^w \, k^{-x}$$

Dimensional analysis:

$$ET = \left(L^\alpha T^{-1}\right)^u \left(E^{\frac{2-n}{2}} \, T^{-\frac{n}{2}} L^{\left(\frac{n-2}{2}\right)d+\gamma_n}\right)^v \left(ET^{-1}L^{-d}\right)^w L^x$$

Get 3 equations for 4 unknowns, u, v, w and x:

$$\begin{aligned} \left(\frac{2-n}{2}\right) v + w &= 1 \\ -u - \frac{n}{2} v - w &= 1 \\ \alpha\, u + \left(\left(\frac{n-2}{2}\right) d + \gamma_n\right) v - d\, w &= -x. \end{aligned}$$

Kolmogorov dimensional argument is under-determined for wave turbulence without additional constraints.

Dimensionally consistent spectra

We can solve these equations for any fixed x:

$$u = \frac{2\gamma_n + (n-1)d - (n-1)x}{(n-1)\alpha - \gamma_n}$$

$$v = -\frac{2\alpha + d - x}{(n-1)\alpha - \gamma_n}$$

$$w = \frac{(n-2)x + 2\alpha - 2\gamma_n - (n-2)d}{2((n-1)\alpha - \gamma_n)}.$$

In principle, any scaling is possible but special spectra appear as limiting cases.

Special cases:

- $u = 0$ - no c dependence:

$$x = \frac{2\gamma_n}{n-1} + d$$

This is the Kolmogorov-Zakharov spectrum.

- $w = 0$ - no J dependence:

$$x = \frac{2\gamma_n - 2\alpha}{n-2} + d.$$

This is the Generalised Phillips (or "Critical balance") spectrum.

Timescales and the weakly nonlinear regime

If $n_{\mathbf{k}} \sim k^{-x}$ then

$$a_{\mathbf{k}} \sim k^{-\frac{x+d}{2}}.$$

There are two timescales in Eq. (8). A linear timescale results from comparing $\partial_t a_{\mathbf{k}}$ to $\omega_{\mathbf{k}} a_{\mathbf{k}}$:

$$\tau_{\mathrm{L}} \sim k^{-\alpha} \tag{9}$$

A non–linear timescale results from comparing $\partial_t a_{\mathbf{k}}$ to the interaction term:

$$\tau_{\mathrm{NL}} \sim k^{-\gamma_n + \left(\frac{n}{2}-1\right)(x-d)}. \tag{10}$$

If the $\tau_{\mathrm{L}} \ll \tau_{\mathrm{NL}}$ as $k \to \infty$, the nonlinear evolution becomes slow as we enter the inertial range:

$$\frac{\tau_{\mathrm{L}}}{\tau_{\mathrm{NL}}} \sim k^{-\alpha + \gamma_n - \frac{n-2}{2}(x-d)} \ll 1$$

This is equivalent to assuming the nonlinearity becomes weak.

Consistency of weak nonlinearity: weak vs strong wave turbulence

KZ spectrum:
If $x = \frac{2\gamma_n}{n-1} + d$,

$$\frac{\tau_{\mathrm{L}}}{\tau_{\mathrm{NL}}} \sim k^{\frac{\gamma_n}{n-1}-\alpha} \ll 1.$$

is violated as $k \to \infty$ if
$\gamma_n - (n-1)\alpha > 0$.
This is the "breakdown criterion" of Newell et al. (2001)
If $\gamma_n - (n-1)\alpha > 0$, the KZ spectrum is inconsistent with weak nonlinearity.

Generalised Phillips' spectrum:
If $x = \frac{2\gamma_n - \alpha}{n-2} + d$

$$\frac{\tau_{\mathrm{L}}}{\tau_{\mathrm{NL}}} \sim 1.$$

Hence the term "critical balance". Generalised Phillips scaling is believed to be relevant to strong wave turbulence.

Finite and infinite capacity cascades

If forcing supplies energy at a constant rate, $E \sim Jt$, how long does it take to "fill" the KZ spectrum?

The energy of the system is

$$\begin{aligned} E &\sim \int_{k_F}^{k_D} \omega_k n_k k^{d-1} dk \\ &\sim \int_{k_F}^{k_D} k^{\alpha} k^{-\frac{2\gamma_n}{n-1}-d} k^{d-1} dk \\ &\sim \int_{k_F}^{k_D} k^{\left(\alpha-\frac{2\gamma_n}{n-1}\right)-1} dk \\ &\sim k_D^{\alpha-\frac{2\gamma_n}{n-1}} \end{aligned}$$

Infinite capacity if

$$\alpha - \frac{2\gamma_n}{n-1} > 0.$$

Finite capacity if

$$\alpha - \frac{2\gamma_n}{n-1} < 0.$$

Finite capacity systems might be expected to exhibit a *dissipative anomaly* since the capacity of the spectrum will be exceeded in finite time even as $k_D \to \infty$.